Lecture 20 (April 10)

More on derived equivalences between abelian varieties. Let X and Y be two abelian varieties (of the same dimension g). Last time, we associated to any equivalence $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ an isomorphism of abelian varieties

$$\varphi_E \colon X \times \hat{X} \to Y \times \hat{Y}.$$

The construction used the following commutative diagram:

$$\begin{tikzcd} \mathrm{D}^b(X \times \hat{X}) \arrow[r, "F_E"] \arrow[d, "\mathbf{R}\Phi_{A(X)}"] & \mathrm{D}^b(Y \times \hat{Y}) \\ \mathrm{D}^b(X \times X) \arrow[r, "\mathbf{R}\Phi_E \times \mathbf{R}\Phi_E^{-1}"] & \mathrm{D}^b(Y \times Y) \arrow[u, "\mathbf{R}\Phi_{A(Y)}^{-1}"'] \end{tikzcd} \tag{20.1}$$

Here $\mathbf{R}\Phi_{A(X)}$ is the equivalence that takes the structure sheaf of a closed point $(x, \alpha) \in X(k) \times \hat{X}(k)$ to the object $(t_x, \mathrm{id})_* P_\alpha$ on $X \times X$, which is the kernel of the auto-equivalence

$$T_{(x,\alpha)} \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X), \quad T_{(x,\alpha)}(K) = L \otimes t_x^* K.$$

We showed that the equivalence F_E, defined as in the diagram above, has the form

$$F_E(K) = \mathbf{R}(\varphi_E)_*(N_E \otimes K)$$

for a line bundle $N_E \in \mathrm{Pic}(X \times \hat{X})$. The isomorphism φ_E records how $\mathbf{R}\Phi_E$ interacts with translations and tensor product: one has $\varphi_E(x, \alpha) = (y, \beta)$ iff

$$\mathbf{R}\Phi_E \circ T_{(x,\alpha)} \cong T_{(y,\beta)} \circ \mathbf{R}\Phi_E.$$

Today, we are going to investigate this construction a bit further.

Property 1. The first observation is that the construction of F_E (and of φ_E) is compatible with composition, in the following sense. Suppose that

$$\begin{tikzcd} \mathrm{D}^b(X) \arrow[r, "\mathbf{R}\Phi_E"] \arrow[rr, bend left, "\mathbf{R}\Phi_{E*G}"] & \mathrm{D}^b(Y) \arrow[r, "\mathbf{R}\Phi_G"] & \mathrm{D}^b(Z) \end{tikzcd}$$

are two equivalences of derived categories, with composition $\mathbf{R}\Phi_G \circ \mathbf{R}\Phi_E \cong \mathbf{R}\Phi_{E*G}$, where $E * G$ is the convolution of the two kernels. Then the induced equivalences

$$\begin{tikzcd} \mathrm{D}^b(X \times \hat{X}) \arrow[r, "F_E"] \arrow[rr, bend left, "F_{E*G}"] & \mathrm{D}^b(Y \times \hat{Y}) \arrow[r, "F_G"] & \mathrm{D}^b(Z \times \hat{Z}) \end{tikzcd}$$

are compatible (up to natural isomorphism). Because of the shape of (20.1), this comes down to the identity

$$\big(\mathbf{R}\Phi_G \times \mathbf{R}\Phi_G^{-1}\big) \circ \big(\mathbf{R}\Phi_E \times \mathbf{R}\Phi_E^{-1}\big) \cong \mathbf{R}\Phi_{E*G} \times \mathbf{R}\Phi_{E*G}^{-1}\big),$$

which holds because $\mathbf{R}\Phi_G \circ \mathbf{R}\Phi_E \cong \mathbf{R}\Phi_{E*G}$. By looking at the kernels, we get

$$\varphi_G \circ \varphi_E = \varphi_{E*G},$$

and so the construction of F_E and φ_E respects composition.

