Lecture 17 (April 1)

Grothendieck duality. In additional to the general definitions from last time, we also need two basic tools for actually working with derived categories. The first one is Grothendieck duality. The general theory is fairly complicated, and so we shall only discuss a special case that is sufficient for the purposes of this course.

Let me begin by recalling Serre's duality theorem. It says that if $\mathscr{F}$ is a coherent sheaf on a smooth projective variety X, then

$$\operatorname{Ext}^{n-i}(\mathscr{F}, \omega_X) \simeq \operatorname{Hom}_{\mathbb{C}}\bigl(H^i(X, \mathscr{F}), \mathbb{C}\bigr),$$

where $n = \dim X$ and ω_X denotes the canonical bundle of X. We can reformulate this using the derived category. Because of the relationship between Ext-groups and morphisms in the derived category, we have

$$H^i(X, \mathscr{F}) \simeq \operatorname{Ext}^i(\mathscr{O}_X, \mathscr{F}) \simeq \operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_X)}\bigl(\mathscr{O}_X, \mathscr{F}[i]\bigr)$$
$$\operatorname{Ext}^{n-i}(\mathscr{F}, \omega_X) \simeq \operatorname{Hom}\bigl(\mathscr{F}[i], \omega_X[n]\bigr).$$

Serre duality can therefore be rewritten in the form

$$\operatorname{Hom}\bigl(F, \omega_X[n]\bigr) \simeq \operatorname{Hom}\bigl(\operatorname{Hom}(\mathscr{O}_X, F), \mathbb{C}\bigr),$$

where $F = \mathscr{F}[i]$. Using suitable resolutions, this can be improved to the following general result in the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$.

Theorem 17.1. *Let X be a smooth projective variety, and let F and G be two objects of $\mathrm{D}^b_{coh}(\mathscr{O}_X)$. Then one has an isomorphism of $\mathbb{C}$-vector spaces*

$$\operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_X)}\Bigl(F, G \otimes \omega_X[n]\Bigr) \simeq \operatorname{Hom}\Bigl(\operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_X)}(G, F), \mathbb{C}\Bigr)$$

that is functorial in F and G.

Grothendieck duality is a relative version of Serre duality, where instead of a single variety, one has a proper morphism $f \colon X \to Y$. In Grothendieck's formulation, duality becomes a statement about certain functors: we have the derived pushforward functor $\mathbf{R}f_* \colon \mathrm{D}^b_{coh}(\mathscr{O}_X) \to \mathrm{D}^b_{coh}(\mathscr{O}_Y)$, and the problem is to construct a right adjoint $f^! \colon \mathrm{D}^b_{coh}(\mathscr{O}_Y) \to \mathrm{D}^b_{coh}(\mathscr{O}_X)$, pronounced "$f$-shriek". In other words, we would like to define $f^!$ in such a way that we have functorial isomorphisms

$$\operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_Y)}\bigl(\mathbf{R}f_* F, G\bigr) \simeq \operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_X)}\bigl(F, f^! G\bigr)$$

for $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$. For arbitrary proper morphisms, the construction requires considerable technical effort; it is explained in Hartshorne's book *Residues and Duality.* (There is also a modern treatment by Amnon Neeman, based on the Brown' representability theorem.) But in the special case that both X and Y are smooth projective, there is a much simpler construction due to Alexei Bondal and Mikhail Kapranov.

Theorem 17.2. *If $f \colon X \to Y$ is a morphism between two smooth projective varieties, then*

$$f^! G = \omega_X[\dim X] \otimes \mathbf{L}f^*\bigl(G \otimes \omega_Y^{-1}[-\dim Y]\bigr)$$

for any $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$.

