MATH 132 Solutions to the Final Exam

10 pts 1. Calculate the indefinite integral $\int 2x\sqrt{1+x^2}\,dx$

Solution: We substitute $u = 1 + x^2$ and $du = 2xdx$ to get

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(1+x^2\right)^{3/2} + C.$$

10 pts 2. Calculate the indefinite integral $\int \frac{1}{x(x+1)}\,dx$.

Solution: The partial fraction decomposition is

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1},$$

and so we get

$$\int \frac{1}{x(x+1)}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx = \ln|x| - \ln|x+1| + C.$$

10 pts 3. Calculate the definite integral $\int_0^\pi x\sin x\,dx$. If it does not converge, write "divergent".

Solution: We use integration by parts with $u = x$ and $dv = \sin x\,dx$. Then $du = dx$ and $v = -\cos x$, and therefore

$$\int_0^\pi x\sin x\,dx = -x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx = -\pi\cos(\pi) + \sin x\Big|_0^\pi = \pi.$$

10 pts 4. Calculate the definite integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$. If it does not converge, write "divergent".

Solution: The integral is improper at 0. An antiderivative of $1/\sqrt{x}$ is $2\sqrt{x}$, and because

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left(2 - 2\sqrt{a}\right) = 2,$$

the improper integral converges and is equal to 2.

10 pts 5. Write a power series for the function e^{-x^2}.

Solution: Starting from the familiar Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and replacing x by $-x^2$, we get

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

This series still converges for all real numbers x.

10 pts 6. Solve the differential equation $\frac{dy}{dx} = xe^{-y}$.

Solution: The equation is separable. If we separate the variables, we get

$$\int e^y \, dy = \int x \, dx.$$

After integration, this becomes

$$e^y = \frac{1}{2}x^2 + C \qquad \text{or} \qquad y = \ln\left(\frac{1}{2}x^2 + C\right).$$

10 pts 7. Find the value of the sum $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$.

Solution: This is a geometric series with value $\frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$.

10 pts 8. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges / diverges (circle one) by the ____________________ test.

Justify:

Solution: The series converges by the Alternating Series Test. We can apply this test because the series is alternating, and because the terms $\frac{1}{n}$ are obviously decreasing and going to zero.

10 pts 9. The series $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ converges / diverges (circle one) by the ____________________ test.

Justify:

Solution: The series diverges by the Integral Test. We can apply this test because the function $\frac{1}{x\ln(x)}$ is positive and decreasing. So we need to look at the improper integral

$$\int_2^\infty \frac{1}{x\ln(x)}\,dx.$$

A simple substitution shows that the antiderivative of $\frac{1}{x\ln(x)}$ is $\ln(\ln x)$; therefore

$$\int_2^\infty \frac{1}{x\ln(x)}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\ln(x)}\,dx = \lim_{b\to\infty}\Big(\ln(\ln b) - \ln(\ln 2)\Big) = \infty.$$

Because the integral diverges, the series also diverges.

$\boxed{\text{10 pts}}$ 10. Find the solution to the initial-value problem $\dfrac{dy}{dt} = -2y$, $y(0) = 5$.

Solution: The solution is $y(t) = 5e^{-2t}$.

$\boxed{\text{15 pts}}$ 11. Calculate the sum $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$.

Solution: This is a telescoping series, because we can write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

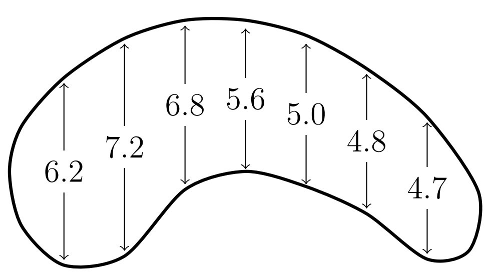
Therefore the partial sums of the series are

$$s_n = \left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

This means that the value of the series is

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1.$$

$\boxed{\text{15 pts}}$ 12. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals, as indicated in the figure. Use Simpson's Rule to estimate the area of the pool.



