Lecture 22 (April 17)

Hodge structures. Let's start with a brief review of Hodge structures, because we are going to need the language. Let H be a finite-dimensional $\mathbb{Q}$-vector space. A *Hodge structure* of weight k on H is a decomposition

$$H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

with the property that $\overline{H^{p,q}} = H^{q,p}$ for all $p+q=k$. We can describe a Hodge structure in terms of its *Hodge filtration* $F^\bullet H_{\mathbb{C}}$; this is the decreasing filtration with

$$F^p H_{\mathbb{C}} = H^{p,k-p} \oplus H^{p+1,k-p+1} \oplus H^{p+2,k-p+2} \oplus \cdots$$

One can recover the Hodge decomposition from the Hodge filtration because

$$H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}}.$$

All Hodge structures that are of interest in geometry are "polarized". By definition, a *polarization* of H is a $(-1)^k$-symmetric bilinear pairing

$$S\colon H \otimes_{\mathbb{Q}} H \to \mathbb{Q}$$

with the property that the hermitian form

$$h(v,w) = \sum_{p+q=k} i^{p-q} S\bigl(v^{p,q}, \overline{w^{p,q}}\bigr)$$

is positive definite and makes the Hodge decomposition into an orthogonal decomposition. Concretely, S is symmetric if k is even, and skew-symmetric if k is odd; $S(H^{p,q}, H^{p',q'}) = 0$ unless $p'=q$ and $q'=p$; and $i^{p-q}S(v,\bar{v}) > 0$ for nonzero $v \in H^{p,q}$. These conditions are coming from the Hodge-Riemann bilinear relations on the cohomology of smooth projective varieties.

Example 22.1. Suppose that $k=2m$ is even. A class $h \in H$ is called a *Hodge class* if $h \in H^{m,m}$. This is equivalent to the condition that $h \in F^m H_{\mathbb{C}}$. Indeed, if we write $h = \sum_{p+q=2m} h^{p,q}$, then $h \in F^m H_{\mathbb{C}}$ means that $h^{p,q} = 0$ for $p < m$. Because $h = \bar{h}$, we also gett $h^{p,q} = \overline{h^{q,p}} = 0$ for $p > m$, and so $h \in H^{m,m}$.

For any integer $\ell \in \mathbb{Z}$, we have Tate's Hodge structure $\mathbb{Q}(\ell)$ on the $\mathbb{Q}$-vector space $\mathbb{Q}(\ell) = (2\pi i)^\ell \mathbb{Q} \subseteq \mathbb{C}$. It has weight -2ℓ: the complexification is $\mathbb{C}$, and the Hodge decomposition is $\mathbb{C} = \mathbb{C}^{-\ell,-\ell}$. For any Hodge structure H, we can then form the *Tate twist* $H(\ell) = H \otimes_{\mathbb{Q}} \mathbb{Q}(\ell)$. The Hodge decomposition of $H(\ell)_{\mathbb{C}} = H_{\mathbb{C}}$ stays the same, but we now view it as a Hodge structure of weight $k-2\ell$ by setting

$$H(\ell)^{p,q} = H^{p+\ell,q+\ell}.$$

Lastly, a *morphism* between two Hodge structures H_1, H_2 of the same weight is a $\mathbb{Q}$-linear mapping $f\colon H_1 \to H_2$ such that $f(H_1^{p,q}) \subseteq H_2^{p,q}$ for all $p+q=k$. In geometry, one often encounters linear mappings that change the weight (such as the Gysin morphism); they are properly considered as morphisms $H_1 \to H_2(\ell)$.

Example 22.2. Let X be a smooth projective variety over $\mathbb{C}$. By Hodge's theorem, each cohomology group $H^k(X,\mathbb{Q})$ has a Hodge structure of weight k. The Hodge filtration

$$F^p H^k(X,\mathbb{C}) = H^{p,k-p}(X) \oplus H^{p+1,k-p-1}(X) \oplus \cdots$$

has an alternative description in terms of de Rham cohomology. Set $n = \dim X$, and consider the holomorphic de Rham complex

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \longrightarrow 0.$$

By the holomorphic Poincaré lemma, the complex $\Omega_X^\bullet$ is a resolution of the constant sheaf $\mathbb{C}_X$, and so

$$H^k(X, \mathbb{C}) \cong H^k(X, \Omega_X^\bullet).$$

We can filter the holomorphic de Rham complex by the family of subcomplexes

$$0 \longrightarrow \Omega_X^p \xrightarrow{d} \Omega_X^{p+1} \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \longrightarrow 0,$$

usually denoted $F^p\Omega_X^\bullet$. A basic result in Hodge theory is that the mapping

$$H^k(X, F^p\Omega_X^\bullet) \to H^k(X, \Omega_X^\bullet)$$

is injective, and that its image is exactly the Hodge filtration $F^pH^k(X, \mathbb{C})$. An equivalent formulation is that the spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Longrightarrow H^{p+q}(X, \mathbb{C})$$

degenerates at E_1, and that the filtration coming from the spectral sequence is the Hodge filtration.

