Lecture 24 (April 24)

Outline of the proof of Deligne's theorem. After these preliminaries about absolute Hodge classes, we can now start talking about Deligne's theorem.

Theorem 24.1 (Deligne). *All Hodge classes on abelian varieties are absolute.*

A key object in the proof are so-called "abelian varieties of CM-type", which are abelian varieties whose (rational) endomorphism algebra contains a CM-field. Let's first recall the necessary definitions.

Definition 24.2. A *CM field* is a number field E, such that for every embedding $s\colon E \hookrightarrow \mathbb{C}$, complex conjugation induces an automorphism of E that is independent of the embedding. In other words, E admits an involution $\iota \in \operatorname{Aut}(E/\mathbb{Q})$, such that for any embedding $s\colon E \hookrightarrow \mathbb{C}$, one has $\bar{s} = s \circ \iota$. Here $\bar{s}$ denotes the composition of the embedding s with complex conjugation on $\mathbb{C}$.

The fixed field of the involution is a totally real field F; concretely, this means that $F = \mathbb{Q}(\alpha)$, where α and all of its conjugates are real numbers. The field E is then of the form $F[x]/(x^2 - f)$, for some element $f \in F$ that is mapped to a negative number under all embeddings of F into $\mathbb{R}$. The simplest example of a CM-field is $\mathbb{Q}(\sqrt{-d})$ for a square-free positive integer d; the involution ι is just complex conjugation.

Definition 24.3. An abelian variety A is said to be *of CM-type* if a CM-field E is contained in $\operatorname{End}(A) \otimes \mathbb{Q}$, and if $H^1(A, \mathbb{Q})$ is one-dimensional as an E-vector space. In that case, we clearly have $2 \dim A = \dim_{\mathbb{Q}} H^1(A, \mathbb{Q}) = [E : \mathbb{Q}]$.

Example 24.4. It is easy to describe elliptic curves of CM-type. Write the elliptic curve as $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where τ is a complex number with $\operatorname{Im} \tau > 0$. Any rational endomorphism can be lifted to an endomorphism of $\mathbb{C}$, and is therefore of the form $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C}$. The lifting needs to preserve $\mathbb{Q} + \mathbb{Q}\tau$, and so we get $\lambda = a\tau + b$ and $\lambda\tau = c\tau + d$ for rational numbers $a, b, c, d \in \mathbb{Q}$. This gives

$$a\tau^2 + (b - c)\tau + d = 0,$$

and because τ has positive imaginary part, we get (for $a \neq 0$) that

$$\tau = \frac{c - b + \sqrt{(b - c)^2 + 4ad}}{2a}.$$

As long as $(b - c)^2 + 4ad < 0$, this is an imaginary quadratic extension of $\mathbb{Q}$, and therefore a CM-field. Observe that there are countably many possible values for τ, which are dense in the upper half-plane; so there are only countably many elliptic curves of CM-type, but they are dense in the space of all elliptic curves.

After this preliminary discussion of abelian varieties of CM-type, we return to Deligne's theorem. Let A be an abelian variety, and let $\alpha \in H^{2p}(A, \mathbb{Q})$ be a Hodge class. The proof consists of the following three steps.

1. The first step is to reduce the problem to abelian varieties of CM-type. This is done by constructing an algebraic family of abelian varieties that links a given A and a Hodge class in $H^{2p}(A, \mathbb{Q})$ to an abelian variety of CM-type and a Hodge class on it, and then applying Principle B.
2. The second step is to show that every Hodge class on an abelian variety of CM-type can be expressed as a sum of pullbacks of so-called split Weil classes. The latter are Hodge classes on certain special abelian varieties, constructed by linear algebra from the CM-field E and its embeddings into $\mathbb{C}$. This part of the proof is a simplification of Deligne's argument, due to Yves André.

3. The last step is to show that all split Weil classes are absolute. For a fixed CM-type, all abelian varieties of split Weil type are naturally parametrized by a certain hermitian symmetric domain; by Principle B, this allows to reduce the problem to split Weil classes on abelian varieties of a very specific form, for which the proof of the result is straightforward.