Solution: If we knew the width of the pool at every point, we could get the area as the integral of the width. We don't, but we can still use Simpson's rule to estimate the integral. The pool is 16 meters long; we divide it into eight parts of length 2 meters. According to Simpson's rule, an estimate for the area (in square meters) is

$$\begin{aligned}&\frac{2}{3}\Big(1\cdot 0+4\cdot 6.2+2\cdot 7.2+4\cdot 6.8+2\cdot 5.6+4\cdot 5.0+2\cdot 4.8+4\cdot 4.7+1\cdot 0\Big)\\&=\frac{2}{3}\Big(4\cdot(6.2+6.8+5.0+4.7)+2\cdot(7.2+5.6+4.8)\Big)=\frac{2}{3}\big(4\cdot 22.7+2\cdot 17.6\big)=84.\end{aligned}$$

Note that we need to include the two ends of the pool, where the width is zero.

$\boxed{\text{15 pts}}$ 13. Use power series to find an antiderivative for the function $\dfrac{\sin x}{x}$.

Solution: We know that

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

After dividing by x, we get

$$\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n+1)!}.$$

If we integrate this power series term by term, we obtain

$$\int\frac{\sin x}{x}\,dx=C+x-\frac{x^3}{3\cdot 3!}+\frac{x^5}{5\cdot 5!}+\cdots=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)\cdot(2n+1)!},$$

where C is an arbitrary constant.

$\boxed{\text{15 pts}}$ 14. Write the first four terms in the Taylor series expansion of $\ln x$ around the point $a=5$.

Solution: In general, the first four terms in the Taylor series expansion are

$$f(a)+f'(a)\cdot(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3.$$

In our case, $f(x)=\ln x$ and $a=5$. We first compute the necessary derivatives:

$$\begin{array}{cccc} f(x)=\ln x & f'(x)=\dfrac{1}{x} & f''(x)=-\dfrac{1}{x^2} & f'''(x)=\dfrac{2}{x^3}\\ f(5)=\ln(5), & f'(5)=\dfrac{1}{5} & f''(5)=-\dfrac{1}{25} & f'''(5)=\dfrac{2}{125}\end{array}$$

Putting it all together, the first four terms in the Taylor series are

$$\ln(5)+\frac{1}{5}(x-5)-\frac{1}{50}(x-5)^2+\frac{1}{375}(x-5)^3.$$

15 pts 15. A tank is filled with 1000 liters of pure water. Brine containing 0.015 kilograms of salt per liter enters the tank at a rate of 10 liters per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt will be in the tank after t minutes?

Solution: We let $S(t)$ be the amount of salt (in kilograms) in the tank after t minutes. To find a differential equation for S, we observe that salt is both coming in (with the brine) and going out (with the excess water), and

$$\frac{dS}{dt} = \big(\text{rate at which salt is coming in}\big) - \big(\text{rate at which salt is going out}\big).$$

Because the concentration of salt in the brine is constant, we have

$$\big(\text{rate at which salt is coming in}\big) = 0.015\frac{\text{kg}}{\ell} \cdot 10\frac{\ell}{\text{min}} = 0.15\frac{\text{kg}}{\text{min}}.$$

Any any given time, the concentration of salt in the tank, and therefore also in the water that is draining, is $S/1000$ kilograms per liter; this means that

$$\big(\text{rate at which salt is going out}\big) = \frac{S}{1000}\frac{\text{kg}}{\ell} \cdot 10\frac{\ell}{\text{min}} = \frac{S}{100}\frac{\text{kg}}{\text{min}}.$$