Proof. This follows very easily from the fact that $\mathbf{L}f^*$ is the left adjoint of $\mathbf{R}f_*$ – if we use Serre duality to interchange left and right. Fix two objects $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$. Applying Serre duality on Y, we get

$$\operatorname{Hom}\bigl(\mathbf{R}f_* F, G \otimes \omega_Y[\dim Y]\bigr) \simeq \operatorname{Hom}\Bigl(\operatorname{Hom}\bigl(G, \mathbf{R}f_* F\bigr), \mathbb{C}\Bigr).$$

Because $\mathbf{L}f^*$ is the left adjoint of $\mathbf{R}f_*$, we have

$$\operatorname{Hom}\bigl(G, \mathbf{R}f_* F\bigr) \simeq \operatorname{Hom}\bigl(\mathbf{L}f^* G, F\bigr).$$

If we now apply Serre duality on X, we get back to

$$\operatorname{Hom}\Big(\operatorname{Hom}\big(\mathbf{L}f^*G, F\big), \mathbb{C}\Big) \simeq \operatorname{Hom}\big(F, \mathbf{L}f^*G \otimes \omega_X[\dim X]\big).$$

Putting all three isomorphisms together, we obtain the desired formula for $f^!G$. □

For a more concise statement, let $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1}$ denote the relative canonical bundle; then the formula in Theorem 17.2 becomes

$$f^! = \omega_{X/Y}[\dim X - \dim Y] \otimes \mathbf{L}f^*.$$

Note that $\dim X - \dim Y$ is simply the relative dimension of the morphism f. To summarize, we have a functorial isomorphism

$$\operatorname{Hom}\Big(\mathbf{R}f_*F, G\Big) \simeq \operatorname{Hom}\Big(F, \omega_{X/Y}[\dim X - \dim Y] \otimes \mathbf{L}f^*G\Big)$$

for $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$. In this form, Grothendieck duality will appear frequently in the derived category calculations below.

Flat base change. Another technical result that we shall use below is the base change theorem. As in the case of Grothendieck duality, there is a very general statement (in the derived category); for our purposes, however, two special cases are enough, and so we shall restrict our attention to those.

The general problem addressed by the base change theorem is the following. Suppose we have a cartesian diagram of schemes:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

We would like to compare the two functors g^*f_* and $f'_*g'^*$; more generally, on the level of the derived category, the two functors $\mathbf{L}g^*\mathbf{R}f_*$ and $\mathbf{R}f'_*\mathbf{L}g'^*$. Using the adjointness of pullback and pushforward, we always have morphisms of functors

$$g^*f_* \to f'_*g'^* \quad \text{and} \quad \mathbf{L}g^*\mathbf{R}f_* \to \mathbf{R}f'_*\mathbf{L}g'^*,$$

but without some assumptions on f or g – or on the sheaves or complexes to which we apply the functors – they are not isomorphisms.

The simplest case where the two functors are isomorphic is when g (and hence also g') is flat. We begin by looking at the case of sheaves.

Lemma 17.3. *Suppose that g is flat, and that f is separated and quasi-compact. Then the base change morphism*

$$g^*f_*\mathscr{F} \to f'_*g'^*\mathscr{F}$$

is an isomorphism for every quasi-coherent sheaf $\mathscr{F}$ on X.

Proof. The statement is local on Y and Y', and so we may assume without loss of generality that $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$ are affine, with A' flat over A. Let $\mathscr{F}' = g'^*\mathscr{F}$; then all sheaves involved are quasi-coherent on Y', and so it suffices to show that

$$\mathscr{F}(X) \otimes_A A' \to \mathscr{F}'(X')$$

is an isomorphism.

We first consider the case when $X = \operatorname{Spec} B$ is also affine; in that case, $X' = \operatorname{Spec} A' \otimes_A B$. We have $\mathscr{F} = \tilde{M}$ for some B-module M; then $g^*f_*\mathscr{F}$ is the quasi-coherent sheaf corresponding to the A'-module

$$A' \otimes_A M_A,$$

while $f'_* g'^* \mathscr{F}$ is the quasi-coherent sheaf corresponding to

$$(A' \otimes_A B) \otimes_B M.$$

The two are evidently isomorphic, which proves the assertion in case X is affine. In general, cover X by finitely many affine open subsets $U_1, \dots, U_n$. Because $\mathscr{F}$ is a sheaf, the complex of A-modules

$$0 \to \mathscr{F}(X) \to \bigoplus_{i=1}^n \mathscr{F}(U_i) \to \bigoplus_{i,j=1}^n \mathscr{F}(U_i \cap U_j)$$

is exact. Now A' is flat over A, and so

$$0 \to \mathscr{F}(X) \otimes_A A' \to \bigoplus_{i=1}^n \mathscr{F}(U_i) \otimes_A A' \to \bigoplus_{i,j=1}^n \mathscr{F}(U_i \cap U_j) \otimes_A A'$$

remains exact. We conclude from the affine case above that the kernel is isomorphic to $\mathscr{F}'(X')$, which is the result we were after. □

In the derived category, we have the following version.