The original proof by Deligne uses Baily-Borel theory to show that certain families of abelian varieties are algebraic. In the presentation below, I am going to replace this by the following two results: the existence of a quasi-projective moduli space for polarized abelian varieties with level structure and the theorem of Cattani-Deligne-Kaplan concerning the algebraicity of Hodge loci.

Abelian varieties of CM-type. To motivate what follows, let's briefly look at a criterion for a simple abelian variety A to be of CM-type that involves the Mumford-Tate group $\mathrm{MT}(A)$. This is a certain algebraic group that serves as a sort of "symmetry group" of the Hodge structure on $H^1(A,\mathbb{Q})$.

Recall that a complex abelian variety A is uniquely determined by the Hodge structure on $H^1(A,\mathbb{Z})$, which is

$$H^1(A,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^1(A,\mathbb{C}) = H^{1,0}(A) \oplus H^{0,1}(A).$$

Indeed, writing $A = V/\Lambda$, we have natural isomorphisms

$$\begin{aligned} V &\cong H^0(A,\mathscr{T}_A) \cong H^0(A,\Omega_A^1)^* \cong H^{1,0}(A)^*, \\ \Lambda &\cong H_1(A,\mathbb{Z}) \cong H^1(A,\mathbb{Z})^*. \end{aligned}$$

Moreover, A is an abelian variety iff A is projective iff the Hodge structure on $H^1(A,\mathbb{Z})$ is polarized. From the rational Hodge structure $H^1(A,\mathbb{Q})$, we can only recover the lattice Λ up to finite index; therefore the Hodge structure on $H^1(A,\mathbb{Q})$ determines the abelian variety A only up to isogeny.

Now let's define the Mumford-Tate group. Suppose that V is a rational Hodge structure of weight n, with Hodge decomposition

$$V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}.$$

We can encode the decomposition into a morphism of real Lie groups

$$h\colon U(1) \to \mathrm{GL}(V_{\mathbb{R}})$$

from the circle group, by letting a complex number z with $|z| = 1$ act on the subspace $V^{p,q}$ as multiplication by $z^{p-q} = z^p|z|^q$. Due to Hodge symmetry, each $h(z)$ is actually a real endomorphism, because

$$\overline{h(z)v} = \overline{z^{p-q}v} = \bar{z}^{p-q}\bar{v} = z^{q-p}\bar{v} = h(z)\bar{v}$$

for $v \in V^{p,q}$. The Hodge decomposition is then exactly the decomposition into common eigenspaces for the commuting endomorphisms $h(z)$ with $z \in U(1)$; these eigenspaces correspond to characters of $U(1)$, which are of the form $z \mapsto z^k$ for $k \in \mathbb{Z}$.

We define the *Mumford-Tate group* $\mathrm{MT}(V)$ as the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ whose set of real points contains the image of h. In other words, we view $\mathrm{GL}(V)$ as an affine variety over $\operatorname{Spec}\mathbb{Q}$, defined by the determinant function on $\mathrm{End}(V)$, and $\mathrm{MT}(V)$ is the Zariski closure of $\operatorname{im} h$.

Lemma 24.5. *The Mumford-Tate group* $\mathrm{MT}(V)$ *is exactly the subgroup of* $\mathrm{GL}(V)$ *that fixes every Hodge class in every tensor product*

$$T^{k,\ell}(V) = V^{\otimes k} \otimes (V^*)^{\otimes \ell}.$$

Proof. One implication is easy. There are countably many Hodge classes in all the tensor products, and their joint stabilizer is defined by countably many algebraic equations with coefficients in $\mathbb{Q}$; therefore it is a $\mathbb{Q}$-algebraic subgroup M of $\mathrm{GL}(V)$. If we have a Hodge class of type (p,p) in some $T^{k,\ell}(V)$, then $h(z)$ acts on it as multiplication by $z^{p-p} = 1$, and so the image of h is contained in M. Because $\mathrm{MT}(V)$ is the Zariski closure of the image, we get $\mathrm{MT}(V) \subseteq M$. The other implication needs a bit of theory of algebraic groups, so I won't present it here. □

The Mumford-Tate group of an abelian variety A is $\mathrm{MT}(A) = \mathrm{MT}\big(H^1(A,\mathbb{Q})\big)$. By the lemma, it is exactly the subgroup of $\mathrm{GL}\big(H^1(A,\mathbb{Q})\big)$ that fixes every Hodge class in every tensor product

$$T^{k,\ell}(A) = H^1(A,\mathbb{Q})^{\otimes k} \otimes H_1(A,\mathbb{Q})^{\otimes \ell}.$$

We have the following nice criterion for simple abelian varieties to be of CM-type.

