MATH 132 Solutions to Midterm 2

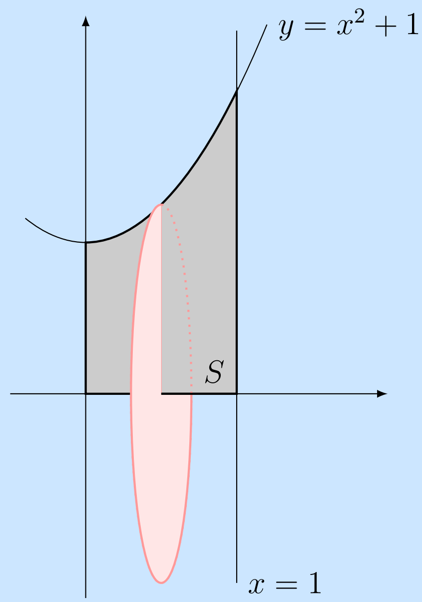
20 pts 1. Compute the average value of the function $\sin x$ on the interval $[0, \pi]$.

Solution: The average value is

$$\frac{1}{\pi - 0}\int_0^\pi \sin x\, dx = \frac{1}{\pi}\Big(-\cos x\Big)\Big|_0^\pi = \frac{2}{\pi}.$$

20 pts 2. Let S be the region bounded by the graph of $y = x^2 + 1$, the x-axis, and the lines $x = 0$ and $x = 1$. Find the volume of the solid obtained by rotating S about the x-axis. State at the beginning which method you are using.

Solution: The sketch below shows the region S together with a typical cross section of the solid. The disk method is clearly the natural choice in this case.



The cross section at distance x from the origin has radius $x^2 + 1$, and so its area is given by $A(x) = \pi(x^2 + 1)^2$. Consequently, the volume of the solid is

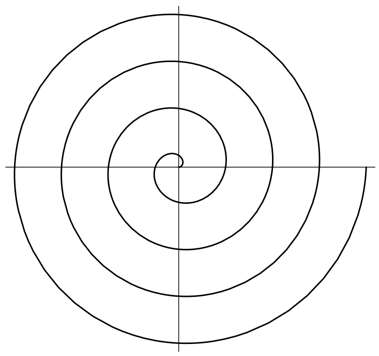
$$\int_0^1 A(x)\, dx = \int_0^1 \pi(x^2+1)^2\, dx = \pi\Big(\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\Big)\Big|_0^1 = \frac{28\pi}{15}.$$

Note: It is also possible (although very cumbersome) to use the shell method. The radius of the cylindrical shells varies between 0 and 2, the latter being the y-coordinate of the topmost corner of S. For $0 \le y \le 1$, the length of the corresponding shell is equal to 1; for $1 \le y \le 2$, that length is $1 - \sqrt{y-1}$. It follows that the volume of the solid is

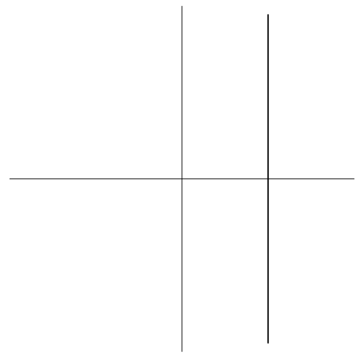
$$\int_0^1 2\pi y\, dy + \int_1^2 2\pi y\big(1 - \sqrt{y-1}\big)^2\, dy,$$

and a lengthy calculation shows that this integral has value $28\pi/15$.

3. Here are two curves, together with four equations in polar coordinates:



Curve **A**



Curve **B**

(1) $r = \cos(2\theta)$

(2) $r = \theta$

(3) $r = \dfrac{1}{\cos\theta}$

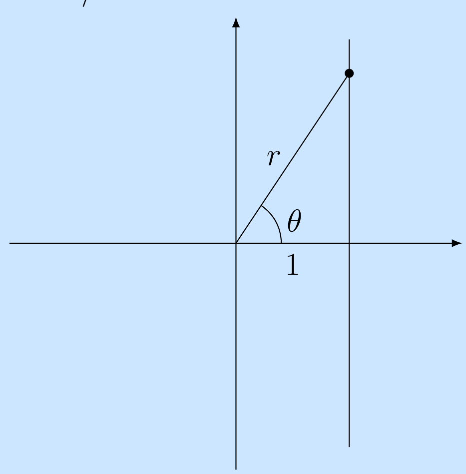
(4) $r = 1$

10 pts (a) Which of the four equations describes Curve A? To receive credit, justify your choice, for example by citing some feature(s) of the curve or of the equations.