Example 22.3. Let's also quickly review the formulas for the polarization. Let L be an ample line bundle on X, and set $\omega = c_1(L)$, which is a Hodge class in $H^2(X, \mathbb{Z}(1))$. The Hard Lefschetz theorem says that for $0 \leq k \leq n$, the mapping

$$\omega^{n-k} \colon H^k(X, \mathbb{Q}) \to H^{2n-k}\big(X, \mathbb{Q}(n-k)\big)$$

is an isomorphism of Hodge structures. (The Tate twist is needed to make the weight of the second Hodge structure equal to k.) The primitive cohomology is

$$H_0^k(X, \mathbb{Q}) = \ker\Big(\omega^{n-k+1} \colon H^k(X, \mathbb{Q}) \to H^{2n-k+2}\big(X, \mathbb{Q}(n-k+1)\big)\Big).$$

According to the Hodge-Riemann bilinear relations, the pairing

$$S_k(\alpha, \beta) = (-1)^{k(k-1)/2} \frac{1}{(2\pi i)^n} \int_X \alpha \wedge \beta \wedge \omega^{n-k}$$

is a polarization of the Hodge structure on $H_0^k(X, \mathbb{Q})$. (The sign and the factor i^{p-q} show up because the associated hermitian form is, up to a positive constant, exactly the hermitian inner product on $H_0^k(X, \mathbb{C})$ induced by the Kähler metric.)

One can get a polarization on all of $H^k(X, \mathbb{Q})$ by using the Lefschetz decomposition. Because of the Hard Lefschetz theorem, we only need to consider $0 \leq k \leq n$. In that case, the Lefschetz decomposition is

$$H^k(X, \mathbb{Q}) = H_0^k(X, \mathbb{Q}) \oplus \omega H_0^{k-2}(X, \mathbb{Q}(-1)) \oplus \omega^2 H_0^{k-4}(X, \mathbb{Q}(-2)) \oplus \cdots$$

and one can show that $(-1)^\ell S_k$ polarizes the summand $\omega^\ell H_0^{k-2\ell}(X, \mathbb{Q}(-\ell))$. If we define an involution σ of $H^k(X, \mathbb{Q})$ that acts on the ℓ-th summand as $(-1)^\ell$, this means that the bilinear form

$$(\alpha, \beta) \mapsto S_k\big(\alpha, \sigma(\beta)\big)$$

is a polarization for the Hodge structure on $H^k(X, \mathbb{Q})$.

Algebraic de Rham cohomology. The theory of absolute Hodge classes is based on the observation that one can compute the cohomology of a smooth projective variety algebraically. Let X be a smooth projective variety over $\operatorname{Spec} k$, where k is a field containing $\mathbb{Q}$. (For example, k could be a finitely-generated extension of $\mathbb{Q}$, or $k = \mathbb{C}$.) From the sheaf of Kähler differentials $\Omega_{X/k}^1$ and its wedge powers, one can form the *algebraic de Rham complex*

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{d} \Omega_{X/k}^1 \xrightarrow{d} \Omega_{X/k}^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{X/k}^n \longrightarrow 0.$$

Its hypercohomology groups

$$H_{dR}^i(X/k) = H^i(X, \Omega_{X/k}^\bullet)$$

are called the *algebraic de Rham cohomology* of X. They are finite-dimensional k-vector spaces. As before, we can filter the algebraic de Rham complex by the family of subcomplexes

$$0 \longrightarrow \Omega^p_{X/k} \xrightarrow{d} \Omega^{p+1}_{X/k} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{X/k} \longrightarrow 0,$$

denoted $F^p\Omega^\bullet_{X/k}$, and define the *Hodge filtration* as

$$F^p H^i_{dR}(X/k) = \operatorname{im}\Big(H^i(X, F^p\Omega^\bullet_{X/k}) \to H^i(X, \Omega^\bullet_{X/k}\Big).$$