Proposition 24.6. *A simple abelian variety is of CM-type if and only if its Mumford-Tate group* $\mathrm{MT}(A)$ *is an abelian group.*

Proof. Let's look at the proof of the interesting direction, namely that $\mathrm{MT}(A)$ abelian implies that A is of CM-type. Let $H = H^1(A,\mathbb{Q})$. The abelian variety A is simple, which implies that $E = \mathrm{End}(A) \otimes \mathbb{Q}$ is a division algebra. (This is just Schur's lemma: because A does not have nontrivial abelian subvarieties, any endomorphism must be surjective with finite kernel, hence an isogeny.) It is also the space of Hodge classes in $\mathrm{End}_{\mathbb{Q}}(H)$, and therefore consists exactly of those endomorphisms that commute with $\mathrm{MT}(A)$. Because the Mumford-Tate group is abelian, its action splits $H^1(A,\mathbb{C})$ into a direct sum of character spaces

$$H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\chi} H_{\chi},$$

where $m \cdot h = \chi(m)h$ for $h \in H_{\chi}$ and $m \in \mathrm{MT}(A)$. Now any endomorphism of H_{χ} obviously commutes with $\mathrm{MT}(A)$, and is therefore contained in $E \otimes_{\mathbb{Q}} \mathbb{C}$. By counting dimensions, we find that

$$\dim_{\mathbb{Q}} E \geq \sum_{\chi} \big(\dim_{\mathbb{C}} H_{\chi}\big)^2 \geq \sum_{\chi} \dim_{\mathbb{C}} H_{\chi} = \dim_{\mathbb{Q}} H.$$

On the other hand, we have $\dim_{\mathbb{Q}} E \leq \dim_{\mathbb{Q}} H$; indeed, since E is a division algebra, the map $E \to H$, $e \mapsto e \cdot h$, is injective for every nonzero $h \in H$. Therefore $[E : \mathbb{Q}] = \dim_{\mathbb{Q}} H = 2 \dim A$; moreover, each character space H_{χ} is one-dimensional, and this implies that E is commutative, hence a field. To construct the involution $\iota \colon E \to E$ that makes E into a CM-field, choose a polarization $\psi \colon H \times H \to \mathbb{Q}$, and define ι by the condition that, for every $h, h' \in H$,

$$\psi(e \cdot h, h') = \psi\big(h, \iota(e) \cdot h'\big).$$

The fact that $i\psi$ is positive definite on the subspace $H^{1,0}(A)$ can then be used to show that ι is nontrivial, and that $\bar{s} = s \circ \iota$ for any embedding of E into the complex numbers. (We'll prove this below.) □

Hodge structures of CM-type. When A is an abelian variety of CM-type, $H^1(A,\mathbb{Q})$ is an example of a Hodge structure of CM-type. We now undertake a more careful study of this class of Hodge structures. Let V be a rational Hodge structure of weight n, with Hodge decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}.$$

Because the weight n is fixed, there is a one-to-one correspondence between such decompositions and group homomorphisms $h\colon U(1) \to \mathrm{GL}(V_{\mathbb{R}})$, with $h(z)$ acting as multiplication by $z^{p-q} = z^{2p-n}$ on the subspace $V^{p,q}$.

Definition 24.7. We say that V is a *Hodge structure of CM-type* if the following two equivalent conditions are satisfied:

(a) The group of real points of $\mathrm{MT}(V)$ is a compact torus.
(b) $\mathrm{MT}(V)$ is abelian and V is polarizable.

