LECTURE 27 (MAY6)

Split Weil classes are absolute. The third step in the proof of Deligne's theorem is to show that split Weil classes are absolute. We begin by describing a special class of abelian varieties of split Weil type where this can be proved directly.

Let V_0 be a rational Hodge structure of even rank d and type $\{(1,0),(0,1)\}$. Let ψ_0 be a Riemann form that polarizes V_0, and W_0 an isotropic subspace of dimension $d/2$. (For example, $V_0^{1,0}$ is an isotropic subspace of dimension $d/2$ over $\mathbb{C}$, and because ψ_0 is defined over $\mathbb{Q}$, it will also have isotropic subspaces of the same dimenension over $\mathbb{Q}$.) We also fix an element $\zeta \in E^\times$ with $\bar{\zeta} = -\zeta$.

Now set $V = V_0 \otimes_\mathbb{Q} E$, with Hodge structure induced by the isomorphism

$$V \otimes_\mathbb{Q} \mathbb{C} \simeq V_0 \otimes_\mathbb{Q} \left(E \otimes_\mathbb{Q} \mathbb{C}\right) \simeq \bigoplus_{s \in S} V_0 \otimes_\mathbb{Q} \mathbb{C}.$$

Define a $\mathbb{Q}$-bilinear form $\psi \colon V \times V \to \mathbb{Q}$ by the formula

$$\psi(v_0 \otimes e, v_0' \otimes e') = \mathrm{Tr}_{E/\mathbb{Q}}\left(e\overline{e'}\right) \cdot \psi_0(v_0, v_0').$$

This is a Riemann form on V, for which $W = W_0 \otimes_\mathbb{Q} E$ is an isotropic subspace of dimension $d/2$. By Lemma 26.6, there is a unique E-hermitian form $\phi \colon V \times V \to E$ such that $\psi = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta\phi)$; clearly W is a totally isotropic subspace of dimension $d/2$ for ϕ. By Corollary 26.4, (V, ϕ) is split, and V is therefore of split Weil type. Let A_0 be an abelian variety with $H^1(A_0, \mathbb{Q}) = V_0$. The integral lattice of V_0 induces an integral lattice in $V = V_0 \otimes_\mathbb{Q} E$. We denote by $A_0 \otimes_\mathbb{Q} E$ the corresponding abelian variety. It is of split Weil type since V is.

The next result is the key to proving that split Weil classes are absolute.

Proposition 27.1. *Let A_0 be an abelian variety with $H^1(A_0, \mathbb{Q}) = V_0$ as above, and define $A = A_0 \otimes_\mathbb{Q} E$. Then the subspace $\bigwedge_E^d H^1(A, \mathbb{Q})$ of $H^d(A, \mathbb{Q})$ consists entirely of absolute Hodge classes.*

Proof. We have $H^d(A, \mathbb{Q}) \simeq \bigwedge_\mathbb{Q}^d H^1(A, \mathbb{Q})$, and the subspace

$$\bigwedge_E^d H^1(A, \mathbb{Q}) \simeq \bigwedge_E^d V_0 \otimes_\mathbb{Q} E \simeq \left(\bigwedge_\mathbb{Q}^d V_0\right) \otimes_\mathbb{Q} E \simeq H^d(A_0, \mathbb{Q}) \otimes_\mathbb{Q} E$$

consists entirely of Hodge classes by Proposition 26.9. But since $\dim A_0 = d/2$, the space $H^d(A_0, \mathbb{Q})$ is generated by the fundamental class of a point, which is clearly absolute. This implies that every class in $\bigwedge_E^d H^1(A, \mathbb{Q})$ is absolute. $\square$

The following theorem, together with Principle B (from Theorem 23.1), completes the proof of Deligne's theorem.

