The Killing form and Cartan's criteria

MAT 552

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Brief review: solvable and semisimple

Our current topic is the structure of (abstract) Lie algebras. We had introduced two important classes:

1. $\mathfrak{g}$ is solvable if the chain

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supseteq \cdots$$

   goes to zero. Example: upper triangular matrices

2. $\mathfrak{g}$ is semisimple if the only solvable ideal is the zero ideal. Example: $\mathfrak{sl}(n, \mathbb{C})$

Brief review: radical

The radical $\operatorname{rad}(\mathfrak{g})$ is the largest solvable ideal in $\mathfrak{g}$.

- The quotient $\mathfrak{g}/\operatorname{rad}(\mathfrak{g})$ is semisimple.
- Lie's theorem: $\mathfrak{g} \cong \operatorname{rad}(\mathfrak{g}) \oplus \mathfrak{g}/\operatorname{rad}(\mathfrak{g})$.
- Every Lie algebra has a solvable and a semisimple part.

$\mathfrak{g}$ is reductive if $\operatorname{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$, the center of $\mathfrak{g}$.

- Example: $\mathfrak{gl}(n, \mathbb{R})$
- $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is semisimple
- $\mathfrak{g} \cong$ (abelian Lie algebra) $\oplus$ (semisimple Lie algebra)

Brief review: invariant bilinear forms

Before the break, we showed that the classical Lie algebras are reductive/semisimple using invariant bilinear forms.
Given a representation $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$, define

$$B_V(x, y) = \mathrm{tr}\Big(\rho(x) \circ \rho(y)\Big).$$

The bilinear form B_V is

- symmetric: $B_V(x, y) = B_V(y, x)$
  Reason: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
- invariant: $B_V([z, x], y) + B_V(x, [z, y]) = 0$
  Reason: $\mathrm{tr}\Big((CA - AC)B) + A(CB - BC)\Big) = 0$

Theorem

If B_V is nondegenerate (for some V), then $\mathfrak{g}$ is reductive.

The Killing form

Most important case: the adjoint representation

$$\operatorname{ad}\colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g}), \quad \operatorname{ad} x.y = [x, y]$$

The bilinear form

$$K(x, y) = K^{\mathfrak{g}}(x, y) = \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y)$$

is called the Killing form of $\mathfrak{g}$. It is symmetric and invariant.

Homework problem

If $I \subseteq \mathfrak{g}$ is an ideal, then $K^I = K^{\mathfrak{g}}|_{I \times I}$.

The Killing form

Example

Take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. In the basis e, h, f, we have

$$\operatorname{ad} e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \operatorname{ad} f = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix},$$

$$\operatorname{ad} h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Thus $K(h, h) = 8$, $K(e, f) = 4$, $K(e, e) = K(f, f) = 0$.
In fact, $K(x, y) = 4 \operatorname{tr}(xy)$. (Since $\mathfrak{sl}(2, \mathbb{C})$ is simple, the invariant bilinear form is unique up to scaling.)

Cartan's criteria

The Killing form tells us the structure of $\mathfrak{g}$.

Theorem (Cartan's criterion for solvability)

A Lie algebra $\mathfrak{g}$ is solvable if and only if $K([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) = 0$.

Theorem (Cartan's criterion for semisimplicity)

A Lie algebra $\mathfrak{g}$ is semisimple if and only if K is nondegenerate.

Many theoretical consequences (next time). Today, the proof.

Jordan decomposition

Main tool: Jordan decomposition (= fancy version of JCF)
Let V be a finite-dimensional $\mathbb{C}$-vector space.
Every $A \in \mathrm{End}(V)$ can be uniquely written as

$$A = A_s + A_n,$$

with A_n nilpotent and A_s semisimple (= diagonalizable).

- A_s and A_n commute.
- A_s and A_n can be written as polynomials in A.

Jordan decomposition: proof

Let $A \in \mathrm{End}(V)$. Put A in Jordan canonical form.

$$\begin{array}{|ccc|c|cc|}
\hline
\lambda_1 & 1 & & & & \\
& \lambda_1 & 1 & & & \\
& & \lambda_1 & & & \\
\hline
& & & \lambda_2 & & \\
\hline
& & & & \lambda_3 & 1 \\
& & & & & \lambda_3 \\
\hline
\end{array}$$

Let A_s be the diagonal part, and $A_n = A - A_s$.
Then A_n is clearly nilpotent.

Jordan decomposition: proof

Decomposition into (generalized) eigenspaces

$$V = V_1 \oplus \cdots \oplus V_n,$$

with A_s acting as multiplication by λ_j on V_j.
Want a polynomial $P(t) \in \mathbb{C}[t]$ such that $A_s = P(A)$.

- Choose $P(t) \in \mathbb{C}[t]$ such that

$$P(t) \equiv \lambda_j \mod (t - \lambda_j)^{\dim V_j}.$$

- This exists by the Chinese Remainder Theorem.
- Since $(A - \lambda_j \operatorname{id})^{\dim V_j} = 0$ on V_j, the matrix $P(A)$ acts as multiplication by λ_j on V_j.
- Therefore $P(A) = A_s$.

