1. Compute the average value of the function $\sin x$ on the interval $[0, \pi]$.

**Solution:** The average value is

$$\frac{1}{\pi - 0} \int_{0}^{\pi} \sin x \, dx = \frac{1}{\pi} \left( -\cos x \right) \bigg|_{0}^{\pi} = \frac{2}{\pi}.$$ 

2. Let $S$ be the region bounded by the graph of $y = x^2 + 1$, the $x$-axis, and the lines $x = 0$ and $x = 1$. Find the volume of the solid obtained by rotating $S$ about the $x$-axis. State at the beginning which method you are using.

**Solution:** The sketch below shows the region $S$ together with a typical cross section of the solid. The disk method is clearly the natural choice in this case.

The cross section at distance $x$ from the origin has radius $x^2 + 1$, and so its area is given by $A(x) = \pi(x^2 + 1)^2$. Consequently, the volume of the solid is

$$\int_{0}^{1} A(x) \, dx = \int_{0}^{1} \pi(x^2 + 1)^2 \, dx = \pi \left( \frac{1}{3} x^5 + \frac{2}{3} x^3 + x \right) \bigg|_{0}^{1} = \frac{28\pi}{15}.$$ 

**Note:** It is also possible (although very cumbersome) to use the shell method. The radius of the cylindrical shells varies between 0 and 2, the latter being the $y$-coordinate of the topmost corner of $S$. For $0 \leq y \leq 1$, the length of the corresponding shell is equal to 1; for $1 \leq y \leq 2$, that length is $1 - \sqrt{y - 1}$. It follows that the volume of the solid is

$$\int_{0}^{1} 2\pi y \, dy + \int_{1}^{2} 2\pi y (1 - \sqrt{y - 1})^2 \, dy,$$

and a lengthy calculation shows that this integral has value $28\pi/15$. 

3. Here are two curves, together with four equations in polar coordinates:

<table>
<thead>
<tr>
<th>Curve A</th>
<th>Curve B</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $r = \cos(2\theta)$</td>
<td>(1) $r = \cos(2\theta)$</td>
</tr>
<tr>
<td>(2) $r = \theta$</td>
<td>(2) $r = \theta$</td>
</tr>
<tr>
<td>(3) $r = \frac{1}{\cos \theta}$</td>
<td>(3) $r = \frac{1}{\cos \theta}$</td>
</tr>
<tr>
<td>(4) $r = 1$</td>
<td>(4) $r = 1$</td>
</tr>
</tbody>
</table>

(a) Which of the four equations describes Curve A? To receive credit, justify your choice, for example by citing some feature(s) of the curve or of the equations.

**Solution:** The answer is $r = \theta$. As we go around the spiral in a counterclockwise direction (with $\theta$ increasing from 0 to $2\pi$ to $4\pi$ and so on), the radius $r$ is increasing at a constant rate; this is consistent with $r = \theta$.

(b) Which of the four equations describes Curve B? To receive credit, justify your choice, for example by citing some feature(s) of the curve or of the equations.

**Solution:** The answer is $r = \frac{1}{\cos \theta}$.

From the triangle in the figure, we get $1 = r \cos \theta$, or $r = \frac{1}{\cos \theta}$.

4. In polar coordinates, a curve is often given by an equation $r = f(\theta)$.

(a) We can parametrize the curve by $x = f(\theta) \cos \theta$ and $y = f(\theta) \sin \theta$, with $\theta$ playing the role of the parameter. Show that $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (f')^2 + f^2$.

**Solution:** We have

\[
\frac{dx}{d\theta} = f' \cos \theta - f \sin \theta \quad \text{and} \quad \frac{dy}{d\theta} = f' \sin \theta + f \cos \theta,
\]
and so we get
\[
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (f')^2 \cos^2 \theta - f' f \cos \theta \sin \theta + f^2 \sin^2 \theta + (f')^2 \sin^2 \theta + f' f \sin \theta \cos \theta + f^2 \cos^2 \theta = (f')^2 + f^2,
\]
after using the identity \(\cos^2 \theta + \sin^2 \theta = 1\).

(b) Find the length of the curve \(r = e^\theta\), for \(0 \leq \theta \leq 2\pi\).

**Solution:** Using the result of (a), we have
\[
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (e^\theta)^2 + (e^\theta)^2 = 2e^{2\theta}.
\]
We get the length of the curve by integrating the square root of that expression:
\[
\int_0^{2\pi} \sqrt{2e^\theta} \, d\theta = \sqrt{2} \left[ e^\theta \right]_0^{2\pi} = \sqrt{2} (e^{2\pi} - 1).
\]

5. Waiting times are modelled by a random variable \(T\) with probability density function
\[
f(t) = \begin{cases} 
0 & \text{for } t < 0, \\
c e^{-ct} & \text{for } t \geq 0,
\end{cases}
\]
where \(c > 0\) is a constant.

(a) Let \(\mu\) be the mean of the random variable \(T\). Show that \(\mu = c^{-1}\).