Proposition 17.4. *Suppose that g is flat, and the f is separated and quasi-compact. Then for any $F \in \mathrm{D}^+\big(\mathrm{QCoh}(X)\big)$, the base change morphism*

$$\mathbf{L}g^* \mathbf{R}f_* F \to \mathbf{R}f'_* \mathbf{L}g'^* F$$

is an isomorphism.

Proof. After replacing F by an injective resolution, we may assume without loss of generality that F is a complex of injective quasi-coherent sheaves. The result now follows by applying Lemma 17.3 termwise. □

Mukai's Fourier transform. From now on, let's write $\mathrm{D}^b(X)$ for the bounded derived category of coherent sheaves on X. It may happen that two smooth projective varieties X and Y have isomorphic derived categories, without X and Y themselves being isomorphic.[1] The first interesting example of this was discovered by Mukai: if X is an abelian variety, and $\hat{X}$ the dual abelian variety, then

$$\mathrm{D}^b(X) \cong \mathrm{D}^b(\hat{X}).$$

Here "isomorphic" means that there is an exact k-linear equivalence between the two categories. This equivalence comes from the Poincaré bundle P on the product $X \times \hat{X}$, using the projections to the two factors:

$$\begin{tikzcd} X \times \hat{X} \arrow[r, "p_2"] \arrow[d, "p_1"] & \hat{X} \\ X \end{tikzcd}$$

Given a complex $K \in \mathrm{D}^b(X)$, we can define its "Fourier transform"

$$\mathbf{R}\Phi_P(K) = \mathbf{R}(p_2)_* \big(\mathbf{L}p_1^* K \otimes P\big)$$

which is an object in $\mathrm{D}^b(\hat{X})$. Because p_1 is flat, the functor p_1^* is already exact; similarly, P is a line bundle, and so the tensor product with P is also exact. So the only genuinely derived functor is $\mathbf{R}(p_2)_*$, and so $\mathbf{R}\Phi_P$ really is the derived functor of the naive functor $\mathscr{F} \mapsto (p_2)_*(p_1^* \mathscr{F} \otimes P)$ on sheaves. Mukai called this the "Fourier transform" because of its formal similarities with the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx$$

[1] Bondal and Orlov proved that if the (anti-)canonical bundle of X is ample, then any isomorphism $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ comes from an isomorphism $X \cong Y$.

for L^1-functions on $\mathbb{R}$. (In this analogy, complexes of sheaves are functions; tensoring with P is multiplication by the exponential function; and the direct image along p_2 is integration along the fibers.)

With that in mind, Mukai's theorem is as follows.

Theorem 17.5 (Mukai)**.** *Let X be an abelian variety, and let $\hat{X}$ be the dual abelian variety. Then the Fourier transform*

$$\mathbf{R}\Phi_P \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})$$

is an equivalence of categories.

Note that X and $\hat{X}$ are usually not isomorphic; but they are nevertheless related on the level of the derived category.

General integral transforms. The Fourier transform is an example of what people call an "integral transform" (or "Fourier-Mukai transform") between derived categories. Suppose that X and Y are two smooth projective varieties, and that $E \in \mathrm{D}^b(X \times Y)$ is an object on the product. We can then define an exact functor

$$\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$$

by the same formula as above:

$$\mathbf{R}\Phi_E(K) = \mathbf{R}(p_2)_*\big(\mathbf{L}p_1^* \overset{\mathbf{L}}{\otimes} E\big),$$

but now the tensor product is also derived (because E is no longer locally free). The object E is called the "kernel" of the transform; the name again comes from integral transforms on function spaces (where the kernel is some kind of function or distribution on the product).

Example 17.6. A basic example is $E = \Delta_* \mathscr{O}_X$, the structure sheaf of the diagonal on the product $X \times X$. In this case, $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ is the identity.

