SOLUTIONS TO THE MIDTERM

1. Determine the connected components of the space $\mathbb{R}_\ell$.

Solution. Every connected component of $\mathbb{R}_\ell$ consists of just one point, because no subset $X \subseteq \mathbb{R}_\ell$ with at least two points can be connected. Indeed, if $a, b \in X$ are distinct points, say with $a < b$, then $U = X \cap (-\infty, b)$ and $V = X \cap [b, \infty)$ are disjoint open subsets with $a \in U$ and $b \in V$. They form a separation of $X$, which means that $X$ is not connected. \(\square\)

2. (a) Prove that every compact metric space is second countable. (b) Give an example of a metric space that is not second countable.

Solution. We first prove (a). For every integer $n \geq 1$, the collection of all open balls of radius $1/n$ is an open covering of $X$. Since $X$ is compact, finitely many of these open balls cover $X$; let us denote this finite collection by $B_n$. Their union

$$B = \bigcup_{n=1}^{\infty} B_n$$

is clearly countable. I claim that it is a basis for the metric topology, which is therefore second countable. To prove that $B$ is a basis, it suffices to show that every open set $U \subseteq X$ is a union of open balls in $B$. Let $x \in U$ be an arbitrary point. By definition of the metric topology, there is some $r > 0$ with $B_r(x) \subseteq U$. Take any integer $n$ with $1/n < r/2$. Since the finitely many open balls in $B_n$ cover $X$, we can find some $y \in X$ with

$$x \in B_{1/n}(y) \in B_n.$$  

Now $d(x, y) < 1/n < r/2$, and by the triangle inequality, we get $B_{1/n}(y) \subseteq B_r(x) \subseteq U$. This proves that $U$ is a union of open balls in $B$.

For (b), we can take any uncountable set $X$ with the metric $d(x, y) = 0$ if $x = y$, and $d(x, y) = 1$ if $x \neq y$. In this metric, every one-point set is open; every basis for the topology has to contain all these sets, and must therefore be uncountable. \(\square\)

3. Let $X$ be a connected and normal topological space with at least two points. Show that $X$ must be uncountable.

Solution. Let $a, b \in X$ be two distinct points. Since $X$ is normal, Urysohn’s lemma gives us a continuous function $f : X \to [0, 1]$ with $f(a) = 0$ and $f(b) = 1$. Now suppose for contradiction that $X$ is countable. Then $f(X)$ is countable, too; because $[0, 1]$ is uncountable, there is a real number $c \in (0, 1)$ with $c \not\in f(X)$. But now

$$X = f^{-1}[0, c] \cup f^{-1}(c, 1]$$

is a separation of $X$, contradicting the fact that $X$ is connected. \(\square\)
4. Let $X$ and $Y$ be topological spaces, with $Y$ compact. Show that the projection $p: X \times Y \to X$ is closed.

Solution. Let $A \subseteq X \times Y$ be a closed subset. We have to show that $p(A)$ is closed, or equivalently, that $X \setminus p(A)$ is open. Let $x \in X \setminus p(A)$ be an arbitrary point; we shall argue that some neighborhood of $x$ is contained in $X \setminus p(A)$.

Since $x \notin p(A)$, we must have $(x, y) \notin A$ for every $y \in Y$; in other words, $(x, y)$ is a point of the open set $X \times Y \setminus A$. By definition of the product topology, there is a basic open set with

$$(x, y) \in U(y) \times V(y) \subseteq X \times Y \setminus A.$$

Note that we have $x \in U(y)$ and $y \in V(y)$. The collection of open sets $V(y)$ covers $Y$, and so by compactness, there are finitely many points $y_1, \ldots, y_n \in Y$ with

$$Y = V(y_1) \cup \cdots \cup V(y_n).$$

If we define $U = U(y_1) \cap \cdots \cap U(y_n)$, then $x \in U$. I claim that $U$ is contained in $X \setminus p(A)$. Indeed, any point $x' \in U \cap p(A)$ would be the projection of a point $(x', y') \in A$; but then $y' \in V(y_i)$ for some $i$, and so

$$(x', y') \in U \times V(y_i) \subseteq U(y_i) \times V(y_i) \subseteq X \times Y \setminus A,$$

which is absurd. □