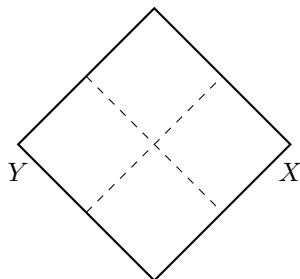


LECTURE 10 (FEBRUARY 27)

Last time, someone asked where the name “seesaw theorem” comes from. In one of the explanatory paragraphs in his collected works, André Weil writes that he introduced the name in a course on abelian varieties that he taught at the University of Chicago in 1954/55. Unfortunately, he does not explain why the theorem made him think of a seesaw. Ravi Vakil (in *The Rising Sea*) says that he has no idea why it is called the seesaw theorem. Herbert Lange (in his book *Complex Abelian Varieties*) says that it is “called the seesaw theorem for obvious reasons”. Perhaps the reason is that if we draw $X \times Y$ like this



then the two slices $\{x_0\} \times Y$ and $X \times \{y_0\}$ look like the two opposite positions of a seesaw. But your guess is as good as mine.

Anyway, here is a useful corollary.

Corollary 10.1. *Let L be a line bundle on $X \times Y$, where X, Y are varieties, and X is complete. If $L|_{X \times \{y\}}$ is trivial for every $y \in Y$, and if $L|_{\{x_0\} \times Y}$ is trivial for some point $x_0 \in X$, then L is trivial.*

Proof. By the seesaw theorem, we have $L \cong p_2^*M$ for a line bundle M on Y ; now restrict to $\{x_0\} \times Y$ to conclude that M is trivial. \square

The theorem of the cube. Our main topic today is the “theorem of the cube”, which is a result about line bundles on $X \times Y \times Z$. It is the crucial ingredient in proving results about line bundles on abelian varieties. Here is the statement.

Theorem 10.2. *Let L be a line bundle on $X \times Y \times Z$, where X, Y, Z are varieties, and X and Y are complete. Suppose that there are points $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$ such that the three line bundles*

$$L|_{\{x_0\} \times Y \times Z}, \quad L|_{X \times \{y_0\} \times Z}, \quad L|_{X \times Y \times \{z_0\}}$$

are trivial. Then L is trivial.

Note that this only works for three or more factors: a line bundle on $X \times Y$ can be trivial on $\{x_0\} \times Y$ and on $X \times \{y_0\}$ without being trivial. We can get some intuition for the statement from the case of complex manifolds. If $\text{Pic}(X)$ denotes the group of holomorphic line bundles, we have an exact sequence

$$0 \longrightarrow \text{Pic}^0(X) \longrightarrow \text{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

where $\text{Pic}^0(X)$ means line bundles with trivial first Chern class. Now consider a holomorphic line bundle L on $X \times Y \times Z$, say with X, Y, Z connected. By the Künneth formula, we have

$$\begin{aligned} H^2(X \times Y \times Z, \mathbb{Z}) \cong & H^2(X, \mathbb{Z}) \oplus H^2(Y, \mathbb{Z}) \oplus H^1(X, \mathbb{Z}) \otimes H^1(Y, \mathbb{Z}) \oplus \\ & H^1(X, \mathbb{Z}) \otimes H^1(Z, \mathbb{Z}) \oplus H^1(Y, \mathbb{Z}) \otimes H^1(Z, \mathbb{Z}) \oplus H^2(Z, \mathbb{Z}). \end{aligned}$$

Each summand involves at most two factors of the product, because we are looking at H^2 . If the restriction of L to all three slices $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$ and

$X \times Y \times \{z_0\}$ is trivial, it follows from this that $c_1(L) = 0$. Because $\text{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$, we also get from the Künneth formula that

$$\text{Pic}^0(X \times Y \times Z) \cong \text{Pic}^0(X) \times \text{Pic}^0(Y) \times \text{Pic}^0(Z),$$

and so a line bundle $L \in \text{Pic}^0(X \times Y \times Z)$ that is trivial on all three slices is trivial.

Before giving the proof, let's first deduce the following nice corollary.