Property 2. The next question is which homomorphisms $\varphi\colon X\times\hat{X}\to Y\times\hat{Y}$ can show up. We can write such a homomorphism as a matrix

$$\varphi = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

where $\alpha\colon X\to Y$, $\beta\colon\hat{X}\to Y$, $\gamma\colon X\to\hat{Y}$, and $\delta\colon\hat{X}\to\hat{Y}$. Each of these four homomorphisms has a dual homomorphism: $\hat{\alpha}\colon\hat{Y}\to\hat{X}$, $\hat{\beta}\colon\hat{Y}\to X$, $\hat{\gamma}\colon Y\to\hat{X}$, and $\hat{\delta}\colon Y\to X$. We can put these together into a sort of "adjoint" matrix

$$\varphi^* = \begin{pmatrix}\hat{\delta} & -\hat{\beta}\\ -\hat{\gamma} & \hat{\alpha}\end{pmatrix}$$

which then defines a homomorphism $\varphi^*\colon Y\times\hat{Y}\to X\times\hat{X}$.

Example 20.2. In the case of the Fourier transform $\mathbf{R}\Phi_P$, we had

$$\varphi_P\colon X\times\hat{X}\to\hat{X}\times X,\quad \varphi_P(x,\alpha)=(\alpha,-x).$$

Here we get $\varphi_P^* = -\varphi_P$ because

$$\varphi_P = \begin{pmatrix}0 & \mathrm{id}\\ -\mathrm{id} & 0\end{pmatrix}$$

Inside the group of all isomorphisms from $X\times\hat{X}$ to $Y\times\hat{Y}$, consider the subset

$$U(X\times\hat{X},Y\times\hat{Y}) = \left\{\,\varphi\colon X\times\hat{X}\to Y\times\hat{Y}\;\middle|\;\varphi^*\circ\varphi=\mathrm{id}\,\right\}.$$

It turns out that when $\mathbf{R}\Phi_E\colon \mathrm{D}^b(X)\to\mathrm{D}^b(Y)$ is an equivalence, then the associated isomorphism φ_E must lie in this set.

Lemma 20.3. *One has* $\varphi_E\in U(X\times\hat{X},Y\times\hat{Y})$.

Proof. For any closed point $(x,\alpha)\in X(k)\times\hat{X}(k)$, we have the auto-equivalence $T_{(x,\alpha)}$ of $\mathrm{D}^b(X)$, and we let $F_{(x,\alpha)}$ be the associated auto-equivalence of $\mathrm{D}^b(X\times\hat{X})$. One can check that $\varphi_{(x,\alpha)}=\mathrm{id}$ and $N_{(x,\alpha)}=P_\alpha\boxtimes\hat{P}_{-x}$; in fact, we did half of this computation last time, when we looked at tensor products by line bundles.

The idea behind the proof is to use the fact that Orlov's construction respects compositions. Suppose that $\varphi_E(x,\alpha)=(y,\beta)$. Then

$$\mathbf{R}\Phi_E\circ T_{(x,\alpha)}\cong T_{(y,\beta)}\circ\mathbf{R}\Phi_E,$$

and therefore $F_E\circ F_{(x,\alpha)}\cong F_{(y,\beta)}\circ F_E$. Writing out both sides explicitly, we get

$$\mathbf{R}(\varphi_E)_*\bigl(N_E\otimes N_{(x,\alpha)}\otimes -\bigr)\cong N_{(y,\beta)}\otimes\mathbf{R}(\varphi_E)_*\bigl(N_E\otimes -\bigr)$$

and therefore $\varphi_E^*N_{(y,\beta)}\cong N_{(x,\alpha)}$ (by the projection formula). This gives

$$\varphi_E^*\bigl(P_\beta\boxtimes\hat{P}_{-y}\bigr)\cong P_\alpha\boxtimes\hat{P}_{-x},$$

or in terms of the dual homomorphism $\hat{\varphi}_E\colon\hat{Y}\times Y\to\hat{X}\times X$,

$$\hat{\varphi}_E(\beta,-y)=(\alpha,-x).$$

Because of how we defined φ_E^*, this becomes $\varphi_E^*(y,\beta)=(x,\alpha)$, and so $\varphi_E^*\circ\varphi_E$ is indeed the identity. □

Property 3. In fact, Polishchuk and Orlov showed that *every* $\varphi\in U(X\times\hat{X},Y\times\hat{Y})$ is equal to φ_E for some equivalence $\mathbf{R}\Phi_E\colon\mathrm{D}^b(X)\to\mathrm{D}^b(Y)$. To prove this, one has to construct sufficiently many kernels on $X\times Y$; in fact, they are all of the form, a vector bundle supported on an abelian subvariety of $X\times Y$.

