Problem 1. Let \( h(t) = \frac{\sin^3(\pi t)}{\arctan(1 - t^2) - t} + e^{\sqrt{t}} \) and let \( G(x) = \int_0^x h(t) \, dt \).

(a) Find \( \int_0^1 h'(t) \, dt \).

**Answer:**

By the fundamental theorem of calculus, we have

\[
\int_0^1 h'(t) \, dt = h(1) - h(0) = \frac{\sin^3(\pi) + e}{\arctan(0) - 1} - \frac{\sin^3(0) + e^0}{\arctan(1) - 0} = e - \frac{1}{\pi} = -\frac{4}{\pi} - e.
\]

(b) Find \( G'(0) \).

**Answer:**

By the fundamental theorem of calculus (the other part) we have \( G'(0) = h(0) = -\frac{1}{\pi} \).
Problem 2. Determine whether the following integrals converge or diverge. Explain your answer completely. Find the exact answer if possible.

(a) \( \int_{2}^{\infty} \frac{dx}{x \ln(x)} \)

**Answer:**

Diverges. We integrate explicitly (using \( u \)-substitution):

\[
\int_{2}^{\infty} \frac{dx}{x \ln(x)} = \lim_{b \to \infty} \int_{2}^{b} \frac{dx}{x \ln(x)} = \lim_{b \to \infty} \ln(\ln(x)) \bigg|_{2}^{b} = \lim_{b \to \infty} \ln(\ln(b)) - \ln(\ln(2)) = \infty.
\]

(b) \( \int_{0}^{\infty} \frac{dx}{1 + x^3} \)

**Answer:**

Converges. Note that \( \int_{0}^{\infty} \frac{dx}{1 + x^3} = \int_{1}^{\infty} \frac{dx}{1 + x^3} + \int_{1}^{\infty} \frac{dx}{1 + x^3} \) and \( \int_{0}^{1} \frac{dx}{1 + x^3} \) is finite. Hence, \( \int_{0}^{\infty} \frac{dx}{1 + x^3} \) converges if \( \int_{1}^{\infty} \frac{dx}{1 + x^3} \) converges. We now show that \( \int_{1}^{\infty} \frac{dx}{1 + x^3} \) converges by the comparison theorem. We have

\[
0 < \frac{1}{1 + x^3} < \frac{1}{1 + x^2} < \frac{1}{x^2} \text{ for all } x > 1.
\]

Since \( \int_{1}^{\infty} \frac{1}{x^2} = \frac{1}{3} \) which converges, the comparison theorem says that \( \int_{1}^{\infty} \frac{dx}{1 + x^3} \) converges.

(c) \( \int_{0}^{\infty} \frac{x}{e^x} \, dx \)

**Answer:**

Converges. We integrate explicitly (using integration by parts):

\[
\int_{0}^{\infty} \frac{x}{e^x} \, dx = \lim_{b \to \infty} \int_{0}^{b} \frac{x}{e^x} \, dx = \lim_{b \to \infty} \frac{-1 - x}{e^x} \bigg|_{0}^{b} = \lim_{b \to \infty} \frac{-1 - b}{e^b} - \frac{-1}{1} = 1.
\]

(d) \( \int_{0}^{2} \frac{dx}{\sqrt{2 - x}} \)

**Answer:**

Converges. We integrate explicitly.

\[
\int_{0}^{2} \frac{dx}{\sqrt{2 - x}} = \lim_{b \to 2} \int_{0}^{b} \frac{dx}{\sqrt{2 - x}} = \lim_{b \to 2} -2\sqrt{2 - x} \bigg|_{0}^{b} = 2\sqrt{2}.
\]
Problem 3.

(a) Use the left hand rule with \( n = 5 \) to approximate \( \int_{2}^{12} \frac{8x + 3}{x^2 - 5x + 9} \, dx \).

**Answer:**

Let’s denote (for this part and every part) \( \frac{8x + 3}{x^2 - 5x + 9} \) by \( f(x) \). Then, the left hand rule with \( n = 5 \) gives

\[
\int_{2}^{12} \frac{8x + 3}{x^2 - 5x + 9} \, dx \approx \frac{12 - 2}{5} (f(2) + f(4) + f(6) + f(8) + f(10))
\]

\[
= 2 \left( \frac{19}{3} + \frac{17}{5} + \frac{67}{33} + \frac{83}{59} \right) = 40.3408.
\]

(b) Use the right hand rule with \( n = 5 \) to approximate \( \int_{2}^{12} \frac{8x + 3}{x^2 - 5x + 9} \, dx \).