Solution: The answer is $r = \theta$. As we go around the spiral in a counterclockwise direction (with θ increasing from 0 to 2π to 4π and so on), the radius r is increasing at a constant rate; this is consistent with $r = \theta$.

10 pts (b) Which of the four equations describes Curve B? To receive credit, justify your choice, for example by citing some feature(s) of the curve or of the equations.

Solution: The answer is $r = 1/\cos\theta$.



From the triangle in the figure, we get $1 = r\cos\theta$, or $r = 1/\cos\theta$.

4. In polar coordinates, a curve is often given by an equation $r = f(\theta)$.

10 pts (a) We can parametrize the curve by $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$, with θ playing the role of the parameter. Show that $\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2 = (f')^2 + f^2$.

Solution: We have

$$\frac{dx}{d\theta} = f'\cos\theta - f\sin\theta \qquad \text{and} \qquad \frac{dy}{d\theta} = f'\sin\theta + f\cos\theta,$$

and so we get

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (f')^2\cos^2\theta - f'f\cos\theta\sin\theta + f^2\sin^2\theta$$
$$+ (f')^2\sin^2\theta + f'f\sin\theta\cos\theta + f^2\cos^2\theta = (f')^2 + f^2,$$

after using the identity $\cos^2\theta + \sin^2\theta = 1$.

10 pts

(b) Find the length of the curve $r = e^\theta$, for $0 \le \theta \le 2\pi$.

Solution: Using the result of (a), we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(e^\theta\right)^2 + \left(e^\theta\right)^2 = 2e^{2\theta}.$$

We get the length of the curve by integrating the square root of that expression:

$$\int_0^{2\pi} \sqrt{2}e^\theta\,d\theta = \sqrt{2}e^\theta\Big|_0^{2\pi} = \sqrt{2}\big(e^{2\pi} - 1\big).$$

5. Waiting times are modelled by a random variable T with probability density function

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ ce^{-ct} & \text{for } t \ge 0, \end{cases}$$

where $c > 0$ is a constant.

10 pts

(a) Let μ be the mean of the random variable T. Show that $\mu = c^{-1}$.

Solution: The mean of T is given by the formula

$$\mu = \int_{-\infty}^{\infty} tf(t)\,dt = \int_0^\infty tce^{-ct}\,dt.$$

After the substitution $u = tc$ and $du = c\,dt$, which leaves the limits of the integral unchanged, we obtain

$$\mu = \int_0^\infty ue^{-u}c^{-1}\,du = c^{-1}\int_0^\infty ue^{-u}\,du.$$

To evaluate the indefinite integral, we use integration by parts:

$$\int ue^{-u}\,du = -ue^{-u} + \int e^{-u}\,du = -ue^{-u} - e^{-u} + C = -(u+1)e^{-u} + C.$$

Consequently,

$$\int_0^\infty ue^{-u}\,du = \lim_{L\to\infty}\int_0^L ue^{-u}\,du$$
$$= \lim_{L\to\infty}\Big(-(u+1)e^{-u}\Big)\Big|_0^L = \lim_{L\to\infty}\Big(1 - (L+1)e^{-L}\Big) = 1,$$

which shows that $\mu = c^{-1}$.

10 pts

(b) Suppose that, for phone calls to the DMV, the average time callers have to wait on hold is 10 minutes. What percentage of callers is waiting between 0 and 20 minutes? (Maybe it is useful to know that $e^{-1} \approx 0.367$, $e^{-2} \approx 0.135$, $e^{-3} \approx 0.049$; maybe not.)

Solution: We use the above model with $\mu = 10$; the probability density function is

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ \frac{1}{10} e^{-t/10} & \text{for } t \geq 0. \end{cases}$$

We are looking for the probability $P(0 \leq T \leq 20)$ that a caller has to wait between 0 and 20 minutes. Using the probability density function, we get

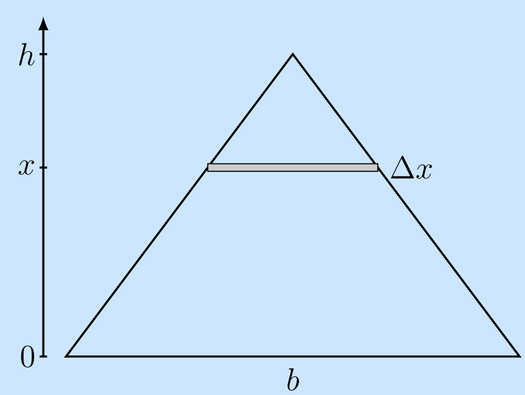
$$P(0 \leq T \leq 20) = \int_0^{20} f(t)\, dt = \int_0^{20} \frac{1}{10} e^{-t/10}\, dt = \left(-e^{-t/10}\right)\Big|_0^{20} = 1 - e^{-2}.$$

The table of values shows that this is approximately $1 - 0.135 = 0.865$; in other words, 86.5% of callers have to wait between 0 and 20 minutes.