We mostly use (b) in what follows; the equivalence between (a) and (b) needs a bit of structure theory for algebraic groups, and so we'll skip it.

It is not hard to see that any Hodge structure of CM-type is a direct sum of irreducible Hodge structures of CM-type. Indeed, since V is polarizable, it admits a finite decomposition $V = V_1 \oplus \dots \oplus V_r$, with each V_i irreducible. As subgroups of $\mathrm{GL}(V) = \mathrm{GL}(V_1) \times \dots \times \mathrm{GL}(V_r)$, we then have $\mathrm{MT}(V) \subseteq \mathrm{MT}(V_1) \times \dots \times \mathrm{MT}(V_r)$, and since the projection to each factor is surjective, it follows that $\mathrm{MT}(V_i)$ is abelian. But this means that each V_i is again of CM-type. It is therefore sufficient to concentrate on irreducible Hodge structures of CM-type. For those, there is a nice structure theorem that we shall now explain.

Let V be an irreducible Hodge structure of weight n that is of CM-type, and as above, denote by $M = \mathrm{MT}(V)$ its Mumford-Tate group. Because V is irreducible, its algebra of endomorphisms

$$E = \mathrm{End}_{\mathbb{Q}\text{-HS}}(V)$$

must be a division algebra. In fact, since the endomorphisms of V as a Hodge structure are exactly the Hodge classes in $\mathrm{End}_{\mathbb{Q}}(V)$, we see that E consists of all rational endomorphisms of V that commute with $\mathrm{MT}(V)$. If $T_E = E^{\times}$ denotes the algebraic torus in $\mathrm{GL}(V)$ determined by E, then we get $\mathrm{MT}(V) \subseteq T_E$ because $\mathrm{MT}(V)$ is commutative by assumption.

Since $\mathrm{MT}(V)$ is commutative, it acts on $V \otimes_{\mathbb{Q}} \mathbb{C}$ by characters, and so we get a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\chi} V_{\chi},$$

where $m \in \mathrm{MT}(V)$ acts on $v \in V_{\chi}$ by the rule $m \cdot v = \chi(m) v$. Any endomorphism of V_{χ} therefore commutes with $\mathrm{MT}(V)$, and so $E \otimes_{\mathbb{Q}} \mathbb{C}$ contains the spaces $\mathrm{End}_{\mathbb{C}}(V_{\chi})$. As before, this leads to the inequality

$$\dim_{\mathbb{Q}} E \geq \sum_{\chi} \bigl(\dim_{\mathbb{C}} V_{\chi}\bigr)^2 \geq \sum_{\chi} \dim_{\mathbb{C}} V_{\chi} = \dim_{\mathbb{Q}} V.$$

On the other hand, we have $\dim_{\mathbb{Q}} V \leq \dim_{\mathbb{Q}} E$ because every nonzero element in E is invertible. It follows that each V_{χ} is one-dimensional, that E is commutative, and therefore that E is a field of degree $[E : \mathbb{Q}] = \dim_{\mathbb{Q}} V$. In particular, V is one-dimensional as an E-vector space.

The decomposition into character spaces can be made more canonical in the following way. Let $S = \mathrm{Hom}(E, \mathbb{C})$ denote the set of all complex embeddings of E; its cardinality is $[E : \mathbb{Q}]$. Then

$$E \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{s \in S} \mathbb{C}, \quad e \otimes z \mapsto \sum_{s \in S} s(e) z,$$

is an isomorphism of E-vector spaces; E acts on each summand on the right through the corresponding embedding s. This decomposition induces an isomorphism

$$V \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{s \in S} V_s,$$

where $V_s = V \otimes_{E,s} \mathbb{C}$ is a one-dimensional complex vector space on which E acts via s. The induced homomorphism $U(1) \to \mathrm{MT}(V) \to E^\times \to \mathrm{End}_{\mathbb{C}}(V_s)$ is a character of $U(1)$, hence of the form $z \mapsto z^k$ for some integer k. Solving $k = p - q$ and $n = p + q$, we find that $k = 2p - n$, which means that V_s is of type $(p, n-p)$ in the Hodge decomposition of V. Now define a function $\varphi \colon S \to \mathbb{Z}$ by setting $\varphi(s) = p$; then any choice of isomorphism $V \simeq E$ puts a Hodge structure of weight n on E, whose Hodge decomposition is given by

$$E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{s \in S} \mathbb{C}^{\varphi(s), n-\varphi(s)}.$$

