Problem 1. The picture below shows the graph of a function $g$.

(a) [10 points] Find $\int_{-1}^{5} g'(t) \, dt$.

**Answer:**

By the fundamental theorem of calculus (the “evaluation theorem” in the text), we have:

$$\int_{-1}^{5} g'(t) \, dt = g(5) - g(-1) = 15 - (-5) = 20.$$

(b) [10 points] Let $A(x) = \int_{1}^{x} g(t) \, dt$. Find $A'(2)$.

**Answer:**

Here, we use the other part of the fundamental theorem of calculus:

$$A'(x) = \frac{d}{dx} \left( \int_{1}^{x} g(t) \, dt \right) = g(x)$$

and so $A'(2) = g(2) = -5$. 
Problem 2. [20 points] Determine whether \( \int_1^\infty \frac{\sin^2(x)}{x^3} \, dx \) converges or diverges. Justify your answer completely.

**Answer:**
We use the comparison theorem to show that \( \int_1^\infty \frac{\sin^2(x)}{x^3} \, dx \) converges.

First, observe that \( \int_1^\infty \frac{dx}{x^3} \) converges:

\[
\int_1^\infty \frac{dx}{x^3} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^3} = \lim_{b \to \infty} \left[ -\frac{1}{2x^2} \right]_1^b = \lim_{b \to \infty} \left( -\frac{1}{2b^2} - \left( -\frac{1}{2(1)^2} \right) \right) = \frac{1}{2}.
\]

Now, since \(-1 \leq \sin(x) \leq 1\) for any \(x\), we have

\[
0 < \frac{\sin^2(x)}{x^3} < \frac{1}{x^3}.
\]

This inequality, together with the fact that \( \int_1^\infty \frac{dx}{x^3} \) converges, implies that \( \int_1^\infty \frac{\sin^2(x)}{x^3} \, dx \) converges by the comparison theorem.
Problem 3. Below is a sketch of $y = f(x)$. The polygonal paths may make it easier to approximate $\int_1^3 f(x)dx$.

(a) [10 points] Use the trapezoid rule with $n = 4$ to approximate $\int_1^3 f(x)dx$.

**Answer:**

The trapezoid rule, with $n = 4$ gives:

$$\int_1^3 f(x)dx \approx \frac{\Delta x}{2} \left( f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + f(3) \right)$$

$$= \frac{1}{4} \left( 1.5 + 1.5 + 2.5 + 3.5 \right) = \frac{9}{2}.$$

(b) [10 points] Use the midpoint rule with $n = 4$ to approximate $\int_1^3 f(x)dx$.

**Answer:**

The midpoint rule, with $n = 4$ gives:

$$\int_1^3 f(x)dx \approx \Delta x \left( f(1.25) + f(1.75) + f(2.25) + f(2.75) \right)$$

$$= \frac{1}{2} \left( 1.5 + 1.5 + 2.5 + 3.5 \right) = \frac{9}{2}.$$
Problem 4. [20 points] Consider the region trapped by the two curves

\[ y = \frac{2}{1 + x^2} \text{ and } y = 2 - \sqrt{x} \]

between the points \((0, 2)\) and \((1, 1)\). Here is a sketch showing the region:

Use an integral to express the volume of the solid formed by rotating this region around the \(y\)-axis. Do not evaluate the integral.

**Answer:**

Using shells:

\[
V \approx \sum_{i=1}^{n} 2\pi rh \Delta x \Rightarrow V = \int_{x=0}^{x=1} 2\pi x \left( \frac{2}{1 + x^2} - (2 - \sqrt{x}) \right) dx.
\]

Using washers is a little harder—we need to solve for \(x\) in terms of \(y\), which we’ll do now:

\[ y = \frac{2}{1 + x^2} \Rightarrow x = \sqrt{\frac{2}{y} - 1} \quad \text{and} \quad y = 2 - \sqrt{x} \Rightarrow x = (2 - y)^2. \]

Now,

\[
V \approx \sum_{i=1}^{n} \pi R^2 - \pi r^2 \Delta y \Rightarrow V = \int_{y=1}^{y=2} \pi \left( \sqrt{\frac{2}{y} - 1} \right)^2 - \pi \left( (2 - y)^2 \right)^2 \right) dy = \int_{y=1}^{y=2} \pi \left( \frac{2}{y} - 1 - (2 - y)^4 \right) dy.
\]

It wasn’t part of the question, but just for practice, we’ll compute these integrals:

\[
\int_{x=0}^{x=1} 2\pi x \left( \frac{2}{1 + x^2} - (2 - \sqrt{x}) \right) dx = 2\pi \left[ \ln(1 + x^2) + \frac{2}{5} x^2 - x^2 \right]_{0}^{1} = \left( \ln(4) - \frac{6}{5} \right) \pi \text{ and }
\]

\[
\int_{y=1}^{y=2} \pi \left( \frac{2}{y} - 1 - (2 - y)^4 \right) dy = \pi \left[ -17y + 16y^2 - 8y^3 + 2y^4 - \frac{y^5}{5} + 2 \ln(y) \right]_{1}^{2} = \left( \ln(4) - \frac{6}{5} \right) \pi.
\]
Problem 5. [5 points each] Matching. Put the letter that matches the answer on the line. You need not show your work.

• (c) \[ \int_{-1}^{3} \frac{dx}{x^2} \]

• (d) \[ \int_{0}^{1} x\sqrt{1-x^2}dx \]

• (a) \[ \int_{-\frac{1}{2}}^{0} 3y e^{-2y} dy \]

• (b) \[ \int_{-1}^{1} \sqrt{1-t^2} dt \]

(a) $\frac{3}{4}$  
(b) $\frac{\pi}{2}$  
(c) $\infty$  
(d) $\frac{1}{3}$