Lecture 25 (April 29)

Last time, we talked about abelian varieties and Hodge structures of CM-type. Let E be a number field, and let $S = \operatorname{Hom}(E, \mathbb{C})$ be the set of its complex embeddings. The cardinality of S is equal to $[E : \mathbb{Q}]$. For every $s \in S$, we denote by $\bar{s}$ the conjugate embedding, meaning the composition of s with complex conjugation on $\mathbb{C}$. Recall that E is called a CM-field if there is an involution $\iota \in \operatorname{Aut}(E/\mathbb{Q})$ such that $\bar{s} = s \circ \iota$ for every $s \in S$; in other words, ι corresponds to complex conjugation under every embedding of E. From now on, we adopt the simplified notation

$$\bar{e} = \iota(e);$$

then we have $\overline{s(e)} = s(\bar{e})$ for every embedding $s \in S$. An abelian variety A is of CM-type if $\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a CM-field E and if $V = H^1(A, \mathbb{Q})$ is 1-dimensional as an E-vector space; this is equivalent to the condition that the Mumford-Tate group $\operatorname{MT}(A)$ is abelian.

We also constructed all Hodge structures of CM-type explicitly, starting from the following class of examples. Recall that

$$E \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{s \in S} \mathbb{C}, \quad e \otimes z \mapsto \sum_{s \in S} s(e) \cdot z,$$

is an isomorphism; the summand corresponding to an embedding $s \in S$ consists of all elements of $E \otimes_{\mathbb{Q}} \mathbb{C}$ on which every $e \in E$ acts as multiplication by the complex number $s(e)$. (In other words, this is exactly the decomposition into common eigenspaces for the action by E on itself.) By definition, a CM-type of E is a function $\varphi \colon S \to \{0, 1\}$ with the property that $\varphi(s) + \varphi(\bar{s}) = 1$ for every $s \in S$. It determines a Hodge structure E_φ of weight 1 on the $\mathbb{Q}$-vector space E by setting

$$E_\varphi \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{s \in S} \mathbb{C}^{\varphi(s), \varphi(\bar{s})}.$$

These are exactly the Hodge structures that appear as $H^1(A, \mathbb{Q})$, where A is an abelian variety of CM-type, with the CM-field being E. We proved last time that every Hodge structure of CM-type can be obtained (by duals, tensor products, and direct sums) from these basic Hodge structures of CM-type.

Moduli of abelian varieties. The proof of Deligne's theorem involves the construction of algebraic families of abelian varieties, in order to apply Principle B. For this, we shall use the existence of a fine moduli space for polarized abelian varieties with level structure.

Suppose that we are looking at moduli of some class of smooth projective varieties (such as abelian varieties). We would like to have a moduli space M whose points correspond to isomorphism classes of abelian varieties; and over M, there should be a universal family, such that every family of abelian varieties over some base S is the pullback of the universal family along a morphism $S \to M$. Because all abelian varieties have nontrivial isomorphisms, such a universal family cannot exist: from a nontrivial automorphism, one can construct a locally trivial family over $\mathbb{C}^*$ that is not globally trivial, and so this family cannot come from a morphism to M. The solution is to add some extra data that eliminates all nontrivial automorphisms.

Recall that if A is an abelian variety of dimension g, the subgroup $A[N]$ of its N-torsion points is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^{\oplus 2g}$. A *level N-structure* is a choice of symplectic isomorphism $A[N] \simeq (\mathbb{Z}/N\mathbb{Z})^{\oplus 2g}$. If we write $A \cong V/\Lambda$, then we have

$$A[N] \cong \frac{1}{N}\Lambda/\Lambda \cong \Lambda/N\Lambda,$$

and so we can also think of $A[N] \cong H_1(A, \mathbb{Z}/N\mathbb{Z})$ as the first homology of A with coefficients in $\mathbb{Z}/N\mathbb{Z}$.