Corollary 10.3. *If X and Y are complete varieties, then every line bundle on $X \times Y \times Z$ is isomorphic to a line bundle of the form*

$$p_{12}^*L_{12} \otimes p_{13}^*L_{13} \otimes p_{23}^*L_{23},$$

where L_{12}, L_{13}, L_{23} are line bundles on the three double products.

Proof. Choose three points $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$. Let M_1 denote the restriction of L^{-1} to $X \times \{y_0\} \times \{z_0\}$, and define M_2 and M_3 similarly. After replacing L by the tensor product

$$L \otimes p_1^*M_1 \otimes p_2^*M_2 \otimes p_3^*M_3,$$

we can assume without loss of generality that L is trivial on those three subvarieties. Now suppose that L_{12} is a line bundle on $X \times Y$ that is trivial on $\{x_0\} \times Y$ and on $X \times \{y_0\}$, and similarly for L_{13} and L_{23} . The condition that

$$M = L^{-1} \otimes p_{12}^*L_{12} \otimes p_{13}^*L_{13} \otimes p_{23}^*L_{23}$$

should be trivial on $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$ and $X \times Y \times \{z_0\}$ then uniquely determines L_{12} , L_{13} , and L_{23} . For example, we have

$$M|_{X \times Y \times \{z_0\}} \cong L^{-1}|_{X \times Y \times \{z_0\}} \otimes L_{12},$$

because L_{13} is trivial on $X \times \{z_0\}$ and L_{23} is trivial on $Y \times \{z_0\}$; therefore we can set $L_{12} = L|_{X \times Y \times \{z_0\}}$. With these choices, M is trivial on all three slices. The theorem of the cube implies that M is trivial, and this gives the desired result. \square

Proof of the theorem. Let L be a line bundle on $X \times Y \times Z$ such that

$$L|_{\{x_0\} \times Y \times Z}, \quad L|_{X \times \{y_0\} \times Z}, \quad L|_{X \times Y \times \{z_0\}}$$

are trivial. We want to prove that L itself must be trivial. The proof will hopefully make it clear why we need X and Y to be complete.

Step 1. To get started, we observe that it is enough to prove that $L|_{\{x\} \times Y \times \{z\}}$ is trivial for every $(x, z) \in X \times Z$. This is because of the seesaw theorem: Y is complete, and if L is trivial on every fiber of $p_{13}: X \times Y \times Z \rightarrow X \times Z$, it is the pullback of a line bundle from $Y \times Z$; but that line bundle must be trivial because we are assuming that L is trivial on $\{x_0\} \times Y \times Z$.

Step 2. This observation allows us to reduce the problem to the case where X is a nonsingular curve. Let $x \in X$ be an arbitrary point. Choose a complete irreducible curve $C \subseteq X$ that passes through the two points x_0 and x . (Such a curve clearly exists when X is projective; and by Chow's lemma, any complete variety admits a surjective map from a projective variety.) Let $f: \tilde{C} \rightarrow C$ be the normalization; then \tilde{C} is nonsingular and irreducible. Consider the pullback $M = (f \times \text{id} \times \text{id})^*L$ of the line bundle along the morphism

$$f \times \text{id} \times \text{id}: \tilde{C} \times Y \times Z \rightarrow X \times Y \times Z.$$

It still satisfies the assumptions in the theorem of the cube, but now on the product $\tilde{C} \times Y \times Z$. If we can show that M is trivial on every subvariety of the form $\{c\} \times Y \times \{z\}$, then M is trivial; and because x is in the image of f , this then implies that L is trivial on $\{x\} \times Y \times \{z\}$. So if we can prove the theorem of the cube when $\dim X = 1$ and X is nonsingular, then it will hold in general.

Remark. We don't actually need Chow's lemma here. For fixed $z \in Z$, the set of points $x \in X$ such that L is trivial on $\{x\} \times Y \times \{z\}$ is closed (by Theorem 9.10), and so it is enough to prove this for all x in an affine open neighborhood of the point x_0 . But any two points in an affine variety can clearly be connected by an irreducible curve.