The point is of course that this agrees with the definitions we gave earlier. To see why, suppose that $k = \mathbb{C}$. For the sake of clarity, let's denote the compact complex manifold associated to the smooth projective variety X by the symbol X^{an}. The analytification of $\Omega^1_{X/\mathbb{C}}$ is the sheaf of holomorphic 1-forms $\Omega^1_{X^{an}}$, and the analytification of the complex $\Omega^\bullet_{X/\mathbb{C}}$ is the holomorphic de Rham complex $\Omega^\bullet_{X^{an}}$. By Serre's GAGA theorem, we get a natural isomorphism

$$H^i_{dR}(X/\mathbb{C}) = H^i(X, \Omega^\bullet_{X/\mathbb{C}}) \cong H^i(X^{an}, \Omega^\bullet_{X^{an}}) \cong H^i(X^{an}, \mathbb{C}).$$

This isomorphism takes the subspace $F^p H^i_{dR}(X/\mathbb{C})$ to the subspace $F^p H^i(X^{an}, \mathbb{C})$, hence to the usual Hodge filtration.

Note. We can not get the rational cohomology $H^i(X^{an}, \mathbb{Q})$ in this way; for that, we need the underlying topological space of the complex manifold X^{an}.

In general, we can take any embedding $\sigma\colon k \hookrightarrow \mathbb{C}$, and consider the base change

$$\begin{tikzcd} X_{\mathbb{C}} \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r, "\sigma^*"] & \operatorname{Spec} k. \end{tikzcd}$$

We have $\Omega^1_{X_{\mathbb{C}}/\mathbb{C}} \cong f^*\Omega^1_{X/k}$ because Kähler differentials are compatible with base change; we therefore get a natural isomorphism

$$H^i_{dR}(X_{\mathbb{C}}/\mathbb{C}) \cong H^i_{dR}(X/k) \otimes_k \mathbb{C}.$$

So both the algebraic de Rham cohomology, and the Hodge filtration on it, are actually defined over the field k.

Conjugate varieties. We can now give a precise definition of conjugating a variety by an automorphism of $\mathbb{C}$. Let X be a smooth projective variety over $\operatorname{Spec}\mathbb{C}$, and let $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q})$ be an automorphism. We define the *conjugate variety* X^σ as the base change

$$\begin{tikzcd} X^\sigma \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r, "\sigma^*"] & \operatorname{Spec}\mathbb{C} \end{tikzcd}$$

While X and X^σ are isomorphic as abstracts schemes (or as schemes over $\operatorname{Spec}\mathbb{Q}$), they are *not* isomorphic as schemes over $\operatorname{Spec}\mathbb{C}$. Because algebraic de Rham cohomology is compatible with base change, we have

$$H^i_{dR}(X^\sigma/\mathbb{C}) \cong H^i_{dR}(X/\mathbb{C}) \otimes_{\mathbb{C},\sigma} \mathbb{C}$$

with the tensor product being taken over the isomorphism $\sigma\colon \mathbb{C} \to \mathbb{C}$. We therefore get a natural isomorphism

$$H^i\big((X^\sigma)^{an}, \mathbb{C}\big) \cong H^i(X^{an}, \mathbb{C}) \otimes_{\mathbb{C},\sigma} \mathbb{C},$$

and this allows us to associate to every cohomology class $\alpha \in H^i(X^{an}, \mathbb{C})$ a conjugate α^σ in the i-th cohomology of $(X^\sigma)^{an}$.

The small problem is that the only automorphisms of $\mathbb{C}$ that one can write down (without the axiom of choice) are the identity and complex conjugation. A simpler way to accomplish the same thing is to start from a smooth projective variety X defined over a field k; in practice, k is going to be finitely-generated over $\mathbb{Q}$. For any embedding $\sigma \colon k \hookrightarrow \mathbb{C}$, we can define $X^\sigma_{\mathbb{C}}$ as the base change

$$\begin{tikzcd} X^\sigma_{\mathbb{C}} \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r, "\sigma^*"] & \operatorname{Spec} k. \end{tikzcd}$$

As before, we get a natural isomorphism

$$H^i\big((X^\sigma_{\mathbb{C}})^{an}, \mathbb{C}\big) \cong H^i_{dR}(X^\sigma_{\mathbb{C}}/\mathbb{C}) \cong H^i_{dR}(X/k) \otimes_k \mathbb{C}.$$

This again allows us to take any class in the cohomology of one complex manifold $(X^\sigma_{\mathbb{C}})^{an}$, and transport it in a canonical way to the cohomology of any other conjugate. The advantage is that embeddings of a finitely-generated field k into the complex numbers are easy to describe.

Note. When $k = \mathbb{Q}$, we get two different $\mathbb{Q}$-structures on the $\mathbb{C}$-vector space

$$H^i(X^{an}, \mathbb{C}) \cong H^i(X^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^i_{dR}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},$$

one coming from singular cohomology, the other from algebraic de Rham cohomology. The relation between the two is the subject of Grothendieck's period conjecture: roughly speaking, it says that unless there is a geometric reason, the entries of the matrix relating the two $\mathbb{Q}$-structures are as transcendental as possible.