20 pts

6. Triangle College is installing a triangle-shaped sculpture on its campus. The sculpture is made of glass of a constant thickness, and has a mass of m kilograms; when installed, it is b meters wide and h meters tall. How much work will it take to raise the sculpture from the ground into a vertical position? (Use the letter g for the gravitational constant.)

Solution: We divide the sculpture into thin strips of height Δx, and approximate each strip by a rectangle. Consider the rectangle at height x, as in the picture below.



By similar triangles, the rectangle is $b(h-x)/h$ units wide, and so its area is equal to $b(h-x)/h \cdot \Delta x$. Since the entire sculpture has area $\frac{1}{2}bh$ and mass m, and since the glass is of uniform thickness, it follows that the mass of the rectangle is

$$\Delta m = \frac{b(h-x)/h \cdot \Delta x}{\frac{1}{2}bh} \cdot m = 2m \frac{h-x}{h^2} \Delta x.$$

When raising up the sculpture, our rectangle has to be lifted a distance of x units; the required work is the product of the gravitational force times the distance, or

$$\Delta W = \underbrace{(\Delta m \cdot g)}_{\text{force}} \cdot x = 2m \frac{(h-x)}{h^2} \Delta x \cdot g \cdot x = 2mg \cdot \frac{x}{h} \left(1 - \frac{x}{h}\right) \Delta x.$$

The total amount of work W needed is approximately the sum of the individual ΔW. If we let $\Delta x \to 0$, we obtain the following integral:

$$W = \int_0^h 2mg \cdot \frac{x}{h}\left(1 - \frac{x}{h}\right)\, dx.$$

To find its value, we substitute $u = x/h$ and $du = dx/h$ to obtain

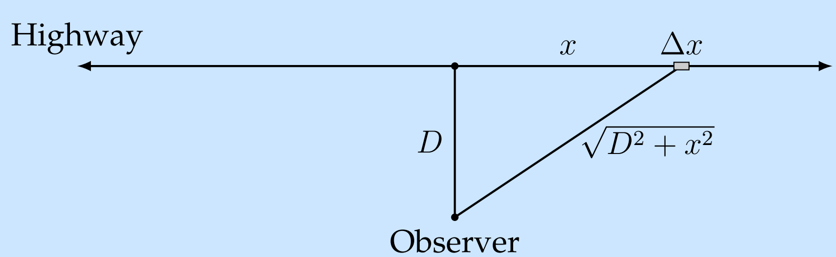
$$W = 2mgh \cdot \int_0^1 u(1-u)\, du = 2mgh \cdot \left(\frac{1}{2}u^2 - \frac{1}{3}u^3\right)\Big|_0^1 = 2mgh \cdot \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}mgh.$$

Note: An alternative solution is to show that the center of mass of the triangle is located at height $h/3$; consequently, the work needed is $W = \frac{1}{3}mgh$.

20 pts 7. When a car is going by at a certain distance r (in miles), the noise from the car is inversely proportional to the square of the distance; in other words, the intensity of the noise (in suitable units) is given by C/r^2, where C is a constant that depends on the circumstances.

Use this fact to find a formula for the intensity of the noise D miles away from a busy freeway. To simplify the problem, assume that the freeway is perfectly straight and infinitely long; that all cars on the freeway are identical and are going at the same speed; and that there are ρ cars per mile of road. Briefly explain your approach!

Solution: We put the x-axis along the highway, and assume that we are measuring the intensity of the noise at the point opposite the origin.



Consider a short stretch of the highway, of length Δx. Each car there is approximately $\sqrt{D^2 + x^2}$ miles away from the observer, and therefore contributes $C/(D^2 + x^2)$ to the measurement. Since the number of cars on that portion of the highway is $\rho\Delta x$, the contribution to the noise intensity is

$$\Delta I = \frac{C}{D^2 + x^2} \cdot \rho\Delta x.$$

If we add up the contributions from all the little pieces, and then let $\Delta x \to 0$, we see that the intensity of the noise D miles from the freeway is given by the integral

$$I = \int_{-\infty}^{\infty} \frac{C\rho}{D^2 + x^2}\, dx.$$

Using the symmetry of the integrand, we get

$$I = 2\int_0^\infty \frac{C\rho}{D^2 + x^2}\,dx = 2\lim_{L\to\infty} \frac{C\rho}{D}\arctan\frac{x}{D}\bigg|_0^L = \frac{\pi C\rho}{D}.$$