Theorem 27.2. *Let E be a CM-field, and let A be an abelian variety of split Weil type (relative to E). Then there exists a family $\pi \colon \mathcal{A} \to B$ of abelian varieties, with B irreducible and quasi-projective, such that the following three things are true:*

(a) $\mathcal{A}_0 = A$ for some point $0 \in B$.
(b) For every $t \in B$, the abelian variety $\mathcal{A}_t = \pi^{-1}(t)$ is of split Weil type (relative to E).
(c) The family contains an abelian variety of the form $A_0 \otimes_\mathbb{Q} E$.

In the remainder of the lecture, we are going to prove Theorem 27.2. Throughout, we let $V = H^1(A, \mathbb{Q})$, which is an E-vector space of some even dimension d. The polarization on A corresponds to a Riemann form $\psi \colon V \times V \to \mathbb{Q}$, with the property that the Rosati involution acts as complex conjugation on E. Fix a totally imaginary element $\zeta \in E^\times$; then $\psi = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta\phi)$ for a unique E-hermitian form ϕ by Lemma 26.6. Since A is of split Weil type, the pair (V, ϕ) is split.

As before, let D be the period domain, whose points parametrize Hodge structures of type $\{(1,0),(0,1)\}$ on V that are polarized by the form ψ. Let $D^{\mathrm{sp}} \subseteq D$ be the subset of those Hodge structures that are of split Weil type (relative to E, and with polarization given by ψ). Our first task is to show that D^{sp} is a complex manifold (and, in fact, a hermitian symmetric domain).

We begin by observing that there are essentially $2^{[E:\,\mathbb{Q}]}/2$ many different choices for the totally imaginary element ζ, up to multiplication by totally positive elements in $F^\times$. Indeed, if we fix a choice of $i = \sqrt{-1}$, and define $\varphi_\zeta \colon S \to \{0,1\}$ by the rule

(27.3)
$$\varphi_\zeta(s) = \begin{cases} 1 & \text{if } s(\zeta)/i > 0, \\ 0 & \text{if } s(\zeta)/i < 0, \end{cases}$$

then $\varphi_\zeta(s) + \varphi_\zeta(\bar{s}) = 1$ because $\bar{s}(\zeta) = -s(\zeta)$, and so φ_ζ is a CM-type for E. If we change ζ by a totally positive element $f \in F^\times$, then φ_ζ does not change (because $s(f) > 0$ for every $s \in S$). Conversely, one can show that any CM-type of the CM-field E is obtained in this way. Indeed, for a given CM-type $\varphi \colon S \to \{0,1\}$, we are looking for an element $f \in F^\times$ with the property that $s(f) > 0$ if $\varphi(s) = \varphi_\zeta(s)$, and $s(f) < 0$ if $\varphi(s) \neq \varphi_\zeta(s)$, because then $\varphi = \varphi_{f\zeta}$. The existence of such an element $f \in F^\times$ is an exercise in field theory.

Exercise 27.1. Let F be a totally real number field, and let $S = \mathrm{Hom}(F, \mathbb{R})$ be the set of all embeddings of F. Then for any function $\varphi \colon S \to \{-1, +1\}$, there is an element $f \in F^\times$ such that $\varphi(s) = \mathrm{sgn}\, s(f)$.

Lemma 27.4. *The subset D^{sp} of the period domain D is a hermitian symmetric domain; in fact, it is isomorphic to the product of $|S| = [E : \mathbb{Q}]$ many copies of Siegel upper halfspace.*

Proof. Recall that V is an E-vector space of even dimension d, and that the Riemann form is equal to $\psi = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta\phi)$ for a split E-hermitian form $\phi \colon V \times V \to E$ and a totally imaginary $\zeta \in E^\times$. The Rosati involution corresponding to ψ induces complex conjugation on E; this means that $\psi(ev, w) = \psi(v, \bar{e}w)$ for every $e \in E$.

By definition, D^{sp} parametrizes all Hodge structures of type $\{(1,0),(0,1)\}$ on V that admit ψ as a Riemann form and are of split Weil type relative to the given CM-field E. Such a Hodge structure amounts to a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

with $V^{0,1} = \overline{V^{1,0}}$, with the following two properties:

  (a) The action by E preserves $V^{1,0}$ and $V^{0,1}$.
  (b) The form $i\psi(x, \bar{y}) = \psi\big(h(i)x, \bar{y}\big)$ is positive definite on $V^{1,0}$.

