**Math 534**
**Problem Set 4**

due Thursday, September 27, 2018

Midterm 1 will be held in class on Thursday, September 27.

1. A *characteristic subgroup* of a group is a subgroup $H \subseteq G$ such that $\alpha(H) = H$ for every automorphism $\alpha \in \mathrm{Aut}(G)$. Show that the center $Z$ and the commutator subgroup $G'$ are characteristic subgroups of $G$.

2. Let $\phi \colon G \to \mathrm{Aut}(G)$ be the map that associates to any $g \in G$ the corresponding inner automorphism: $\phi(g) = \alpha_g$, where $\alpha_g(x) = gxg^{-1}$.

   (a) Show that $\phi$ is a homomorphism.

   (b) Show that the kernel of $\phi$ is the center $Z \subseteq G$.

   (c) Let $\mathrm{Inn}(G)$ be the image of $\phi$. Show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

3. Let $n \geq 2$. Show that $A_n$ is the only subgroup of $S_n$ of index 2.

4. Let $n \geq 3$, and let $N \subseteq A_n$ be the subgroup generated by all 3-cycles.

   (a) Show that $N$ is a normal subgroup of $A_n$.

   (b) Show that $N = A_n$.

5. Let $n \geq 4$. Show that there exists an injective group homomorphism $S_n \to \mathrm{Aut}(A_n)$. What happens for $n = 3$?

6. Suppose that $G$ is a simple group of order 60.

   (a) Show that $G$ has exactly six Sylow 5-subgroups.

   (b) Show that the action of $G$ on its Sylow 5-subgroups (by conjugation) gives an injective homomorphism $\phi \colon G \to S_6$, and that the image of $\phi$ is a subgroup of $A_6$ of index 6.

   (c) Show that $G \cong A_5$.

7. In this problem, we construct an exotic automorphism of $S_6$.

   (a) Show that $S_5$ has exactly 6 Sylow 5-subgroups.

   (b) Show that the action of $S_5$ on its Sylow 5-subgroups gives an injective homomorphism $\phi \colon S_5 \to S_6$, whose image $H$ is a subgroup of $S_6$ of index 6.

(c) Let $H_k \subseteq S_6$ be the stabilizer of $k \in \{1, 2, \ldots, 6\}$. Show that $H$ is *not* one of the subgroups $H_1, \ldots, H_6$.

(d) Show that the action of $S_6$ on the cosets of $H$ gives an automorphism $\alpha\colon S_6 \to S_6$.

(e) Show that $\alpha$ is *not* an inner automorphism.

8. Let $G$ be a solvable finite group. Show that there is a chain

$$G = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_r = \{1\},$$

such that $N_{i+1} \triangleleft N_i$, and $N_i/N_{i+1}$ is cyclic, for every $0 \leq i \leq r-1$.

9. Let $G$ be a solvable group.

(a) Show that every subgroup of $G$ is solvable.

(b) Let $\phi\colon G \to H$ be a homomorphism. Show that $\phi(G)$ is solvable.

10. Let $N$ be a normal subgroup of a group $G$. Show that $G$ is solvable if and only if both $N$ and $G/N$ are solvable.