Corollary 20.4. *One has* $\mathrm{D}^b(X)\cong\mathrm{D}^b(Y)$ *iff* $U(X\times\hat{X},Y\times\hat{Y})\neq\emptyset$.

Property 4. Let's prove that the kernel of any derived equivalence $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ between abelian varieties must be a vector bundle supported on an abelian subvariety. The first step is to compute the kernel of the induced equivalence F_E using convolutions, as in (20.1). We can then push this object forward along the projection $p_{13} \colon X \times \hat{X} \times Y \times \hat{Y} \to X \times Y$. At the same time, we know that the kernel is isomorphic to $(\mathrm{id}, \varphi_E)_* N_E$. If we use the diagram

$$\begin{tikzcd} X \times \hat{X} \arrow[r, "(\mathrm{id},\varphi_E)"] \arrow[dr, bend right, "f_E"'] & X \times \hat{X} \times Y \times \hat{Y} \arrow[d, "p_{13}"] \\ & X \times Y \end{tikzcd}$$

to define a homomorphism $f_E = p_{13} \circ (\mathrm{id}, \varphi_E)$ from $X \times \hat{X}$ to $X \times Y$, then the result of this (big) computation is that

$$\mathbf{R}(f_E)_* N_E = \mathbf{R}(p_{13})_* \big((\mathrm{id}, \varphi_E)_* N_E\big) \cong E \otimes_k E^\vee|_{(0,0)}.$$

Here $E^\vee|_{(0,0)}$ means that we take the dual complex $E^\vee = \mathbf{R}\mathcal{H}om(\mathscr{E}, \mathscr{O}_{X \times Y})$ and restrict it to the closed point $(0,0)$; this is just a complex of finite-dimensional k-vector spaces. So up to this small "error term", we can recover the kernel object E from the isomorphism φ_E and the line bundle N_E.

Example 20.5. The Fourier transform $\mathbf{R}\Phi_P$ had $\varphi_P(x, \alpha) = (\alpha, -x)$, and $N_P = P$. Here f_P is the identity, because $(\mathrm{id}, \varphi_P)(x, \alpha) = (x, \alpha, \alpha, -x)$.

$$\begin{tikzcd} X \times \hat{X} \arrow[r, "(\mathrm{id},\varphi_P)"] \arrow[dr, bend right, "\mathrm{id}"'] & X \times \hat{X} \times \hat{X} \times X \arrow[d, "p_{13}"] \\ & X \times \hat{X} \end{tikzcd}$$

So it is indeed the case that P is the pushforward of N_P.

We can now prove the following result, originally due to Orlov.

Proposition 20.6. *Suppose that $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is an equivalence. Then up to a shift, E is a locally free sheaf supported on an abelian subvariety of $X \times Y$.*

We are going to abstract a bit, in order to simplify the notation. Consider a homomorphism $f \colon X \to Y$ between two abelian varieties of the same dimension g. Suppose that $\ker f$ has dimension n, so that $Z = \operatorname{im} f \subseteq Y$ is an abelian subvariety of codimension n. We are going to use the following morphisms:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[rr, bend left, "f"] & Z \arrow[r, "i"] & Y \end{tikzcd}$$

Suppose that $E \in \mathrm{D}^b(Y)$ is an object in the derived category, such that

$$E \otimes E^\vee|_0 \cong \mathbf{R}f_* L$$

for a line bundle $L \in \mathrm{Pic}(X)$. Then we claim that, up to a shift, E must be a locally free sheaf supported on Z.

Proof. After a shift, we may assume that $\mathcal{H}^i E = 0$ for $i > 0$, but that $\mathcal{H}^0 E \neq 0$. On a suitable affine open neighborhood of the point $0 \in Y(k)$, we can find a minimal locally free resolution for E, of the form

$$0 \to \mathscr{E}_p \to \mathscr{E}_{p-1} \to \cdots \to \mathscr{E}_0 \to 0.$$

The dual complex $E^\vee$ is then

$$0 \to (\mathscr{E}_0)^* \to (\mathscr{E}_1)^* \to \cdots \to (\mathscr{E}_p)^* \to 0,$$

and by minimality, $E^\vee|_0$ is the complex with terms $(\mathscr{E}_i)^*|_0$ and trivial differentials. So in the derived category of k-vector spaces, $E^\vee|_0$ decomposes as

$$E^\vee|_0 \cong \bigoplus_{i=0}^{p} \mathcal{H}^i\big(E^\vee|_0\big)[-i] \cong \bigoplus_{i=0}^{p} V_i[-i].$$