Let $S = \mathrm{Hom}(E, \mathbb{C})$, and consider the isomorphism

$$V \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{s \in S} V_s,$$

where $V_s = V \otimes_{E,s} \mathbb{C}$. Since V_s is exactly the subspace on which $e \in E$ acts as multiplication by $s(e) \in \mathbb{C}$, the condition in (a) is equivalent to demanding that each complex vector space V_s decomposes as $V_s = V_s^{1,0} \oplus V_s^{0,1}$.

On the other hand, ϕ induces a hermitian form ϕ_s on each V_s, and we have

$$\psi(v, w) = \mathrm{Tr}_{E/\mathbb{Q}}\big(\zeta\phi(v, w)\big) = \sum_{s \in S} s(\zeta)\phi_s(v \otimes 1, w \otimes 1).$$

Therefore ψ polarizes the Hodge structure $V^{1,0} \oplus V^{0,1}$ if and only if $i\psi(x, \bar{x}) > 0$ for every nonzero $x \in V_s^{1,0}$. Writing

$$x = \sum_j v_j \otimes z_j \in V \otimes_{\mathbb{Q}} \mathbb{C},$$

we computed last time that

$$i\psi(x,\bar{x}) = is(\zeta)\phi_s(x,x).$$

Remembering the definition of φ_ζ in (27.3), we see that this will be positive definite exactly when the hermitian form $(-1)^{\varphi_\zeta(s)}\phi_s$ is positive definite on $V_s^{1,0}$.

In summary, Hodge structures of split Weil type on V for which ψ is a Riemann form are parametrized by a choice of $d/2$-dimensional subspace $V_s^{1,0} \subseteq V_s$, one for each $s \in S$, with the property that the hermitian form $x \mapsto (-1)^{\varphi_\zeta(s)}\phi_s(x,x)$ is positive definite on $V_s^{1,0}$. This information determines the subspace $V_s^{0,1}$ as the orthogonal complement of $V_s^{1,0}$ with respect to ϕ_s. Since we have $a_s = b_s = d/2$ for every $s \in S$ (by Corollary 26.4), the hermitian form ϕ_s has signature $(d/2, d/2)$; this implies that the space

$$D_s = \left\{ W \in \mathrm{Grass}_{d/2}(V_s) \mid (-1)^{\varphi_\zeta(s)}\phi_s(x,x) > 0 \text{ for } 0 \neq x \in W \right\}$$

is isomorphic to the usual Siegel upper halfspace. The parameter space D^{sp} for our Hodge structures is therefore the hermitian symmetric domain

$$D^{\mathrm{sp}} \simeq \prod_{s \in S} D_s.$$

In particular, it is a connected complex manifold. $\qquad\square$

To be able to satisfy the final condition in Theorem 27.2, we need to know that D^{sp} contains Hodge structures of the form $V_0 \otimes_{\mathbb{Q}} E$. This is the content of the following lemma.

Lemma 27.5. *With notation as above, there is a rational Hodge structure V_0 of weight one, such that $V_0 \otimes_{\mathbb{Q}} E$ belongs to D^{sp}.*

Proof. Since the pair (V, ϕ) is split, there is a totally isotropic subspace $W \subseteq V$ of dimension $\dim_E W = d/2$. Arguing as in the proof of Corollary 26.4, we can therefore find a basis $v_1, \ldots, v_d$ for the E-vector space V, with the property that

$$\phi(v_i, v_{i+d/2}) = \zeta^{-1} \quad \text{for } 1 \leq i \leq d/2,$$
$$\phi(v_i, v_j) = 0 \quad \text{for } |i - j| \neq d/2.$$