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend right, "\mathrm{id}"'] & X \times X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \\
& X &
\end{tikzcd}$$

Indeed, for any object $K \in \mathrm{D}^b(X)$, we have

$$\mathbf{L}p_1^* K \overset{\mathbf{L}}{\otimes} \Delta_* \mathscr{O}_X \cong \mathbf{R}\Delta_* K$$

by the projection formula, and therefore

$$\mathbf{R}\Phi_E(K) \cong \mathbf{R}(p_2)_* \mathbf{R}\Delta_* K \cong K.$$

Example 17.7. More generally, take a morphism $f \colon X \to Y$, look at its graph

$$\Gamma_f \colon X \to X \times Y, \quad \Gamma_f(x) = \big(x, f(x)\big),$$

and use the object $E = (\Gamma_f)_* \mathscr{O}_X$ on the product $X \times Y$ as the kernel of an integral transform. The following diagram shows the relevant morphisms:

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f"] \arrow[rr, bend left, "f"] \arrow[dr, bend right, "\mathrm{id}"'] & X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\
& X &
\end{tikzcd}$$

By exactly the same computation as above, we have $\mathbf{R}\Phi_E = \mathbf{R}f_* \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$. If we swap the roles of X and Y, and denote by

$$\mathbf{R}\Psi_E \colon \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$$

the integral transform with kernel E going the other way, then we have

$$\mathbf{L}p_2^* K \overset{\mathbf{L}}{\otimes} (\Gamma_f)_* \mathscr{O}_X \cong \mathbf{R}\Delta_* \mathbf{L}f^* K,$$

and therefore $\mathbf{R}\Psi_E(K) \cong \mathbf{R}(p_1)_* \mathbf{R}\Delta_* \mathbf{L}f^* K \cong \mathbf{L}f^* K$. So both $\mathbf{R}f_*$ and $\mathbf{L}f^*$ are special cases of integral transforms.

Let's check that the composition of two integral transforms is again an integral transform. Say $E \in \mathrm{D}^b(X \times Y)$ and $F \in \mathrm{D}^b(Y \times Z)$ are two kernels. Consider the composition

$$\mathbf{R}\Phi_F \circ \mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Z).$$

To work out what this does, we are going to use the following big diagram:

$$\begin{tikzcd}
X \times Y \times Z \arrow[r, "p_{23}"] \arrow[d, "p_{12}"] \arrow[rr, bend left, "p_3"] \arrow[dd, bend right, "p_1"'] & Y \times Z \arrow[r, "p_2"] \arrow[d, "p_1"] & Z \\
X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y & \\
X & &
\end{tikzcd}$$

Let $K \in \mathrm{D}^b(X)$ be any object. Then

$$\mathbf{R}\Phi_E(K) = \mathbf{R}(p_2)_* \big(\mathbf{L}p_1^* K \overset{\mathbf{L}}{\otimes} E\big).$$

In order to compute $\mathbf{L}p_1^*$ of this complex, we can use flat base change (along the projection $p_1 \colon Y \times Z \to Y$). This gives

$$\mathbf{L}p_1^* \mathbf{R}\Phi_E(K) \cong \mathbf{R}(p_{23})_* \big(\mathbf{L}p_{12}^*(\mathbf{L}p_1^* K \overset{\mathbf{L}}{\otimes} E)\big) \cong \mathbf{R}(p_{23})_* \big(\mathbf{L}p_1^* K \overset{\mathbf{L}}{\otimes} \mathbf{L}p_{12}^* E\big).$$

Tensoring by F and pushing forward to Z then produces

$$\begin{aligned}
\mathbf{R}\Phi_F \mathbf{R}\Phi_E(K) &\cong \mathbf{R}(p_2)_* \Big(\mathbf{R}(p_{23})_* \big(\mathbf{L}p_1^* K \otimes \mathbf{L}p_{12}^* E\big) \overset{\mathbf{L}}{\otimes} F\Big) \\
&\cong \mathbf{R}(p_2)_* \mathbf{R}(p_{23})_* \Big(\mathbf{L}p_1^* K \otimes \mathbf{L}p_{12}^* E \overset{\mathbf{L}}{\otimes} \mathbf{L}p_{23}^* F\Big) \\
&\cong \mathbf{R}(p_3)_* \Big(\mathbf{L}p_1^* K \otimes \mathbf{L}p_{12}^* E \overset{\mathbf{L}}{\otimes} \mathbf{L}p_{23}^* F\Big);
\end{aligned}$$