In particular, the 0-th cohomology V_0 of the complex $E^\vee|_0$ is nontrivial. Because tensor product over k is exact, it follows that

$$R^i f_* L \cong \mathcal{H}^i\big(E \otimes E^\vee|_0\big)$$

contains $\mathcal{H}^i E \otimes V_0$ as a direct summand. For obvious reasons, $R^i f_* L = 0$ for $i < 0$, and therefore $\mathcal{H}^i E = 0$ for $i < 0$; this means that E is isomorphic to a sheaf in degree 0. Since

$$\mathbf{R} f_* L \cong i_* \mathbf{R} p_* L,$$

this sheaf is isomorphic to a direct summand of $i_* \mathbf{R} p_* L$, and so it is annihilated by the ideal sheaf $\mathcal{I}_Z$. Consequently, $E \cong i_* \mathscr{F}$, where $\mathscr{F}$ is a coherent sheaf on Z. Moreover, $\mathscr{F}$ is isomorphic to a direct summand of $\mathbf{R} p_* L \in \mathrm{D}^b(Z)$.

Now we argue that Z is actually locally free. Since Z is smooth projective of dimension $g - n$, we can find a locally free resolution

$$0 \to \mathscr{E}_{g-n} \to \cdots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0.$$

Consider the dual complex $\mathbf{R}\mathcal{H}om_{\mathscr{O}_Z}(\mathscr{F}, \mathscr{O}_Z)$, which is isomorphic to

$$0 \to (\mathscr{E}_0)^* \to \cdots \to (\mathscr{E}_{g-n})^* \to 0.$$

We are going to argue that the dual complex is a sheaf. The key point is that

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_Z}(\mathbf{R} p_* L, \mathscr{O}_Z) \cong \mathbf{R} p_* \mathcal{H}om_{\mathscr{O}_X}(L, p^! \mathscr{O}_Z) \cong \mathbf{R} p_* L^{-1}[n],$$

because $p\colon X \to Z$ is smooth of relative dimension n and the canonical bundles of X and Z are both trivial. For dimension reasons, the complex $\mathbf{R} p_* L^{-1}[n]$ is concentrated in degrees $-n, \dots, 0$. Because $\mathscr{F}$ is a direct summand of $\mathbf{R} p_* L$, it follows that $\mathbf{R}\mathcal{H}om_{\mathscr{O}_Z}(\mathscr{F}, \mathscr{O}_Z)$ is a direct summand of $\mathbf{R} p_* L^{-1}[n]$, and therefore also concentrated in deegree $-n, \dots, 0$. It follows that $\mathbf{R}\mathcal{H}om_{\mathscr{O}_Z}(\mathscr{F}, \mathscr{O}_Z)$ is a single locally free sheaf in degree 0; dualizing back, we find that $\mathscr{F}$ is itself locally free. $\square$

Property 5. One can use the results above to classify all auto-equivalences of the derived category $\mathrm{D}^b(X)$. Let's write $\operatorname{Aut} \mathrm{D}^b(X)$ for the group of all auto-equivalences. We showed above that the function

$$\operatorname{Aut} \mathrm{D}^b(X) \to U(X \times \hat{X}, X \times \hat{X}), \quad \mathbf{R}\Phi_E \mapsto \varphi_E,$$

is a group homomorphism; we also know (from Property 3) that it is surjective. One can show that the kernel consists exactly of the auto-equivalences $T_{(x,\alpha)}$, with $(x, \alpha) \in X(k) \times \hat{X}(k)$, and of the shift functors $[n]$ with $n \in \mathbb{Z}$. This makes precise the heuristic from last time that $X(k) \times \hat{X}(k)$ is the neutral component of the automorphism group of $\mathrm{D}^b(X)$.

Mukai's $\mathrm{SL}_2(\mathbb{Z})$-action. Suppose now that X is a principally polarized abelian variety; this means that we have an ample line bundle L such that $h^0(X, L) = 1$. Equivalently, the morphism

$$\phi_L\colon X \to \hat{X}, \quad t_x^* L \otimes L^{-1} \cong P_{\phi_L(x)},$$

is an isomorphism. (It is surjective by Theorem 11.7 and has degree $h^0(X, L)^2 = 1$.) In this case, we get several interesting auto-equivalences of the derived category, and Mukai noticed that they determine an action of the group $\mathrm{SL}_2(\mathbb{Z})$ on $\mathrm{D}^b(X)$.