From the fact that $\overline{e \otimes z} = e \otimes \bar{z}$, we deduce that

$$\overline{\sum_{s \in S} z_s} = \sum_{s \in S} \overline{z_{\bar{s}}}.$$

Since complex conjugation has to interchange $\mathbb{C}^{p,q}$ and $\mathbb{C}^{q,p}$, this implies that $\varphi(\bar{s}) = n - \varphi(s)$, and hence that $\varphi(s) + \varphi(\bar{s}) = n$ for every $s \in S$.

Definition 24.8. Let E be a number field, and $S = \mathrm{Hom}(E, \mathbb{C})$ the set of its complex embeddings. Any function $\varphi \colon S \to \mathbb{Z}$ with the property that $\varphi(s) + \varphi(\bar{s}) = n$ defines a *Hodge structure E_φ of weight* n on the $\mathbb{Q}$-vector space E, whose Hodge decomposition is given by

$$E_\varphi \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{s \in S} \mathbb{C}^{\varphi(s), \varphi(\bar{s})}.$$

By construction, the action of E on itself respects this decomposition.

In summary, we have $V \simeq E_\varphi$, which is an isomorphism both of E-modules and of Hodge structures of weight n. Next, we would like to prove that in all interesting cases, E must be a CM-field. Recall from Definition 24.2 that a field E is called a *CM-field* if there exists a nontrivial involution $\iota \colon E \to E$, such that complex conjugation induces ι under any embedding of E into the complex numbers. In other words, we must have $s(\iota e) = \bar{s}(e)$ for any $s \in S$ and any $e \in E$. We usually write $\bar{e}$ in place of ιe, and refer to it as complex conjugation on E. The fixed field of E is then a totally real subfield F, and E is a purely imaginary quadratic extension of F.

To prove that E is either a CM-field or $\mathbb{Q}$, we choose a polarization ψ on E_φ. We then define the so-called *Rosati involution* $\iota \colon E \to E$ by the condition that

$$\psi(e \cdot x, y) = \psi(x, \iota e \cdot y)$$

for every $x, y, e \in E$. Denoting the image of $1 \in E$ by $\sum_{s \in S} 1_s$, we have

$$\sum_{s \in S} \psi(1_s, 1_{\bar{s}}) s(e \cdot x) \bar{s}(y) = \sum_{s \in S} \psi(1_s, 1_{\bar{s}}) s(x) \bar{s}(\iota e \cdot y),$$

which implies that $s(e) = \bar{s}(\iota e)$. Now there are two cases: Either ι is nontrivial, in which case E is a CM-field and the Rosati involution is complex conjugation. Or ι is trivial, which means that $\bar{s} = s$ for every complex embedding. In the second case, we see that $\varphi(s) = n/2$ for every s, and so the Hodge structure must be $\mathbb{Q}(-n/2)$, being irreducible and of type $(n/2, n/2)$. This implies that $E = \mathbb{Q}$.

From now on, we exclude the trivial case $V = \mathbb{Q}(-n/2)$ and assume that E is a CM-field.

Definition 24.9. A *CM-type* of E is a mapping $\varphi \colon S \to \{0, 1\}$ with the property that $\varphi(s) + \varphi(\bar{s}) = 1$ for every $s \in S$.

When φ is a CM-type, E_φ is a polarizable rational Hodge structure of weight 1. As such, it is the rational Hodge structure of an abelian variety with complex multiplication by E. This variety is unique up to isogeny. In general, we have the following structure theorem.

