HOMEWORK 2, SOLUTION TO #2

**Problem.** Let $X$ be the set of all functions $f : \mathbb{N} \to \mathbb{R}$; such a function is of course the same thing as the sequence of real numbers $f(0), f(1), \ldots$

(a) For every $g \in X$ and every sequence of positive real numbers $\varepsilon_0, \varepsilon_1, \ldots$, we can form the following subset of $X$:

$$\{ f \in X \mid |f(n) - g(n)| < \varepsilon_n \text{ for every } n = 0, 1, \ldots \}$$

Prove that the collection of all such subsets is a basis for a topology on $X$. Show that this topology does not satisfy the first countability axiom.

(b) Find an example of a subset of $X$ that is sequentially closed but not closed.

**Proof.** Step 1. To show that the collection $\mathcal{B}$ of subsets as above is a basis for a topology on $X$, we can use Proposition 2.8 from the notes. On the one hand, the union of all the sets in $\mathcal{B}$ is equal to $X$: given $f \in X$, we set $\varepsilon_n = |f(n)| + 1$, and then $f$ belongs to the basic open set

$$\{ f \in X \mid |f(n)| < \varepsilon_n \text{ for every } n = 0, 1, \ldots \}.$$

On the other hand, the intersection of two basic open sets is either empty or again a basic open set: if one open set is defined by $g \in X$ and $\varepsilon_0, \varepsilon_1, \ldots$, and the other by $g' \in X$ and $\varepsilon'_0, \varepsilon'_1, \ldots$, then their intersection consists of all functions $f \in X$ that satisfy

$$f(n) \in (g(n) - \varepsilon_n, g(n) + \varepsilon_n) \cap (g'(n) - \varepsilon'_n, g'(n) + \varepsilon'_n) \quad (\text{for all } n \geq 0).$$

Because the intersection of two open intervals is either empty or an open interval, this again defines a basic open set. Proposition 2.8 shows that $\mathcal{B}$ is a basis for a topology on $X$; of course, this topology is nothing but the box topology on the infinite product $X = \mathbb{R}^\mathbb{N}$.

Step 2. Now we show that the constant function $0$ has no countable neighborhood basis; this means that the topology is not first countable. Suppose to the contrary that there were countably many open sets $U_1, U_2, \ldots$ such that every neighborhood of $0$ contained at least one $U_k$. Since every $U_k$ is itself a union of basic open sets, we can replace $U_k$ by a basic open subset containing $0$, and assume without loss of generality that

$$U_k = \{ f \in X \mid |f(n)| < \varepsilon_{k,n} \text{ for every } n = 0, 1, \ldots \}$$

for some collection of positive real numbers $\varepsilon_{k,n}$. Now we can use a version of the diagonal argument to construct a neighborhood of $0$ that does not contain any $U_k$. Define $\varepsilon'_n = \frac{1}{2} \varepsilon_{n,n}$ and consider the basic open set

$$\{ f \in X \mid |f(n)| < \varepsilon'_n \text{ for every } n = 0, 1, \ldots \}.$$

It is a neighborhood of $0$, but does not contain $U_k$ for any $k$; this contradicts our original assumption.

Step 3. Before we describe a subset that is sequentially closed but not closed, let us think about what it means for a sequence $f_1, f_2, \ldots \in X$ to converge to a point $f \in X$. Since the coordinate projections $p_n : X \to \mathbb{R}$ are continuous, convergence
in $X$ implies that $\lim_{k \to \infty} f_k(n) = f(n)$ for every $n \in \mathbb{N}$. But I claim that much more is true: if the sequence converges, then there is an integer $N \geq 0$ such that

$$f_k(n) = f(n) \text{ for every } k, n \geq N.$$  

(In fact, one can show that both conditions together are equivalent to convergence.) Suppose, by way of contradiction, that this was not the case. This would mean that, no matter how large $N$ is, we can always find $k, n \geq N$ with $f_k(n) \neq f(n)$. We can use this to pick out a subsequence that does not converge to $f$, a contradiction. Start by choosing $k_1, n_1 \geq 1$ with $f_{k_1}(n_1) \neq f(n_1)$. Next, choose $k_2, n_2 \geq \max(k_1, n_1) + 1$ with $f_{k_2}(n_2) \neq f(n_2)$. Then choose $k_3, n_3 \geq \max(k_2, n_2) + 1$ with $f_{k_3}(n_3) \neq f(n_3)$. Continuing in this manner, we obtain a subsequence

$$f_{k_1}, f_{k_2}, \ldots$$

of our original sequence, with the property that

$$f_{k_i}(n_i) \neq f(n_i)$$

for natural numbers $n_1 < n_2 < \cdots$. Now we can again employ a diagonal argument to show that this subsequence does not converge to $f$. Let us define

$$\varepsilon_n = \frac{1}{2} |f_{k_i}(n_i) - f(n_i)|$$

if $n = n_i$ for some $i = 1, 2, \ldots$; otherwise, set $\varepsilon_n = 1$. Then the basic open set

$$\{ g \in X \mid |g(n) - f(n)| < \varepsilon_n \text{ for every } n = 0, 1, \ldots \}$$

does not contain any term of the subsequence, which is absurd.

Step 4. Now we can write down a sequentially closed subset that is not closed. One example was mentioned on Friday: the set of all functions $f \in X$ such that $f(n) = 0$ for only finitely many values of $n \in \mathbb{N}$. Here is an even wilder set:

$$Y = \{ f \in X \mid f(n) \in \mathbb{Q} \text{ for only finitely many } n \in \mathbb{N} \}$$

I claim that $Y$ is sequentially closed, but that the closure of $Y$ is equal to all of $X$. The first half is easy: if a sequence $f_1, f_2, \ldots \in Y$ converges to some $f \in X$, then Step 3 shows that there is some $N$ such that $f(n) = f_k(n)$ for every $k, n \geq N$; this clearly means that $f \in Y$. For the second half, it will be enough to show that every basic open set contains at least one point of $Y$: since $X \setminus \overline{Y}$ is open, this forces $\overline{Y} = X$. Given a basic open set

$$\{ f \in X \mid |f(n) - g(n)| < \varepsilon_n \text{ for every } n = 0, 1, \ldots \},$$

we simply choose one irrational number $f(n)$ in the interval $(g(n) - \varepsilon_n, g(n) + \varepsilon_n)$ for every $n \in \mathbb{N}$; the resulting function $f$ belongs to $Y$. $\square$