Note. Somebody asked about an example of two conjugate varieties that are not homeomorphic. I recommend the paper "Conjugate varieties with distinct real cohomology algebras" by François Charles, which you can find here:

`https://www.math.ens.psl.eu/~charles/crll5855.pdf`

Absolute Hodge classes. We can now give a precise definition of absolute Hodge classes. One formulation starts from algebraic de Rham cohomology.

Definition 22.4. Let X be a smooth projective variety over a field k that is finitely-generated over $\mathbb{Q}$. A class $\alpha \in F^p H^{2p}_{dR}(X/k) \otimes_k \mathbb{C}$ is called an *absolute Hodge class* if, for every embedding $\sigma \colon k \hookrightarrow \mathbb{C}$, the image of α under the isomorphism

$$H^{2p}_{dR}(X/k) \otimes_k \mathbb{C} \cong H^{2p}\big((X^\sigma_{\mathbb{C}})^{an}, \mathbb{C}\big)$$

belongs to the subspace $H^{2p}\big((X^\sigma_{\mathbb{C}})^{an}, \mathbb{Q}\big)$, and is therefore a Hodge class on $(X^\sigma_{\mathbb{C}})^{an}$.

In practice, we are usually starting from a smooth projective variety over $\operatorname{Spec} \mathbb{C}$ and are interested in classes in the rational cohomology of X^{an}. So here is an equivalent definition that is closer to what we said last time.

Definition 22.5. Let X be a smooth projective variety over $\operatorname{Spec} \mathbb{C}$. A Hodge class $\alpha \in H^{2p}(X^{an}, \mathbb{Q}) \cap F^p H^{2p}(X^{an}, \mathbb{C})$ is called an *absolute Hodge class* if, for every automorphism $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q})$, the image of α under the isomorphism

$$F^p H^{2p}(X^{an}, \mathbb{C}) \otimes_{\mathbb{C},\sigma} \mathbb{C} \cong F^p H^{2p}\big((X^\sigma)^{an}, \mathbb{C}\big)$$

belongs to the subspace $H^{2p}\big((X^\sigma)^{an}, \mathbb{Q}\big)$, and is therefore a Hodge class on $(X^\sigma)^{an}$.

The main example are of course fundamental classes of algebraic subvarieties. Let's start with a simpler example.

Example 22.6. Let L be a line bundle on the smooth projective variety X. Let L^{an} denote the associated holomorphic line bundle on the complex manifold X^{an}. Then $c_1(L^{an})$ is an absolute Hodge class in $H^2(X^{an}, \mathbb{Q}(1))$. To see why, we need to look at the construction of the first Chern class, especially in algebraic de Rham cohomology. This will also explain where the $2\pi i$ comes from.

In the analytic topology, we can use the exponential sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathscr{O}_{X^{an}} \xrightarrow{\exp} \mathscr{O}_{X^{an}}^{\times} \longrightarrow 0.$$

Here $\mathbb{Z}(1) = 2\pi i \mathbb{Z} \subseteq \mathbb{C}$ shows up as the kernel of the exponential function. The first Chern class is the connecting homomorphism

$$c_1 \colon \operatorname{Pic}(X^{an}) \cong H^1\bigl(X^{an}, \mathscr{O}_{X^{an}}^{\times}\bigr) \to H^2\bigl(X^{an}, \mathbb{Z}(1)\bigr).$$

To compute $c_1(L^{an})$, we take a good covering by contractible open subsets U_i on which L^{an} is trivial. The transition functions $g_{ij} \in \Gamma\bigl(U_i \cap U_j, \mathscr{O}_{X^{an}}^{\times}\bigr)$ form a 1-cocycle. We then write

$$g_{ij} = e^{f_{ij}}$$

for holomorphic functions $f_{ij} \in \Gamma(U_i \cap U_j, \mathscr{O}_{X^{an}})$, and then the 2-cocycle

$$f_{jk} - f_{ik} + f_{ij} \in \mathbb{Z}(1)$$

represents $c_1(L^{an}) \in H^2(X^{an}, \mathbb{Z}(1))$. To convert this class to de Rham cohomology, we consider the holomorphic 1-forms

$$df_{ij} = \frac{dg_{ij}}{g_{ij}} \in \Gamma(U_i \cap U_j, \Omega^1_{X^{an}}).$$

They are closed, and form a 1-cocycle, and so they determine a class in $H^2(X^{an}, F^1\Omega^\bullet_{X^{an}})$ (using Čech cohomology). Since this computes $F^1 H^2(X^{an}, \mathbb{C})$, the first Chern class of L^{an} is a Hodge class. Note that it naturally lives in $H^2(X^{an}, \mathbb{Q}(1))$, which is a Hodge structure of weight zero (because of the Tate twist).