Jordan decomposition: consequences

Define $\operatorname{ad} A\colon \operatorname{End}(V) \to \operatorname{End}(V)$ by $B \mapsto AB - BA$. Then

- $(\operatorname{ad} A)_s = \operatorname{ad} A_s$
- $\operatorname{ad} A_s = P(\operatorname{ad} A)$, for some $P(t) \in t\,\mathbb{C}[t]$.

Define $\bar{A}_s \in \operatorname{End}(V)$: same eigenspaces as A_s, but $\bar{A}_s$ acts as multiplication by $\bar{\lambda}$ on the λ-eigenspace of A_s. Then

- $\operatorname{ad} \bar{A}_s = Q(\operatorname{ad} A)$, for some $Q \in t\,\mathbb{C}[t]$.

Cartan's criterion for solvability: proof

Theorem (Cartan's criterion for solvability)

A Lie algebra $\mathfrak{g}$ is solvable if and only if $K([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) = 0$.

If $\mathfrak{g}$ is real, then $\mathfrak{g}$ is solvable iff $\mathfrak{g}_{\mathbb{C}}$ is solvable.
So only need to consider complex $\mathfrak{g}$.

Cartan's criterion for solvability: proof

"$\Longrightarrow$" Suppose that $\mathfrak{g}$ is solvable.
We need to show that $K([x, y], z) = 0$ for every $x, y, z \in \mathfrak{g}$.

- Since $\mathfrak{g}$ is solvable, there is a basis in $\mathfrak{g}$ in which every $\operatorname{ad} x$ is upper triangular (Lie's theorem).
- Then $\operatorname{ad}[x, y] = [\operatorname{ad} x, \operatorname{ad} y]$ is strictly upper triangular.
- Therefore $\operatorname{ad}[x, y] \circ \operatorname{ad} z$ is also strictly upper triangular.
- It follows that $K([x, y], z) = \operatorname{tr}(\operatorname{ad}[x, y] \circ \operatorname{ad} z) = 0$.

Cartan's criterion for solvability: proof

"$\Longleftarrow$" Suppose that $K([x,y],z) = 0$ for every $x,y,z \in \mathfrak{g}$. We need to show that $\mathfrak{g}$ is solvable.

- Consider $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.
- $\ker \mathrm{ad} = \mathfrak{z}(\mathfrak{g})$ is solvable.
- Thus enough to show that $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is solvable.
- Follows from lemma (with $V = \mathfrak{g}$).

Lemma

Let $\mathfrak{g} \subseteq \mathrm{End}(V)$ be a Lie subalgebra. If $\mathrm{tr}(xy) = 0$ for every $x \in [\mathfrak{g},\mathfrak{g}]$ and every $y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable.

Cartan's criterion for solvability: proof

Now we prove the lemma. Take $x \in [\mathfrak{g}, \mathfrak{g}]$.
Jordan decomposition $x = x_s + x_n$. Then

$$\operatorname{tr}(x\bar{x}_s) = \operatorname{tr}(x_s\bar{x}_s) = \sum_j |\lambda_j|^2,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of x_s.
Since $x \in [\mathfrak{g}, \mathfrak{g}]$, have $x = \sum_k [y_k, z_k]$, hence

$$\operatorname{tr}(x\bar{x}_s) = \sum_k \operatorname{tr}\Big([y_k, z_k]\bar{x}_s\Big) = -\sum_k \operatorname{tr}\Big(z_k[y_k, \bar{x}_s]\Big).$$

But now $[\bar{x}_s, y_k] = \operatorname{ad}(\bar{x}_s).y_k = Q(\operatorname{ad} x).y_k \in [\mathfrak{g}, \mathfrak{g}]$.
By assumption, we get $\operatorname{tr}\Big(z_k[y_k, \bar{x}_s]\Big) = 0$.
Thus $\lambda_1 = \cdots = \lambda_n = 0$, and so x is nilpotent.
By Engel's theorem, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, hence $\mathfrak{g}$ is solvable.

Cartan's criterion for semisimplicity: proof

Theorem (Cartan's criterion for semisimplicity)

A Lie algebra $\mathfrak{g}$ is semisimple if and only if K is nondegenerate.

"$\Longleftarrow$" Suppose that K is nondegenerate.
We need to show that $\mathfrak{g}$ is semisimple.

- The Killing form is $B_{\mathfrak{g}}$ for the adjoint representation.
- So K nondegenerate implies that $\mathfrak{g}$ is reductive.
- Semisimple is equivalent to reductive and $\mathfrak{z}(\mathfrak{g}) = \{0\}$.
- If $x \in \mathfrak{z}(\mathfrak{g})$, then $\operatorname{ad} x = 0$, hence $x \in \ker K$.
- Since K is nondegenerate, get $x = 0$.
- So $\mathfrak{z}(\mathfrak{g}) = \{0\}$, which means $\mathfrak{g}$ is semisimple.

Cartan's criterion for semisimplicity: proof

"$\Longrightarrow$" Suppose that $\mathfrak{g}$ is semisimple.
We need to show that K is nondegenerate.

- Let $I = \ker K$; this is an ideal in $\mathfrak{g}$.
- The Killing form K^I is the restriction of K, hence $K^I = 0$.
- By Cartan's criterion for solvability, I is solvable.
- But $\mathfrak{g}$ is semisimple, and so $I = \{0\}$.