Math 534
Problem Set 9

due Tuesday, November 20, 2018

In problems #1–5, R is always a commutative ring with 1, while M, N, etc. are R-modules. To abbreviate, "module" will always mean R-module, "morphism" will mean morphism of R-modules, and so on.

1. A module M is said to be *simple* if M is not the zero module and the only submodules of M are M and 0. Show that M is a simple module if and only if $M \cong R/I$, where I is a maximal ideal of R. (Here $\cong$ means isomorphic as R-modules.)

2. Suppose that M and N are submodules of a module P. Show that $(M+N)/M \cap N \cong (M/M \cap N) \oplus (N/M \cap N)$.

3. Suppose that R is an integral domain. An element $m \in M$ is said to be a *torsion element* if $rm = 0$ for some $r \neq 0$ in R. (In that case, one also says that r *annihilates* the element m.)

   (a) Let M_{tor} be the set of torsion elements in M. Show that M_{tor} is a submodule of M.

   (b) M is said to be *torsion-free* if $M_{\text{tor}} = 0$. Show that M/M_{tor} is always torsion-free.

   (c) Suppose that $R = \mathbb{Z}$ and $M = \mathbb{R}/\mathbb{Z}$. What is M_{tor} in this case?

4. Let M be a module and let $m \in M$. The *annihilator* of m in R, usually denoted $\operatorname{Ann}(m)$, is the set $\left\{ r \in R \mid rm = 0 \right\}$.

   (a) Show that $\operatorname{Ann}(m)$ is an ideal of R.

   (b) A module M is said to be *cyclic* if it is generated by a single element $m_0 \in M$, in the sense that every element $m \in M$ can be written as $m = rm_0$ for some $r \in R$. Show that

   $$M \text{ is cyclic} \quad \Leftrightarrow \quad M \cong R/I \text{ for some ideal } I \text{ of } R$$

5. Suppose that R is an integral domain, and let I be a nonzero ideal of R. Show that $I \cong R$ if and only if I is a principal ideal of R.

6. Let F be a field, let $F[x]$ be the ring of polynomials over F, and let V be an $F[x]$-module that is finite-dimensional as a vector space over F.

(a) Show that V is a torsion module. (Hint: For a given $v \in V$, consider the sequence of elements $v, xv, x^2v, \ldots$.)

(b) Let $T\colon V \to V$ be the map defined by $T(v) = xv$ for $v \in V$. Show that T is a linear transformation of the F-vector space V.

7. Let F be a field, G a finite group, and $F[G]$ the group ring. Let V be a finitely-generated $F[G]$-module.

   (a) Show that V is a finite-dimensional vector space over F.

   (b) Let $g \in G$, and let $T_g\colon V \to V$ be the map $T_g(v) = gv$. Show that T_g is a linear transformation of the F-vector space V.

   (c) Let $v_1, \ldots, v_n$ be a basis for V over F, and for each $g \in G$, let M_g be the matrix of T_g with respect to the basis $v_1, \ldots, v_n$. Show that $M_g \in \mathrm{GL}_n(F)$, and that the map

   $$\rho\colon G \to \mathrm{GL}_n(F), \quad g \mapsto M_g$$

   is a group homomorphism.

8. Let R be a noetherian ring.

   (a) Let M be a submodule of R^n for some $n \geq 1$. Let $\pi_1\colon R^n \to R$ be the projection to the first coordinate, so $\pi_1(r_1, \ldots, r_n) = r_1$. Show that $\pi_1(M)$ is an ideal in R.

   (b) Let $M_0 = \{\, m \in M \mid \pi_1(m) = 0 \,\}$. Show that there is a finitely-generated submodule $M' \subseteq M$ such that $M = M' + M_0$.

   (c) Use induction on $n \geq 1$ to prove that every submodule of R^n is finitely-generated.

   (d) Deduce that every submodule of a finitely-generated R-module is again finitely-generated.