Suppose that A comes with a polarization, that is with the first Chern class of an ample line bundle L; recall that this determines as isogeny $\theta\colon A \to \hat{A}$ to the dual abelian variety. The degree of the polarization is the degree of this isogeny. The polarization determines a bilinear form $\psi\colon H^1(A,\mathbb{Z}) \otimes H^1(A,\mathbb{Z}) \to \mathbb{Z}$ that polarizes the Hodge structure on $H^1(A,\mathbb{Z})$ (by the Hodge-Riemann bilinear relations). As in Lemma 5.2, we can choose a symplectic basis in $H^1(A,\mathbb{Z})$, and then the polarization has a certain type $m = (m_1, \dots, m_g)$, with $m_1 \mid m_2 \mid \cdots \mid m_g$.

Now say we have an automorphism $f\colon A \to A$ that preserves the given polarization and the given level N-structure. It corresponds to an automorphism

$$f^*\colon H^1(A,\mathbb{Z}) \to H^1(A,\mathbb{Z}),$$

that is an isomorphism of Hodge structures and also compatible with ψ. From the polarization, we construct a hermitian inner product

$$\langle v, w\rangle = \psi\big(h(i)v, w\big),$$

where $h(i)$ acts as i on the subspace $H^{1,0}(A)$, and as $-i$ on the subspace $H^{0,1}(A)$. Because f^* preserves this inner product, it is unitary, and therefore diagonalizable with eigenvalues of absolute value 1. But all eigenvalues are also algebraic integers, and so they are roots of unity (by Kronecker's theorem). Because f preserves the level N-structure, we also get that

$$f^*\colon H^1(A,\mathbb{Z}/N\mathbb{Z}) \to H^1(A,\mathbb{Z}/N\mathbb{Z})$$

is the identity; in other words, f^* is congruent to the identity modulo N. We can now apply the following lemma and conclude that f^* (and hence f) must be the identity.

Lemma 25.1. *Let A be an $n \times n$-matrix with integer entries, all of whose eigenvalues are of absolute value 1. If A is congruent to the identity modulo an integer $N \geq 3$, then all eigenvalues of A are equal to 1.*

Adding a polarization and a level N-structure therefore eliminates all nontrivial automorphisms. One can then prove the following theorem.

Theorem 25.2. *Fix $g \geq 1$ and a type $m = (m_1, \dots, m_g)$. For any $N \geq 3$, there is a smooth quasi-projective variety $\mathcal{M}_{g,m,N}$ that is a fine moduli space for g-dimensional abelian varieties with polarization of type m and level N-structure. In particular, we have a universal family of abelian varieties over $\mathcal{M}_{g,m,N}$.*

The relationship of this result with Hodge theory is the following. Fix an abelian variety A of dimension g, with level N-structure and polarization of type m. The polarization corresponds to an antisymmetric bilinear form $\psi\colon H^1(A,\mathbb{Z}) \otimes H^1(A,\mathbb{Z}) \to \mathbb{Z}$ that polarizes the Hodge structure; we shall refer to ψ as a *Riemann form.* Define $V_\mathbb{Z} = H^1(A,\mathbb{Z})$, and let D be the period domain that parametrizes all possible Hodge structures of type $\{(1,0),(0,1)\}$ on $V_\mathbb{Z}$ that are polarized by the form $\psi\colon V_\mathbb{Z} \otimes V_\mathbb{Z} \to \mathbb{Z}$. The period domain D is an open subset of a certain nonsingular algebraic subvariety of the Grassmannian $G(g, V_\mathbb{C})$. Indeed, a Hodge structure

$$V_\mathbb{C} = V^{1,0} \oplus V^{0,1}$$

on $V_\mathbb{Z}$ is completely determined by the g-dimensional subspace $V^{1,0}$; it has the property that $\psi(v,w) = 0$ for all $v, w \in V^{1,0}$, and that $i\psi(v,\bar{v}) > 0$ for every nonzero $v \in V^{1,0}$. The first condition defines a nonsingular closed subvariety; and the second condition an open subset of this subvariety (in the analytic topology). In particular, D is a complex manifold.