We can imitate this construction in algebraic de Rham cohomology. Indeed, L is an algebraic line bundle, so there is a covering of X by affine open subsets U_i on which L is trivial. Denoting the transition functions again by $g_{ij} \in \Gamma(U_i \cap U_j, \mathscr{O}_X^{\times})$, we get a 1-cocycle consisting of the closed algebraic 1-forms

$$\frac{dg_{ij}}{g_{ij}} \in \Gamma(U_i \cap U_j, \Omega^1_{X/\mathbb{C}}),$$

and using Čech cohomology, this again determines a class in

$$F^1 H^2_{dR}(X/\mathbb{C}) = H^2(X, F^1\Omega^\bullet_{X/\mathbb{C}}).$$

Under the comparison isomorphism, the two classes $c_1(L^{an})$ and $c_1(L)$ then correspond to each other. For every automorphism $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q})$, we can pull L back along the morphism $X^\sigma \to X$ and obtain a line bundle L^σ. By the above, $c_1(L^{an})$ corresponds to the class $c_1(L)$ in de Rham cohomology; under σ, this goes to $c_1(L^\sigma)$, which in turn corresponds to $c_1\bigl((L^\sigma)^{an}\bigr)$. Therefore the conjugate of $c_1(L^{an})$ is again the first Chern class of a line bundle, and this means that $c_1(L^{an})$ is an absolute Hodge class. (With a slightly broader definition of absolute Hodge classes that allows Tate twists.)

More generally, the fundamental class of any closed subvariety is an absolute Hodge class. Let $Z \subseteq X$ be a closed subvariety of codimension p. For the sake of precision, let's denote by $Z^{an} \subseteq X^{an}$ the associated analytic subset of the compact complex manifold X^{an}. The fundamental class

$$[Z^{an}] \in H^{2p}\bigl(X^{an}, \mathbb{Q}(p)\bigr)$$

can be defined using Poincaré duality (which also explain the appearance of the Tate twist). Let $\mu\colon \tilde{Z} \to Z$ be a resolution of singularities, and denote by $f\colon \tilde{Z} \to X$ the composition. The linear functional

$$H^{2n-2p}(X^{an}, \mathbb{Q}) \to \mathbb{Q}, \quad \alpha \mapsto \int_{\tilde{Z}^{an}} f^*\alpha,$$

is represented by a unique cohomology class $\zeta \in H^{2p}(X^{an}, \mathbb{Q})$, which then satisfies

$$\int_{\tilde{Z}^{an}} f^*\alpha = \int_{X^{an}} \zeta \cup \alpha.$$

We saw in the discussion about polarization that it is better to divide the integral over X^{an} by $(2\pi i)^n$. We therefore define the fundamental class of Z as

$$[Z^{an}] = (2\pi i)^p \zeta \in H^{2p}\big(X^{an}, \mathbb{Q}(p)\big),$$

and this turns the identity from above into

$$\frac{1}{(2\pi i)^{n-p}} \int_{\tilde{Z}^{an}} f^*\alpha = \frac{1}{(2\pi i)^n} \int_{X^{an}} [Z^{an}] \cup \alpha.$$

Now we need to define a corresponding class $[Z]$ in the algebraic de Rham cohomology of X. The easiest way to do this is to use Chern classes, which make sense both in usual cohomology and in algebraic de Rham cohomology. Starting from the case of line bundles (for which we have Chern classes in both theories), one first constructs Chern classes for vector bundles (using the splitting principle), and then Chern classes for arbitrary coherent sheaves (using locally free resolutions). Once this theory is in place, the Grothendieck-Riemann-Roch theorem implies that

$$[Z^{an}] = \frac{(-1)^{p-1}}{(p-1)!} c_p\big(\mathscr{O}_{Z^{an}}\big).$$

We can then simply define

$$[Z] = \frac{(-1)^{p-1}}{(p-1)!} c_p\big(\mathscr{O}_Z\big) \in F^p H^{2p}_{dR}(X/\mathbb{C}),$$

and then $[Z^{an}]$ and $[Z]$ correspond to each other under the comparison isomorphism. For every $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we get a conjugate subvariety $Z^\sigma \subseteq X^\sigma$, and just as in the case of line bundles, this implies that $[Z^{an}]$ is an absolute Hodge class.