**Answer:**

\[
\int_{2}^{12} \frac{8x + 3}{x^2 - 5x + 9} \, dx \approx \frac{12 - 2}{5} (f(4) + f(6) + f(8) + f(10) + f(12))
\]

\[
= 2 \left( \frac{17}{5} + \frac{67}{33} + \frac{83}{59} + \frac{33}{31} \right) = 29.8032.
\]
Problem 5. (Continued)

(c) Use the trapezoid hand rule with \( n = 5 \) to approximate \( \int_{12}^{2} \frac{8x + 3}{x^2 - 5x + 9} \, dx \).

**Answer:**

\[
\int_{12}^{2} \frac{8x + 3}{x^2 - 5x + 9} \, dx \approx \frac{12 - 2}{(2)(5)} \left( f(2) + 2f(4) + 2f(6) + 2f(8) + 2f(10) + f(12) \right) \\
= \frac{1}{5} \left( \frac{19}{3} + 2(7) + 2 \left( \frac{17}{5} \right) + 2 \left( \frac{67}{33} \right) + 2 \left( \frac{83}{59} \right) + \frac{33}{31} \right) = 35.072.
\]

(d) Use the midpoint rule with \( n = 5 \) to approximate \( \int_{12}^{2} \frac{8x + 3}{x^2 - 5x + 9} \, dx \).

\[
\int_{12}^{2} \frac{8x + 3}{x^2 - 5x + 9} \, dx \approx \frac{12 - 2}{5} \left( f(3) + f(5) + f(7) + f(9) + f(11) \right) \\
= \frac{2}{5} \left( 9 + \frac{43}{9} + \frac{59}{23} + \frac{5}{3} + \frac{91}{75} \right) = 38.446.
\]

(e) Use Simpson’s rule with \( n = 4 \) to approximate \( \int_{12}^{2} \frac{8x + 3}{x^2 - 5x + 9} \, dx \).

\[
\int_{12}^{2} \frac{8x + 3}{x^2 - 5x + 9} \, dx \approx \frac{12 - 2}{(3)(5)} \left( f(2) + 4f(4) + 2f(6) + 4f(8) + f(10) \right) \\
= \frac{2}{3} \left( \frac{19}{3} + 4(7) + 2 \left( \frac{17}{5} \right) + 4 \left( \frac{67}{33} \right) + \frac{83}{59} \right) = 33.7742
\]

For your information, this integral can be computed exactly (though it’s rather tedious). One obtains the precise antiderivative

\[
\int \frac{8x + 3}{x^2 - 5x + 9} \, dx = \frac{46}{\sqrt{11}} \arctan \left( \frac{2x - 5}{\sqrt{11}} \right) + 4 \ln(x^2 - 5x + 9)
\]

and the integrals (exact to four decimal places)

\[
\int_{12}^{2} \frac{8x + 3}{x^2 - 5x + 9} \, dx = 37.1868 \quad \text{and} \quad \int_{12}^{10} \frac{8x + 3}{x^2 - 5x + 9} \, dx = 34.754
\]
Problem 6. [20 points] Consider the region trapped by the two curves $y = \frac{4}{1+e^x}$ and $y = \sqrt{x} + 2$ and between the lines $x = 0$ and $x = 5$. Here is a picture of the region:

(a) Use an integral to express the volume of the solid formed by rotating this region around the $x$-axis. Do not evaluate the integral.

Answer:

We use “washers.”

$$Volume = \int_{0}^{5} \pi \left( \sqrt{x} + 2 \right)^2 - \pi \left( \frac{4}{1+e^x} \right)^2 dx$$

To use shells here is a little more complicated. First, we solve for $x$ in terms of $y$:

$$y = \frac{4}{1+e^x} \Leftrightarrow x = \ln \left( \frac{4}{y} - 1 \right) \quad \text{and} \quad y = \sqrt{x} + 2 \Leftrightarrow x = (y - 2)^2.$$ 

$$V = \int_{y=0}^{\sqrt{5}+2} 2\pi rh \ dy = \int_{0}^{\sqrt{5}+2} 2\pi \left( 5 - \ln \left( \frac{4}{y} - 1 \right) \right) y \ dy + \int_{2}^{\sqrt{5}+2} 2\pi \left( 5 - (y - 2)^2 \right) y \ dy.$$ 

(b) Use an integral to express the volume of the solid formed by rotating this region around the line $x = 5$. Do not evaluate the integral.

Answer:

Using shells:

$$Volume = \int_{0}^{5} 2\pi (5-x) \left( \sqrt{2} + x - \frac{4}{1+e^x} \right) dx.$$ 

We can use washers (well, discs) here:

$$V = \int_{y=0}^{\sqrt{5}+2} \pi r^2 \ dy = \int_{0}^{\sqrt{5}+2} \pi \left( 5 - \ln \left( \frac{4}{y} - 1 \right) \right)^2 \ dy + \int_{2}^{\sqrt{5}+2} \pi \left( 5 - (y - 2)^2 \right)^2 \ dy.$$