to go from the first to the second line, we used the projection formula (for the morphism $p_{23} \colon X \times Y \times Z \to Y \times Z$. If we now use the factorization

$$\begin{tikzcd}
X \times Y \times Z \arrow[r, "p_{13}"] \arrow[rr, bend left, "p_3"] \arrow[dr, bend right, "p_1"'] & X \times Z \arrow[r, "p_2"] \arrow[d, "p_1"] & Z \\
& X &
\end{tikzcd}$$

and apply the projection formula one more time, we can rewrite this as

$$\mathbf{R}\Phi_F \mathbf{R}\Phi_E(K) \cong \mathbf{R}(p_2)_* \Big(\mathbf{L}p_1^* \overset{\mathbf{L}}{\otimes} \mathbf{R}(p_{13})_* \big(\mathbf{L}p_{12}^* E \overset{\mathbf{L}}{\otimes} \mathbf{L}p_{23}^* F\big)\Big).$$

The composition is therefore again an integral transform, with kernel the object

$$E * F = \mathbf{R}(p_{13})_*\big(\mathbf{L}p_{12}^* E \overset{\mathbf{L}}{\otimes} \mathbf{L}p_{23}^* F)\big) \in \mathrm{D}^b(X \times Z).$$

This object is called the "convolution" of the two kernels $E \in \mathrm{D}^b(X \times Y)$ and $F \in \mathrm{D}^b(Y \times Z)$, again by analogy with the convolution of two functions (which is defined by integration over a common argument). With this notation, we have

$$\mathbf{R}\Phi_F \circ \mathbf{R}\Phi_E \cong \mathbf{R}\Phi_{E*F}, \tag{17.8}$$

where "isomorphism" really means that the two functors are related by a natural isomorphism. This kind of computation – using flat base change and the projection formula – is very typical in the subject.

Example 17.9. In order to show that an integral transform $\mathbf{R}\Phi_E \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is an equivalence, it is enough to find an object $F \in \mathrm{D}^b(Y \times X)$ such that $E * F \cong \Delta_* \mathscr{O}_X$ is the structure sheaf of the diagonal on $X \times X$, and $F * E \cong \Delta_* \mathscr{O}_Y$ is the structure sheaf of the diagonal on $Y \times Y$. The reason is that the structure sheaf of the diagonal represents the identity.

Let me also mention, without proof, the following very nice theorem by Orlov.

Theorem 17.10. *Let X and Y be two smooth projective varieties. Then any (exact and k-linear) equivalence of categories $F \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is of the form $F \cong \mathbf{R}\Phi_E$ for an object $E \in \mathrm{D}^b(X \times Y)$, unique up to isomorphism.*

Thinking of E as the family of objects

$$E_x = E|_{\{x\} \times Y} \in \mathrm{D}^b(Y),$$

parametrized by the closed point $x \in X(k)$, one necessarily has

$$E_x \cong F\big(k(x)\big),$$

the image of the skyscraper sheaf $k(x) \in \mathrm{D}^b(X)$ under the equivalence F. The difficult thing is to show that these objects actually fit together into a complex of coherent sheaves on $X \times Y$.

The symmetric Fourier transform. As a postdoc, when I was doing a lot of computations with Mukai's Fourier transform, I found that I could never remember all the formulas, and so each time I wanted to prove something, I had to go back to Mukai's paper and look up the correct formula. (There are shifts by $\pm \dim X$, signs, and inverses, and it is hard to remember which goes where.) This eventually led me to write a paper with the grand title "The Fourier-Mukai transform made easy", whose main point was that one can change the definition of the Fourier transform very slightly, and make all the formulas easy to remember. The idea is to use the (contravariant) Grothendieck duality functor

$$\mathbf{R}\Delta_X = \mathbf{R}\mathcal{H}om\big(-, \omega_X[\dim X]\big) \colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)^{op},$$

where $\omega_X = \det \Omega^1_{X/k}$ is the canonical bundle of the smooth projective variety X. In the case $X = \operatorname{Spec} k$, we shall use the simplified notation $\mathbf{R}\Delta_k$.