Step 3. From now on, we assume that X is a complete, irreducible, and nonsingular curve. By the same argument as in Step 1, it is enough to prove that the line bundle

$$L_{(y,z)} = L|_{X \times \{y\} \times \{z\}}$$

is trivial for every $(y, z) \in Y \times Z$; in fact, we can even replace Z by a dense open subset, because the set of all such points is closed in $Y \times Z$ by the seesaw theorem.

Let ω_X be the canonical line bundle on the curve X , and let $g = \dim H^0(X, \omega_X)$ be the genus of the curve. We can choose g points $P_1, \dots, P_g \in C$ such that the divisor $D = P_1 + \dots + P_g$ satisfies $\dim H^0(X, \omega_X(-D)) = 0$: take a nontrivial section of ω_X and pick the first point P_1 such that the section does not vanish at P_1 ; then $\dim H^0(X, \omega_X(-P_1)) = g - 1$; and so on. By Serre duality, we get

$$\dim H^1(X, \mathcal{O}_X(D)) = \dim H^0(X, \omega_X(-D)) = 0.$$

We now adjust the line bundle L as follows. Let $p_1: X \times Y \times Z \rightarrow X$ be the first projection, and define

$$L' = L \otimes p_1^* \mathcal{O}_X(D).$$

As before, we set $L'_{(y,z)} = L'|_{X \times \{y\} \times \{z\}}$; evidently,

$$(10.4) \quad L'_{(y,z)} \cong L_{(y,z)} \otimes \mathcal{O}_X(D).$$

Because L is trivial on $X \times Y \times \{z_0\}$, we get $L'_{(y,z_0)} \cong \mathcal{O}_X(D)$ for all $y \in Y$; consequently, the first cohomology $H^1(X, L'_{(y,z_0)}) = 0$.

By Corollary 9.8, the set

$$F = \{ (y, z) \in Y \times Z \mid \dim H^1(X, L'_{(y,z)}) \geq 1 \}$$

is closed in $Y \times Z$. Because Y is proper, the image $p_2(F) \subseteq Z$ is also closed. We have just seen that it does not contain the point z_0 . We can therefore find an open set $Z' \subseteq Z$ containing the point z_0 , such that $p_2(F) \cap Z_0 = \emptyset$. This means concretely that

$$H^1(X, L'_{(y,z)}) = 0$$

for every $(y, z) \in Y \times Z'$. After replacing Z by the dense open subset Z' , we can assume that this holds for every $(y, z) \in Y \times Z$.

Step 4. We can use this to compute the space of global sections. By Corollary 9.8, the Euler characteristic is constant, and so

$$\begin{aligned} \dim H^0(X, L'_{(y,z)}) &= \chi(L'_{(y,z)}) = \chi(L'_{(y,z_0)}) = \chi(X, \mathcal{O}_X(D)) \\ &= \deg D - g + 1 = 1 \end{aligned}$$

by the Riemann-Roch theorem. Every line bundle $L'_{(y,z)}$ therefore has (up to scaling) a unique nontrivial global section, and so it determines a unique effective divisor on X (of degree $g = \deg D$). As we move $(y, z) \in Y \times Z$, these divisors are going to sweep out a divisor \tilde{D} on $X \times Y \times Z$.

To construct \tilde{D} rigorously, we can argue as follows. First, $\dim H^0(X, L'_{(y,z)}) = 1$ is constant, and so Corollary 9.9 implies that the pushforward $(p_{23})_* L'$ is a line bundle on $Y \times Z$. On any open set $U \subseteq Y \times Z$ where this line bundle is trivial, we can choose a nowhere vanishing section $s_U \in H^0(U, (p_{23})_* L')$. By the definition of the pushforward, it comes from a section $\tilde{s}_U \in H^0(X \times U, L')$, and we let \tilde{D}_U be the divisor of \tilde{s}_U . If $V \subseteq Y \times Z$ is another open set of this type, then s_U and

s_V differ from each other by an element of $H^0(U \cap V, \mathcal{O}_X^\times)$, and so \tilde{D}_U and \tilde{D}_V agree on $X \times (U \cap V)$. Consequently, there is a well-defined divisor \tilde{D} such that $\tilde{D}|_U = \tilde{D}_U$. It is clear from the construction that $\tilde{D}|_{X \times \{y\} \times \{z\}}$ is the divisor of the unique nontrivial section of $L'_{(y,z)}$.