**Solution:** The mean of \(T\) is given by the formula
\[
\mu = \int_{-\infty}^{\infty} t f(t) \, dt = \int_{0}^{\infty} t c e^{-ct} \, dt.
\]
After the substitution \(u = tc\) and \(du = c \, dt\), which leaves the limits of the integral unchanged, we obtain
\[
\mu = \int_{0}^{\infty} u e^{-u} c^{-1} \, du = c^{-1} \int_{0}^{\infty} u e^{-u} \, du.
\]
To evaluate the indefinite integral, we use integration by parts:
\[
\int u e^{-u} \, du = -ue^{-u} + \int e^{-u} \, du = -ue^{-u} - e^{-u} + C = -(u+1)e^{-u} + C.
\]
Consequently,
\[
\int_{0}^{\infty} u e^{-u} \, du = \lim_{L \to \infty} \int_{0}^{L} u e^{-u} \, du
\]
\[
= \lim_{L \to \infty} \left[ -(u+1)e^{-u} \right]_0^{L} = \lim_{L \to \infty} \left( 1 - (L+1)e^{-L} \right) = 1,
\]
which shows that \(\mu = c^{-1}\).
(b) Suppose that, for phone calls to the DMV, the average time callers have to wait on hold is 10 minutes. What percentage of callers is waiting between 0 and 20 minutes? (Maybe it is useful to know that $e^{-1} \approx 0.367, e^{-2} \approx 0.135, e^{-3} \approx 0.049$; maybe not.)

**Solution:** We use the above model with $\mu = 10$; the probability density function is

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ \frac{1}{10} e^{-t/10} & \text{for } t \geq 0. \end{cases}$$

We are looking for the probability $P(0 \leq T \leq 20)$ that a caller has to wait between 0 and 20 minutes. Using the probability density function, we get

$$P(0 \leq T \leq 20) = \int_0^{20} f(t) \, dt = \int_0^{20} \frac{1}{10} e^{-t/10} \, dt = \left[-e^{-t/10}\right]_0^{20} = 1 - e^{-2}.$$

The table of values shows that this is approximately $1 - 0.135 = 0.865$; in other words, 86.5% of callers have to wait between 0 and 20 minutes.

6. Triangle College is installing a triangle-shaped sculpture on its campus. The sculpture is made of glass of a constant thickness, and has a mass of $m$ kilograms; when installed, it is $b$ meters wide and $h$ meters tall. How much work will it take to raise the sculpture from the ground into a vertical position? (Use the letter $g$ for the gravitational constant.)

**Solution:** We divide the sculpture into thin strips of height $\Delta x$, and approximate each strip by a rectangle. Consider the rectangle at height $x$, as in the picture below.

By similar triangles, the rectangle is $b(h - x)/h$ units wide, and so its area is equal to $b(h - x)/h \cdot \Delta x$. Since the entire sculpture has area $\frac{1}{2}bh$ and mass $m$, and since the glass is of uniform thickness, it follows that the mass of the rectangle is

$$\Delta m = \frac{b(h - x)}{h} \cdot \Delta x \cdot m = 2m \frac{h - x}{h^2} \Delta x.$$

When raising up the sculpture, our rectangle has to be lifted a distance of $x$ units; the required work is the product of the gravitational force times the distance, or

$$\Delta W = \left(\Delta m \cdot g\right) \cdot x = 2m \frac{(h - x)}{h^2} \Delta x \cdot g \cdot x = 2mg \cdot \frac{x}{h} \left(1 - \frac{x}{h}\right) \Delta x.$$
The total amount of work $W$ needed is approximately the sum of the individual $\Delta W$. If we let $\Delta x \to 0$, we obtain the following integral:

$$W = \int_0^h 2mg \cdot \frac{x}{h} \left(1 - \frac{x}{h}\right) \, dx.$$ 

To find its value, we substitute $u = x/h$ and $du = dx/h$ to obtain

$$W = 2mgh \cdot \int_0^1 u(1 - u) \, du = 2mgh \cdot \left(\frac{1}{2}u^2 - \frac{1}{3}u^3\right)\bigg|_0^1 = 2mgh \cdot \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}mgh.
$$

Note: An alternative solution is to show that the center of mass of the triangle is located at height $h/3$; consequently, the work needed is $W = \frac{1}{3}mgh$.

7. When a car is going by at a certain distance $r$ (in miles), the noise from the car is inversely proportional to the square of the distance; in other words, the intensity of the noise (in suitable units) is given by $C/r^2$, where $C$ is a constant that depends on the circumstances.

Use this fact to find a formula for the intensity of the noise $D$ miles away from a busy freeway. To simplify the problem, assume that the freeway is perfectly straight and infinitely long; that all cars on the freeway are identical and are going at the same speed; and that there are $\rho$ cars per mile of road. Briefly explain your approach!

**Solution:** We put the $x$-axis along the highway, and assume that we are measuring the intensity of the noise at the point opposite the origin.

Consider a short stretch of the highway, of length $\Delta x$. Each car there is approximately $\sqrt{D^2 + x^2}$ miles away from the observer, and therefore contributes $C/(D^2 + x^2)$ to the measurement. Since the number of cars on that portion of the highway is $\rho \Delta x$, the contribution to the noise intensity is

$$\Delta I = \frac{C}{D^2 + x^2} \cdot \rho \Delta x.$$ 

If we add up the contributions from all the little pieces, and then let $\Delta x \to 0$, we see that the intensity of the noise $D$ miles from the freeway is given by the integral

$$I = \int_{-\infty}^{\infty} \frac{C\rho}{D^2 + x^2} \, dx.$$
Using the symmetry of the integrand, we get

\[ I = 2 \int_0^\infty \frac{C \rho}{D^2 + x^2} \, dx = 2 \lim_{L \to \infty} \frac{C \rho}{D} \arctan \frac{x}{D} \bigg|_0^L = \frac{\pi C \rho}{D}. \]