The first auto-equivalence $S\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ is the composition

$$\mathrm{D}^b(X) \xrightarrow{\mathbf{R}\Phi_P} \mathrm{D}^b(\hat{X}) \xrightarrow{\phi_L^*} \mathrm{D}^b(X).$$

Because of the formula $(\mathrm{id} \times \phi_L)^* P \cong m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$, we can write this as

$$S(K) = \phi_L^* \mathbf{R}(p_2)_* (p_1^* K \otimes P) \cong \mathbf{R}(p_2)_* \big(p_1^* K \otimes m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \big).$$

We also have a second auto-equivalence

$$T\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X), \quad T(K) = L \otimes K.$$

Both S and T have associated automorphisms φ_S and φ_T in $\mathrm{Hom}(X \times \hat{X}, X \times \hat{X})$; using the isomorphism ϕ_L between X and $\hat{X}$, we may consider φ_S and φ_T as elements of $\mathrm{Hom}(X \times X, X \times X)$. We showed last time that $\varphi_P(x, \alpha) = (\alpha, -x)$, and after making the identifications, we get

$$\varphi_S = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$$

We also computed last time that $\varphi_T(x, \alpha) = (x, \alpha - \phi_L(x))$; after the appropriate identifications, this tells us that

$$\varphi_T = \begin{pmatrix} \mathrm{id} & 0 \\ -\mathrm{id} & \mathrm{id} \end{pmatrix}.$$

Now we observe that the modular group

$$\mathrm{SL}_2(\mathbb{Z}) = \big\{ A \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}) \; \big| \; \det A = 1 \big\}$$

embeds into $\mathrm{Hom}(X \times X, X \times X)$. Indeed, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det A = ad - bc = 1$, then A defines an automorphism of the abelian variety $X \times X$, represented by the matrix

$$\begin{pmatrix} a_X & b_X \\ c_X & d_X \end{pmatrix};$$

on closed points, the formula is $A \cdot (x, y) = (ax + by, cx + dy)$. So φ_S and φ_T represent the action of the two matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

It is known that these two matrices together generate $\mathrm{SL}_2(\mathbb{Z})$; the relations are

$$S^4 = \mathrm{id} \quad \text{and} \quad (TS)^3 = \mathrm{id}\,.$$

Mukai's observation is that these identities already hold (up to a shift) for the two equivalences S and T. In this sense, the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathrm{D}^b(X)$.

Proposition 20.7. *The two equivalences $S, T\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ satisfy*

$$S^4 \cong [-2g] \quad \textit{and} \quad (T \circ S)^3 \cong [-g].$$

Proof. Let $\mathbf{R}\Phi_P\colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})$ and $\mathbf{R}\Phi_{\hat{P}}\colon \mathrm{D}^b(\hat{X}) \to \mathrm{D}^b(X)$ be the two Fourier transforms. From the proof of Mukai's theorem, we know that

$$\mathbf{R}\Phi_P \circ \mathbf{R}\Phi_{\hat{P}} \cong (-1)^*_{\hat{X}}[-g].$$

The isomorphism $\phi_L\colon X \to \hat{X}$ is self-dual, in the sense that $\hat{\phi}_L = \phi_L$. The identity in (18.5) therefore tells us that the diagram

$$\begin{tikzcd}
\mathrm{D}^b(X) \arrow[r, "\mathbf{R}\Phi_P"] \arrow[d, "(\phi_L)_*"] & \mathrm{D}^b(\hat{X}) \arrow[d, "\phi_L^*"] \\
\mathrm{D}^b(\hat{X}) \arrow[r, "\mathbf{R}\Phi_{\hat{P}}"] & \mathrm{D}^b(X)
\end{tikzcd}$$

is commutative. This gives

$$\begin{aligned}
S \circ S = \phi_L^* \circ \mathbf{R}\Phi_P \circ \phi_L^* \circ \mathbf{R}\Phi_P &\cong \phi_L^* \circ \mathbf{R}\Phi_P \circ \mathbf{R}\Phi_{\hat{P}} \circ (\phi_L)_* \\
&\cong \phi_L^* \circ (-1)_{\hat{X}}^*[-g] \circ (\phi_L)_* \cong (-1)_X^*[-g].
\end{aligned}$$

So clearly $S^4 \cong [-2g]$, which is the first identity.