Step 5. We'll complete the proof by showing that $\tilde{D} = p_1^*(D)$. Observe that

$$\tilde{D}|_{X \times \{y\} \times \{z_0\}} = D \quad \text{and} \quad \tilde{D}|_{X \times \{y_0\} \times \{z\}} = D$$

for every $y \in Y$ and every $z \in Z$; the reason is that $L'_{(y,z_0)} \cong L'_{(y_0,z)} \cong \mathcal{O}_X(D)$. So if we take a point $P \in X$ with $P \neq P_j$ for $j = 1, \dots, g$, then the divisor

$$\tilde{D}_P = \tilde{D}|_{\{P\} \times Y \times Z}$$

does not intersect the two closed subsets $\{P\} \times Y \times \{z_0\}$ and $\{P\} \times \{y_0\} \times Z$. The projection $p_2(\tilde{D}_P) \subseteq Z$ is a closed subset (because Y is complete); because it does not contain the point z_0 , it must be a proper closed subset. For dimension reasons, this implies that the divisor \tilde{D}_P is supported on a finite union of closed subsets of the form $\{P\} \times Y \times T_j$, where $T_j \subseteq Z$ has codimension one. But \tilde{D}_P also does not intersect $\{P\} \times \{y_0\} \times Z$, and this is now only possible if \tilde{D}_P is empty.

Step 6. The conclusion is that \tilde{D} does not intersect the set $\{P\} \times Y \times Z$, and being a divisor, it must therefore be of the form

$$\tilde{D} = \sum_{j=1}^g c_j \cdot \{P_j\} \times Y \times Z$$

for certain integers $c_1, \dots, c_g \in \mathbb{N}$. But $\tilde{D}|_{X \times \{y\} \times \{z_0\}} = D$, and so $c_1 = \dots = c_g = 0$, or equivalently, $\tilde{D} = p_1^*(D)$. This gives $L'_{(y,z)} \cong \mathcal{O}_X(D)$. If we now go back to (10.4), we find that

$$L_{(y,z)} \cong L'_{(y,z)} \otimes \mathcal{O}_X(-D) \cong \mathcal{O}_X,$$

and so L is indeed trivial on all subvarieties of the form $X \times \{y\} \times \{z\}$. As we said above, this is enough to conclude that L is trivial on $X \times Y \times Z$.

Line bundles on abelian varieties. The theorem of the cube has many nice consequences for line bundles on abelian varieties. Let X be an abelian variety, and let L be a line bundle on X . The group operation is $m: X \times X \rightarrow X$, and by extension, we also write

$$m: X \times X \times X \rightarrow X, \quad m(x, y, z) = x + y + z.$$

Denote by $p_{i,j}: X \times X \times X \rightarrow X \times X$ the projections, and set

$$m_{i,j}: X \times X \times X \rightarrow X, \quad m_{i,j} = m \circ p_{i,j}.$$

Consider the line bundle

$$M = m^*L \otimes m_{1,2}^*L^{-1} \otimes m_{1,3}^*L^{-1} \otimes m_{1,3}^*L^{-1} \otimes p_1^*L \otimes p_2^*L \otimes p_3^*L.$$

Because $0 \in X$ is the neutral element, it is easy to see that M is trivial on all three slices $\{0\} \times X \times X$, $X \times \{0\} \times X$, and $X \times X \times \{0\}$. By the theorem of the cube, M is trivial on $X \times X \times X$, and therefore

$$(10.5) \quad m^*L \cong m_{1,2}^*L \otimes m_{1,3}^*L \otimes m_{1,3}^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1} \otimes p_3^*L^{-1}.$$

If we now have three morphisms $f, g, h: T \rightarrow X$ from some other variety T , we can pull back this identity along the mapping $(f, g, h): T \rightarrow X \times X \times X$; this proves the following result.

Corollary 10.6. *Let $f, g, h: T \rightarrow X$ be three morphisms to an abelian variety. For any line bundle L on X , one has*

$$(f + g + h)^*L \cong (f + g)^*L \otimes (f + h)^*L \otimes (g + h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.$$