Each problem is worth 20 points.

(1) If 50 foot-pounds of work is needed to stretch a spring from its natural length of 10 feet to a length of 12 feet, then how much work is needed to stretch the spring from a length of 11 feet to a length of 15 feet. **Show work.**

**Solution** Let $x = 0$ correspond to the rest position. The force needed to overcome the force of the spring at position $x$ is $F(x) = kx$. Thus the work needed to stretch the spring from 0 to 2 is given by $\int_0^2 kx \, dx = 2k = 50$; thus $k = 25$. The work needed to stretch the spring from 11 feet to 15 feet is equal to $\int_1^5 25x \, dx = \frac{25x^2}{2} \bigg|_1^5 = \frac{5^2}{2} - \frac{1}{2}$.

(2) Find the length of curve $y = \int_0^x (u^2e^{u^2} - 1)^{\frac{1}{2}} \, du$, $1 \leq x \leq 3$. **Show work.**

**Solution:** Length $= \int_1^3 (1 + (y')^2)^{\frac{1}{2}} \, dx = \int_1^3 (1 + ((x^2e^{x^2} - 1)^{\frac{1}{2}})^2)^{\frac{1}{2}} \, dx.$

(3) Let $R$ denote the region in the plane bounded by the curves $x = 2$, $x = 9$, $y = \sin(x)$ and $y = 3x$.

(a) Express the area of $R$ as a definite integral.

**Solution:** $\int_2^9 (3x - \sin(x)) \, dx$

(b) Let $S_1$ denote the solid obtained by rotating the region $R$ about the line $y = -2$. Express the volume of $S_1$ as a definite integral.

**Solution:** $\int_2^9 \pi((3x + 2)^2 - (\sin(x) + 2)^2) \, dx$

(c) Let $S_2$ denote the solid obtained by rotating the region $R$ about the $y$-axis. Express the volume of $S_2$ as a definite integral.

**Solution:** $\int_2^9 2\pi x(3x - \sin(x)) \, dx$

(4) A spherical tank of radius 3 meters is buried in the ground so that the top of the tank is 4 meters beneath ground level. The tank is filled with a liquid which has a density of 1500 kilograms/meter$^3$. How much work does it take to pump all of the liquid to ground level? (Recall that gravitational acceleration is 9.8 meters/second$^2$.) **Show work.**

**Solution:** First we must place a coordinate system. I choose to let $x = 0$ correspond to the bottom of the tank, $x = 6$ correspond to the top of the tank.
and $x=10$ correspond to ground level. (There are many ways to impose a coordinate system on the problem; they are all equally correct.)

The radius of a cross section of the spherical tank at level $x_i$ is equal to $(6x_i - (x_i)^2)^{\frac{1}{2}}$. So the volume of the liquid in the tank between level $x_i$ and level $x_i + \Delta x$ is approximately equal to $\pi(6x_i - (x_i)^2)\Delta x$ cubic meters; thus the mass of this liquid is approximately equal to $1500\pi(6x_i - (x_i)^2)\Delta x$ kilograms. The force needed to lift this portion of the liquid to ground level is approximately $9.8(1500)\pi(6x_i - (x_i)^2)\Delta x$ newtons for a distance of approximately $10 - x_i$ meters; thus the work required to lift this portion of the liquid to ground level is approximately $(10 - x_i)9.8(1500)\pi(6x_i - (x_i)^2)\Delta x$ newton-meters=joules. So the total work (in joules) is approximated by the sums of all these portions of work

$$\sum_{i=1}^{n} (10 - x_i)9.8(1500)\pi(6x_i - (x_i)^2)\Delta x.$$  

Taking the limit as $n \to \infty$ we get that the total work is equal to

$$\int_{0}^{6} (10 - x)9.8(1500)\pi(6x - x^2)dx.$$  

(5) Consider the plate (also called a lamina) which occupies the region in the plane lying between the x-axis and the graph of $y = x\ln(x)$, $1 \leq x \leq e$. Suppose that this plate has constant density $\rho$.

(a) Compute the mass of the plate (in terms of $\rho$). Show work.

**Solution:** \text{mass} = \rho \int_{1}^{e} x\ln(x)dx = \left( \frac{x^2\ln(x)}{2} - \frac{x^2}{4} \right) \bigg|_{1}^{e} = \left( e^2 - \frac{e^2}{4} \right) - \left( \frac{1}{4} \right).

(b) Let $m$ denote the mass of the plate and let $(\bar{x}, \bar{y})$ denote the center of mass of the plate. Express each of $m\bar{x}$ and $m\bar{y}$ as a definite integral.

**Solution:** $m\bar{x} = \rho \int_{1}^{e} x(x\ln(x))dx$ and $m\bar{y} = \rho \int_{1}^{e} \frac{1}{2}(x\ln(x))^2dx$. 