Definition 17.11. Let X be an abelian variety, and let $P = P_X$ be the Poincaré bundle on $X \times \hat{X}$. The exact functor

$$\mathsf{FM}_X = \mathbf{R}\Phi_P \circ \mathbf{R}\Delta_X \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})^{op}$$

is called the *symmetric Fourier-Mukai transform.*

Note that FM_X is a *contravariant* functor; this turns out to be quite useful in practice. The following theorem justifies the name "symmetric Fourier-Mukai transform"; it is of course equivalent to Mukai's theorem (because the duality functor is a contravariant equivalence of categories).

Theorem 17.12. *The composed functors* $\mathsf{FM}_{\hat{X}} \circ \mathsf{FM}_X$ *and* $\mathsf{FM}_X \circ \mathsf{FM}_{\hat{X}}$ *are naturally isomorphic to the identity. In other words,*

$$\mathsf{FM}_X \colon \mathrm{D}^b(X) \to \mathrm{D}^b(\hat{X})^{op}$$

is an equivalence of categories, with quasi-inverse $\mathsf{FM}_{\hat{X}}$.

One advantage of the modified definition is that it respects the symmetry between the two abelian varieties X and $\hat{X}$. For example, one can show that

$$\mathsf{FM}_X\bigl(k(0)\bigr) = \mathscr{O}_{\hat{X}} \quad \text{and} \quad \mathsf{FM}_X(\mathscr{O}_X) = k(0). \tag{17.13}$$

Here $k(0) = e_* \mathscr{O}_{\operatorname{Spec} k}$ means the structure sheaf of the closed point $0 \in X(k)$; we use the same notation also on $\hat{X}$.

Let's verify the two identities in (17.13). The first one is very easy: Grothendieck duality, applied to the morphism $e\colon \operatorname{Spec} k \to X$, gives

$$\mathbf{R}\Delta_X\bigl(k(0)\bigr) = e_* \mathbf{R}\Delta_{\operatorname{Spec} k}\bigl(\mathscr{O}_{\operatorname{Spec} k}\bigr) = e_* \mathscr{O}_{\operatorname{Spec} k} = k(0),$$

and so the symmetric Fourier-Mukai transform of $k(0)$ is

$$\mathsf{FM}_X\bigl(k(0)\bigr) = \mathbf{R}\Phi_P\bigl(k(0)\bigr) = \mathscr{O}_X.$$

The second isomorphism comes from the fact that $(e \times \mathrm{id})^* P = P|_{\{0\}\times \hat{X}}$ is trivial. In exactly the same way, one can show that

$$\mathsf{FM}_X\bigl(k(x)\bigr) = P|_{\{x\}\times \hat{X}}.$$

The Fourier-Mukai transform therefore takes structure sheaves of points to line bundles in $\operatorname{Pic}^0(\hat{X})$.

For the second identity in (17.13), we need to compute

$$\mathsf{FM}_X(\mathscr{O}_X) = \mathbf{R}\Phi_P\Bigl(\omega_X[\dim X]\Bigr) = \mathbf{R}(p_2)_*\Bigl(P \otimes p_1^* \omega_X[\dim X]\Bigr).$$

Recall from (15.13) that we have

$$R^i(p_2)_* P \cong \begin{cases} 0 & \text{if } i \neq \dim X, \\ k(0) & \text{if } i = \dim X. \end{cases}$$

In terms of the derived category, this says that $\mathbf{R}(p_2)_* P \cong k(0)[-\dim X]$. If we put this together with the formula above, and remember that ω_X is trivial, we get

$$\mathsf{FM}_X(\mathscr{O}_X) \cong k(0),$$

as required. We will prove later that for any $L \in \operatorname{Pic}^0(X)$, one has

$$\mathsf{FM}_X(L) \cong k(\alpha),$$

where $\alpha \in \hat{X}(k)$ is the unique closed point corresponding to L under the isomorphism of groups $\hat{X}(k) \cong \operatorname{Pic}^0(X)$.