Proposition 24.10. *Any Hodge structure V of CM-type and of even weight $2k$ with $V^{p,q} = 0$ for $p < 0$ or $q < 0$ occurs as a direct factor of $H^{2k}(A, \mathbb{Q})$, where A is a finite product of simple abelian varieties of CM-type.*

Proof. In our classification of irreducible Hodge structures of CM-type above, there were two cases: $\mathbb{Q}(-n/2)$, and Hodge structures of the form E_φ, where E is a CM-field and $\varphi\colon S \to \mathbb{Z}$ is a function satisfying $\varphi(s)+\varphi(\bar{s}) = n$. Clearly φ can be written as a linear combination (with integer coefficients) of CM-types for E. Because of the relations

$$E_{\varphi+\psi} \simeq E_\varphi \otimes_E E_\psi \quad \text{and} \qquad E_{-\varphi} \simeq E_\varphi^\vee,$$

every irreducible Hodge structure of CM-type can thus be obtained from Hodge structures corresponding to CM-types by tensor products, duals, and Tate twists.

As we have seen, every Hodge structure of CM-type is a direct sum of irreducible Hodge structures of CM-type. The assertion follows from this by simple linear algebra. □

To conclude our discussion of Hodge structures of CM-type, we will consider the case when the CM-field E is a Galois extension of $\mathbb{Q}$. In that case, the Galois group $G = \mathrm{Gal}(E/\mathbb{Q})$ acts on the set of complex embeddings of E by the rule

$$(g \cdot s)(e) = s(g^{-1}e).$$

This action is simply transitive. Recall that we have an isomorphism

$$E \otimes_{\mathbb{Q}} E \xrightarrow{\sim} \bigoplus_{g \in G} E, \quad x \otimes e \mapsto g(e)x.$$

For any E-vector space V, this isomorphism induces a decomposition

$$V \otimes_{\mathbb{Q}} E \xrightarrow{\sim} \bigoplus_{g \in G} V, \quad v \otimes e \mapsto g(e)v.$$

When V is an irreducible Hodge structure of CM-type, a natural question is whether this decomposition is compatible with the Hodge decomposition. The following lemma shows that the answer to this question is yes.

Lemma 24.11. *Let E be a CM-field that is a Galois extension of $\mathbb{Q}$, with Galois group $G = \mathrm{Gal}(E/\mathbb{Q})$. Then for any $\varphi\colon S \to \mathbb{Z}$ with $\varphi(s) + \varphi(\bar{s}) = n$, we have*

$$E_\varphi \otimes_{\mathbb{Q}} E \simeq \bigoplus_{g \in G} E_{g\varphi}.$$

Proof. We chase the Hodge decompositions through the various isomorphisms that are involved in the statement. To begin with, we have

$$\big(E_\varphi \otimes_{\mathbb{Q}} E\big) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \big(E_\varphi \otimes_{\mathbb{Q}} \mathbb{C}\big) \otimes_{\mathbb{Q}} E \simeq \bigoplus_{s \in S} \mathbb{C}^{\varphi(s), n-\varphi(s)} \otimes_{\mathbb{Q}} E \simeq \bigoplus_{s,t \in S} \mathbb{C}^{\varphi(s), n-\varphi(s)},$$

and the isomorphism takes $(v \otimes e) \otimes z$ to the element

$$\sum_{s,t \in S} t(e) \cdot z \cdot s(v).$$

On the other hand,

$$\big(E_\varphi \otimes_{\mathbb{Q}} E\big) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{g \in G} E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{g \in G} \bigoplus_{s \in S} \mathbb{C}^{\varphi(s), n-\varphi(s)},$$

and under this isomorphism, $(v \otimes e) \otimes z$ is sent to the element

$$\sum_{g \in G} \sum_{s \in S} s(ge) \cdot s(v) \cdot z.$$

If we fix $g \in G$ and compare the two expressions, we see that $t = sg$, and hence

$$E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{t \in S} \mathbb{C}^{\varphi(s), n-\varphi(s)} \simeq \bigoplus_{t \in S} \mathbb{C}^{\varphi(tg^{-1}), n-\varphi(tg^{-1})}.$$

But since $(g\varphi)(t) = \varphi(tg^{-1})$, this is exactly the Hodge decomposition of $E_{g\varphi}$. $\square$