Example 25.3. When the polarization is principal (which is equivalent to ψ being unimodular), the period domain D is just the Siegel space $\mathcal{H}_g$

One can show that the period domain D is a simply connected Hermitian symmetric domain. In fact, D is isomorphic to the universal covering space of the quasi-projective variety $\mathcal{M}_{g,m,N}$. This is because a polarized abelian variety A' is completely determined by the polarized Hodge structure on $H^1(A', \mathbb{Z})$, but in order to associate to this Hodge structure a point in the period domain, we need to choose an isomorphism $H^1(A', \mathbb{Z}) \cong V_{\mathbb{Z}}$ that takes the given polarization to the pairing ψ. The additional data of such an isomorphism then turns D into an infinite-sheeted covering space of $\mathcal{M}_{g,m,N}$.

Reduction to abelian varieties of CM-type. We now come to the first step in the proof of Deligne's theorem, namely the reduction of the general problem to abelian varieties of CM-type. This is accomplished by the following theorem and Principle B (from Theorem 23.1).

Theorem 25.4. *Let A be an abelian variety, and let $\alpha \in H^{2p}(A, \mathbb{Q})$ be a Hodge class on A. Then there exists a family $\pi\colon \mathcal{A} \to B$ of abelian varieties, with B nonsingular, irreducible, and quasi-projective, such that the following is true:*

(a) $\mathcal{A}_0 \cong A$ for some point $0 \in B$.
(b) There is a Hodge class $\tilde{\alpha} \in H^{2p}(\mathcal{A}, \mathbb{Q})$ whose restriction to A equals α.
(c) For a dense set of $b \in B$, the abelian variety $\mathcal{A}_b = \pi^{-1}(b)$ is of CM-type.

Before giving the proof, let us briefly recall the following useful interpretation of period domains. Say D parametrizes all Hodge structures of weight n on a fixed rational vector space V that are polarized by a given bilinear form ψ. The set of real points of the $\mathbb{Q}$-algebraic group $G = \mathrm{Aut}(V, \psi)$ then acts transitively on D by the rule $(gV)^{p,q} = g \cdot V^{p,q}$ for $g \in G(\mathbb{R})$, and so $D \simeq G(\mathbb{R})/K$. Here K is the stabilizer of any given Hodge structure; this is contained in the unitary group for the inner product $\langle v, w \rangle = \psi(h(i)v, w)$, and therefore compact.

As we said last time, Hodge structures on V that are polarized by the bilinear form ψ are in one-to-one correspondence with homomorphisms of real algebraic groups $h\colon U(1) \to G(\mathbb{R})$; we denote the Hodge structure corresponding to h by the symbol V_h. Then $V_h^{p,q}$ is exactly the subspace of $V \otimes_{\mathbb{Q}} \mathbb{C}$ on which $h(z)$ acts as multiplication by z^{p-q}, and from this, it is easy to verify that $gV_h = V_{ghg^{-1}}$. In other words, the points of the period domain D can be thought of as conjugacy classes of a fixed homomorphism $h\colon U(1) \to G(\mathbb{R})$ under the action by $G(\mathbb{R})$.

Proof of Theorem 25.4. After choosing an ample line bundle on A, we may assume that the Hodge structure on $V = H^1(A, \mathbb{Q})$ is polarized by a Riemann form ψ. Let $G = \mathrm{Aut}(V, \psi)$, and recall that $M = \mathrm{MT}(A)$ is the smallest $\mathbb{Q}$-algebraic subgroup of G whose set of real points $M(\mathbb{R})$ contains the image of the homomorphism $h\colon U(1) \to G(\mathbb{R})$. Let D be the period domain whose points parametrize all possible Hodge structures of type $\{(1,0), (0,1)\}$ on V that are polarized by the form ψ. With $V_h = H^1(A, \mathbb{Q})$ as the base point, we then have $D \simeq G(\mathbb{R})/K$; the points of D are thus exactly the Hodge structures $V_{ghg^{-1}}$, for $g \in G(\mathbb{R})$ arbitrary.