Let V_0 be the $\mathbb{Q}$-linear span of $v_1, \ldots, v_d$; then we have $V = V_0 \otimes_{\mathbb{Q}} E$. Now define $V_0^{1,0} \subseteq V_0 \otimes_{\mathbb{Q}} \mathbb{C}$ as the $\mathbb{C}$-linear span of the vectors $h_k = v_k + iv_{k+d/2}$ for $k = 1, \ldots, d/2$. Evidently, this gives a Hodge structure of weight one on V_0, hence a Hodge structure on $V = V_0 \otimes_{\mathbb{Q}} E$. It remains to show that ψ polarizes this Hodge structure. But we compute that

$$i\psi\left(\sum_{j=1}^{d/2} a_j h_j, \sum_{k=1}^{d/2} \overline{a_k h_k}\right) = \sum_{k=1}^{d/2} |a_k|^2 \psi\big(iv_k - v_{k+d/2}, v_k - iv_{k+d/2}\big)$$

$$= 2 \sum_{k=1}^{d/2} |a_k|^2 \psi(v_k, v_{k+d/2})$$

$$= 2 \sum_{k=1}^{d/2} |a_k|^2 \mathrm{Tr}_{E/\mathbb{Q}}\big(\zeta\phi(v_k, v_{k+d/2})\big) = 2[E:\mathbb{Q}] \sum_{k=1}^{d/2} |a_k|^2,$$

which proves that $x \mapsto i\psi(x, \bar{x})$ is positive definite on the subspace $V_0^{1,0}$. The Hodge structure $V_0 \otimes_{\mathbb{Q}} E$ therefore belongs to D^{sp} as desired. $\qquad\square$

Finishing the proof of Deligne's theorem.

Proof of Theorem 27.2. As in Lecture 25, let $\mathcal{M}$ be the moduli space of abelian varieties of dimension $d/2 \cdot [E \colon \mathbb{Q}]$, with polarization of the same type as ψ, and level 3-structure. Then $\mathcal{M}$ is a quasi-projective complex manifold, and the period domain D is its universal covering space (with the Hodge structure on $H^1(A, \mathbb{Q})$ mapping to the point A). Let $B \subseteq \mathcal{M}$ be the locus of those abelian varieties whose endomorphism algebra contains E. Note that the original abelian variety A is contained in B. Since every element $e \in E$ is a Hodge class in $\mathrm{End}(A) \otimes \mathbb{Q}$, it is clear that B is a Hodge locus; in particular, B is a quasi-projective variety by the theorem of Cattani-Deligne-Kaplan. As before, we let $\pi \colon \mathcal{A} \to B$ be the restriction of the universal family of abelian varieties to B.

Now we claim that the preimage of B in D is precisely the set D^{sp} of Hodge structures of split Weil type. Indeed, the endomorphism ring of any Hodge structure in the preimage of B contains E by construction; since it is also polarized by the form ψ, all the conditions in Definition 26.8 are satisfied, and so the Hodge structure in question belongs to D^{sp}. Because D is the universal covering space of $\mathcal{M}$, this implies in particular that B is connected and smooth, hence a quasi-projective complex manifold.

The first two assertions are obvious from the construction, whereas the third follows from Lemma 27.5. This concludes the proof. $\square$

To complete the proof of Deligne's theorem, we have to show that every split Weil class is an absolute Hodge class. For this, we argue as follows. Consider the family of abelian varieties $\pi \colon \mathcal{A} \to B$ from Theorem 27.2. By Proposition 26.9, the space of split Weil classes $\bigwedge_E^d H^1(\mathcal{A}_t, \mathbb{Q})$ consists of Hodge classes for every $t \in B$. The family also contains an abelian variety of the form $A_0 \otimes_{\mathbb{Q}} E$, and according to Proposition 27.1, all split Weil classes on this particular abelian variety are absolute. But now B is irreducible, and so Principle B applies and shows that for every $t \in B$, all split Weil classes on $\mathcal{A}_t$ are absolute. This finishes the third step of the proof, and finally establishes Deligne's theorem.