Math 534
Problem Set 10

due Thursday, December 6, 2018

1. Describe all finitely generated subgroups of $(\mathbb{Q}, +)$. What are the possible values for the rank of such a subgroup?

2. Let R be a principal ideal domain. Show that every submodule of R^d is isomorphic to R^e for some number $e \leq d$.

3. Let R be an integral domain with fraction field F.

   (a) Show that an isomorphism of R-modules $R^d \cong R^e$ induces an isomorphism of F-vector spaces $F^d \cong F^e$.

   (b) Conclude that $R^d \cong R^e$ implies $d = e$.

4. Let R be a principal ideal domain, and M a nontrivial finitely generated torsion-free R-module. For any element $m \in M$, denote by $\langle m \rangle = \{ rm \mid r \in R \}$ the submodule generated by m.

   (a) Show that if $m_1, m_2 \in M$ are such that $\langle m_1 \rangle \cap \langle m_2 \rangle \neq \{0\}$, then $\langle m_1 \rangle + \langle m_2 \rangle = \langle m \rangle$ for some $m \in M$.

   (b) Show that $M/\langle m \rangle$ is torsion-free if and only if $\langle m \rangle$ is maximal among submodules of this type.

   (c) Show that there exists a nonzero element $m \in M$ such that the R-module $M/\langle m \rangle$ is torsion-free.

5. Let $T\colon V \to V$ be a linear transformation of a finite-dimensional vector space (over F). After choosing a basis $v_1, \ldots, v_n \in V$, we can represent T by an $n \times n$-matrix A. Show that if A' is the $n \times n$-matrix representing T with respect to another basis $v_1', \ldots, v_n' \in V$, then A' is conjugate to A, in the sense that there exists $B \in \mathrm{GL}_n(F)$ with $A' = B^{-1}AB$.

6. Let V be an F-vector space. Denote by $V^* = \mathrm{Hom}(V, F)$ the *dual vector space*, whose elements are the linear functionals $\varphi\colon V \to F$.

   (a) Show that there is a natural linear transformation $V \to V^{**}$.

   (b) If V is finite-dimensional, show that $V \cong V^{**}$.

7. Let V be an F-vector space. The notation $\langle \varphi, v \rangle = \varphi(v)$ is sometimes used to mean the value of a linear functional $\varphi \in V^*$ on a vector $v \in V$.

(a) Show that a linear transformation $T\colon V \to V$ induces a linear transformation $T^*\colon V^* \to V^*$, called the *adjoint* of T, such that $\langle T^*\varphi, v\rangle = \langle \varphi, Tv\rangle$ for all $v \in V$ and all $\varphi \in V^*$.

(b) Choose a basis $v_1, \dots, v_n \in V$. Show that V^* has a unique basis $\varphi_1, \dots, \varphi_n$ with the property that

$$\varphi_i(v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(c) Let A be the matrix representing T with respect to $v_1, \dots, v_n$. Show that the matrix representing T^* with respect to $\varphi_1, \dots, \varphi_n$ is the transpose of the matrix A.

(d) Show that $T^{**} = T$ under the isomorphism in Problem 6.

8. Let $q(x) \in \mathbb{R}[x]$ be an irreducible quadratic polynomial and $e \geq 1$. Find a basis for the $\mathbb{R}$-vector space $\mathbb{R}[x]/(q(x)^e)$ such that the linear transformation corresponding to x is given by the following matrix:

$$\begin{pmatrix} a & b & & & & & \\ -b & a & 1/b & & & & \\ & & a & b & & & \\ & & -b & a & 1/b & & \\ & & & & \ddots & & \\ & & & & & a & b \\ & & & & & -b & a & 1/b \end{pmatrix}$$

Hint: Do the case $e = 1$ first. How do you get a, b from the polynomial?

9. Let V be a finite-dimensional F-vector space. Suppose that the minimal polyonomial of a linear transformation $T\colon V \to V$ can be factored as $(x - \lambda_1)^{e_1} \cdots (x - \lambda_n)^{e_n}$ for distinct elements $\lambda_1, \dots, \lambda_n \in F$. We proved in class that, as $F[x]$-modules,

$$V \cong V_1 \oplus \cdots \oplus V_n,$$

where $V_i \subseteq V$ is the submodule annihilated by $(x - \lambda_i)^{e_i}$. Show that the resulting coordinate projections $\pi_i\colon V \to V_i$ can be written in the form $\pi_i(v) = f_i(T)v$ for certain polynomials $f_i(x) \in F[x]$.