The main idea of the proof is to consider the Mumford-Tate domain

$$D_h = M(\mathbb{R})/K \cap M(\mathbb{R}) \hookrightarrow D.$$

By definition, D_h consists of all Hodge structures of the form $V_{ghg^{-1}}$, for $g \in M(\mathbb{R})$. These are precisely the Hodge structures whose Mumford-Tate group is contained in M: indeed, for $g \in M(\mathbb{R})$, the image of the homomorphism $ghg^{-1}\colon U(1) \to G(\mathbb{R})$ is obviously contained in $M(\mathbb{R})$, and so the Mumford-Tate group of this Hodge structure must be contained in M (because the Mumford-Tate group is defined as the smallest $\mathbb{Q}$-algebraic subgroup containing the image, and M is one such subgroup). Note that D_h is a homogeneous space for the action of the real Lie group $M(\mathbb{R})$, and therefore a complex submanifold of the period domain D.

To find Hodge structures of CM-type in D_h, we appeal to a result by Borel. Since the image of h is abelian, it is contained in a maximal torus T of the real Lie group $M(\mathbb{R})$. One can show that, for a generic element ξ in the Lie algebra $\mathfrak{m}_{\mathbb{R}}$, this torus is the stabilizer of ξ under the adjoint action by $M(\mathbb{R})$. Now $\mathfrak{m}$ is defined over $\mathbb{Q}$, and so there exist elements $g \in M(\mathbb{R})$ arbitrarily close to the identity for which $\operatorname{Ad}(g)\xi = g\xi g^{-1}$ is rational. The stabilizer gTg^{-1} of such a rational point is then a maximal torus in M that is defined over $\mathbb{Q}$. The Hodge structure $V_{ghg^{-1}}$ is a point of the Mumford-Tate domain D_h, and by definition of the Mumford-Tate group, we have $\operatorname{MT}(V_{ghg^{-1}}) \subseteq gTg^{-1}$. In particular, $V_{ghg^{-1}}$ is of CM-type, because its Mumford-Tate group is abelian. This reasoning shows that D_h contains a dense set of points of CM-type.

To obtain an algebraic family of abelian varieties with the desired properties, we can now argue as follows. Let $\mathcal{M}$ be the moduli space of abelian varieties of dimension $\dim A$, with polarization of the same type as ψ, and level 3-structure. Then $\mathcal{M}$ is a smooth quasi-projective variety, and since it is a fine moduli space, it carries a universal family $\pi : \mathcal{A} \to \mathcal{M}$. The period domain D is the universal covering space of $\mathcal{M}$. Now we would like to replace $\mathcal{M}$ by the image of the Mumford-Tate domain D_h.

Recall from last time that the Mumford-Tate group $M = \operatorname{MT}(A)$ is the subgroup of $G = \operatorname{Aut}(V, \psi)$ that fixes all Hodge tensors, meaning all Hodge classes in all tensor powers

$$T^{p,q}(A) = H^1(A, \mathbb{Q})^{\otimes p} \otimes H_1(A, \mathbb{Q})^{\otimes q}.$$

Because M is an algebraic subgroup, we can finitely many such Hodge tensors $\tau_1, \dots, \tau_r$ such that M is exactly the stabilizer of $\tau_1, \dots, \tau_r$. A Hodge structure in D belongs to D_h iff its Mumford-Tate group is contained in M iff $\tau_1, \dots, \tau_r$ remain Hodge tensors. By the theorem of Cattani-Deligne-Kaplan, the subset of $\mathcal{M}$ where each τ_j stays a Hodge class is algebraic. So if we let $B \subseteq \mathcal{M}$ denote the connected component of this subvariety that contains the base point $0 \in \mathcal{M}$, then B is a quasi-projective algebraic variety; it is irreducible and nonsingular, being the image of the complex submanifold $D_h \subseteq D$. Let $\pi \colon \mathcal{A} \to B$ be the restriction of the universal family to B. Then (a) is clearly satisfied for this family.

