HOMEWORK 1

This is the first of your weekly homework assignments. Since they will be graded, please take care to write up your solutions neatly. You may discuss problems with your fellow students, but please keep the following two things in mind: (1) If somebody tells you the solution to a problem before you have thought about it yourself, you miss a chance to learn something. (2) You must write up the solutions by yourself and in your own words.

Each week’s assignment is due the following Monday at the beginning of class. Since September 1 is a holiday, this particular assignment is due on September 3.

PROBLEMS FROM THE TEXTBOOK

§13, Exercises 4, 8(a); §16, Exercises 3, 4, 10; §17, Exercises 10, 11, 13, 21.

OTHER PROBLEMS

1. Prove that the railroad metric is indeed a metric on $\mathbb{R}^2$. What do open balls in this metric look like?

2. Give an example to show that infinite intersections of open sets in a metric space need not be open.

3. Let $X$ be an infinite set. Show that the collection

$$\mathcal{T} = \{ U \subseteq X \mid U = \emptyset \text{ or } X \setminus U \text{ is finite} \}$$

defines a topology on $X$; it is called the cofinite topology.

4. Prove that the two metrics

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \quad \text{and} \quad d'(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

define the same topology on $\mathbb{R}^n$.

5. Hillel Furstenberg found the following cute proof for Euclid’s theorem about prime numbers. An arithmetic progression is a set of the form

$$S(a, b) = \{ an + b \mid n \in \mathbb{Z} \}$$

with $a, b \in \mathbb{Z}$. Let us call a subset $U \subseteq \mathbb{Z}$ open if it is either empty, or a union of arithmetic progressions.

(a) Show that this defines a topology on $\mathbb{Z}$ in which every nonempty open set is infinite.

(b) Prove the identity

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \text{ prime}} S(p, 0),$$

and conclude that there are infinitely many prime numbers.

Due on Wednesday, September 3.