For the second identity, we note that

$$(T \circ S)(K) = L \otimes S(K) \cong \mathbf{R}(p_2)_*\big(p_1^* K \otimes m^* L \otimes p_1^* L^{-1}\big),$$

which means that $T \circ S$ is an integral transform with kernel $m^* L \otimes p_1^* L^{-1}$. The kernel of $(T \circ S)^3$ is therefore given by convolution: concretely, it is

$$\mathbf{R}(p_{14})_*\Big(m_{12}^* L \otimes m_{23}^* L \otimes m_{34}^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \otimes p_3^* L^{-1}\Big),$$

where $p_{14}\colon X \times X \times X \times X \to X \times X$ is the projection to the first and fourth factor, and m_{ij} is the morphism that adds the i-th and j-th coordinates. We can simplify the line bundle in parentheses using the seesaw theorem. For any two closed points $x, y \in X(k)$, its restriction to $\{x\} \times X \times X \times \{y\}$ is

$$p_1^* t_x^* L \otimes m^* L \otimes p_2^* t_y^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \cong m^* L \otimes p_1^* \phi_L(x) \otimes p_2^* \phi_L(y),$$

and under the natural isomorphism $X \times X \times X \times X \cong \hat{X} \times X \times X \times \hat{X}$, this is isomorphic to the restriction of $p_{12}^* P \otimes m_{23}^* L \otimes p_{34}^* P \otimes p_4^* L$. Both bundles also have the same restriction to $X \times \{0\} \times \{0\} \times X$, and so they are isomorphic by the seesaw theorem. Therefore the kernel of $(T \circ S)^3$ is

$$\begin{aligned}
&\mathbf{R}(p_{14})_*\Big(p_{12}^* \hat{P} \otimes p_{23}^* m^* L \otimes p_{34}^* P \otimes p_4^* L\Big) \\
&\qquad \cong p_2^* L \otimes \mathbf{R}(p_{14})_*\Big(p_{12}^* P \otimes p_{23}^* m^* L \otimes p_{34}^* P\Big) \cong p_2^* L \otimes \mathbf{R}\Phi_{P \boxtimes P}(m^* L).
\end{aligned}$$

The second factor is exactly the Fourier-Mukai transform of $m^* L \in \mathrm{D}^b(X \times X)$ using the Poincaré bundle $P \boxtimes P$ on $\hat{X} \times X \times X \times \hat{X}$.

The homomorphism $m\colon X \times X \to X$ is dual to the diagonal $\Delta\colon \hat{X} \to \hat{X} \times \hat{X}$, in the sense that $\hat{m} = \Delta$. Because we know from Proposition 18.4 how the Fourier-Mukai transform interacts with homomorphisms, we get

$$\mathbf{R}\Phi_{P \boxtimes P}(m^* L) \cong \Delta_* \Phi_P(L)[-g] \cong \Delta_* L^{-1}[-g].$$

Therefore the kernel of $(T \circ S)^3$ simplifies to

$$p_2^* L \otimes \Delta_* L^{-1}[-g] \cong \Delta_* \mathscr{O}_{\hat{X}}[-g],$$

which shows that $(T \circ S)^3 \cong [-g]$. $\square$

Exercise 20.1. As an exercise, you can try to figure out what the kernel for the equivalence corresponding to a general matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

might look like. As a starting point, consider the diagram from Proposition 20.6, which now reads (after identifying X and $\hat{X}$)

$$\begin{tikzcd} X \times X \arrow[r, "(\mathrm{id}, A)"] \arrow[drr, bend right, "f_A"'] & X \times X \times X \times X \arrow[d, "p_{13}"] \\ & X \times X. \end{tikzcd}$$

Here $f_A(x, y) = (x, ax + by)$, and if $b \neq 0$, then f_A is an isogeny of degree $\deg b_X = b^{2g}$, and so the kernel object must be a vector bundle on $X \times X$. What is its rank? Can you describe this vector bundle in some cases? What happens when $b = 0$?