Because B is the image of the Mumford-Tate domain D_h, the argument we gave above shows that B contains a dense set of points $b \in B$ such that the Mumford-Tate group of the abelian variety $\mathcal{A}_b = \pi^{-1}(b)$ is abelian; this means that $\mathcal{A}_b$ is an abelian variety of CM-type, and so we get (c). Since B is also contained in the Hodge locus of α, and since the monodromy action of $\pi_1(B, 0)$ on the space of Hodge classes has finite orbits, we can pass to a finite étale cover of B and arrange that α in invariant under monodromy. By the global invariant cycle theorem, it therefore comes from a Hodge class in $H^{2p}(\mathcal{A}, \mathbb{Q})$, and this gives (b). $\square$

Construction of split Weil classes. Let E be a CM-field; as usual, we let $S = \operatorname{Hom}(E, \mathbb{C})$ be the set of complex embeddings; it has $[E : \mathbb{Q}]$ elements.

Let V be a rational Hodge structure of type $\{(1,0), (0,1)\}$ whose endomorphism algebra contains E. We shall assume that $\dim_E V = d$ is an even number. Let $V_s = V \otimes_{E,s} \mathbb{C}$. Corresponding to the decomposition

$$E \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{s \in S} \mathbb{C}, \quad e \otimes z \mapsto \sum_{s \in S} s(e) z,$$

we get a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{s \in S} V_s.$$

The isomorphism is E-linear, where $e \in E$ acts on the complex vector space V_s as multiplication by $s(e)$. Since $\dim_{\mathbb{Q}} V = [E : \mathbb{Q}] \cdot \dim_E V$, each V_s has dimension d

over $\mathbb{C}$. By assumption, E respects the Hodge decomposition on V, and so we get an induced decomposition

$$V_s = V_s^{1,0} \oplus V_s^{0,1}.$$

Note that $\dim_{\mathbb{C}} V_s^{1,0} + \dim_{\mathbb{C}} V_s^{0,1} = d$.

Lemma 25.5. *The rational subspace* $\bigwedge_E^d V \subseteq \bigwedge_{\mathbb{Q}}^d V$ *is purely of type* $(d/2, d/2)$ *if and only if* $\dim_{\mathbb{C}} V_s^{1,0} = \dim_{\mathbb{C}} V_s^{0,1} = d/2$ *for every* $s \in S$.

Proof. We have

$$\left(\bigwedge\nolimits_E^d V\right) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigwedge\nolimits_{E\otimes_{\mathbb{Q}}\mathbb{C}}^d (V \otimes_{\mathbb{Q}} \mathbb{C}) \simeq \bigoplus_{s\in S} \bigwedge\nolimits_{\mathbb{C}}^d V_s \simeq \bigoplus_{s\in S} \left(\bigwedge\nolimits_{\mathbb{C}}^{p_s} V_s^{1,0}\right) \otimes \left(\bigwedge\nolimits_{\mathbb{C}}^{q_s} V_s^{0,1}\right),$$

where $p_s = \dim_{\mathbb{C}} V_s^{1,0}$ and $q_s = \dim_{\mathbb{C}} V_s^{0,1}$. The assertion follows because the Hodge type of each summand is evidently (p_s, q_s). $\square$

Let A be an abelian variety such that $H^1(A, \mathbb{Q}) \cong V$. Assuming that the condition in the lemma is satisfied, we get a subspace in $H^d(A, \mathbb{Q}) \cong \bigwedge_{\mathbb{Q}}^d V$ of dimension $[E : \mathbb{Q}]$ that consists entirely of Hodge classes. These classes are called *Hodge classes of Weil type.* They are not known to be algebraic in general. We will study them in more detail next time.