The differential equation is therefore

$$\frac{dS}{dt} = 0.15 - \frac{S}{100} = \frac{15 - S}{100},$$

and the initial condition is $S(0) = 0$, because there is no salt in the tank at the beginning. The equation is separable, and can be solved as follows:

$$-\ln(15 - S) = \int \frac{dS}{15 - S} = \int \frac{dt}{100} = \frac{t}{100} + C,$$

where C is a constant. If we solve for S, we get

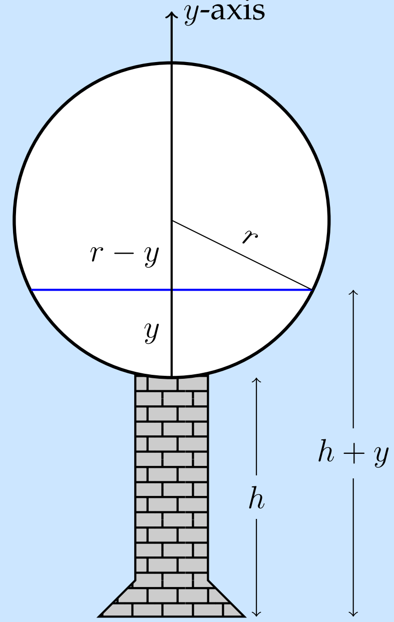
$$15 - S = e^{-t/100 - C} \qquad \text{or} \qquad S = 15 - e^{-t/100-C} = 15 - Ae^{-t/100},$$

where $A = e^{-C}$. The initial condition $S(0) = 0$ gives $A = 15$, and therefore

$$S(t) = 15 - 15e^{-t/100}.$$

15 pts 16. A water tower has a spherical tank of radius r meters, whose bottom sits h meters above the ground. The city wants to pump a liquid of density ρ into the tank until it is half full. Write down an integral that represents the necessary work. (Don't evaluate the integral!)

Solution: We choose our coordinate system so that the bottom of the tank is at $y = 0$, and the center of the tank at $y = r$.



Consider a thin layer of liquid, of thickness Δy, at the vertical position y. Its shape is approximately that of a cylinder of radius

$$\sqrt{r^2-(r-y)^2}=\sqrt{2ry-y^2}$$

and height Δy, and so its volume is approximately

$$\Delta V=\pi(2ry-y^2)\cdot\Delta y,$$

and its mass approximately $\Delta m=\rho\Delta V$. All the liquid in the layer is approximately $h+y$ meters above the ground, and so the work needed to pump it into the tank is

$$\Delta W=\rho\pi(2ry-y^2)g(h+y)\Delta y.$$

We conclude that the work it takes to fill half of the tank is given by the integral

$$W=\int_0^r \rho\pi g(2ry-y^2)(h+y)\,dy.$$

15 pts 17. For what values of x does the series $\displaystyle\sum_{n=1}^{\infty}(\ln x)^n$ converge?

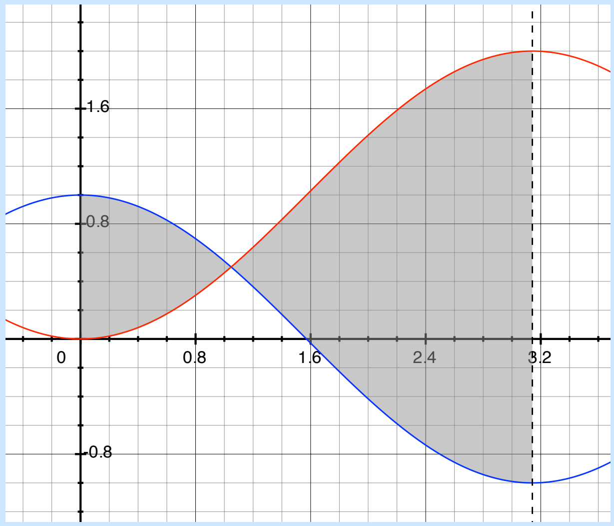
Solution: The series is obtained from the geometric series

$$\sum_{n=1}^{\infty}x^n=x+x^2+x^3+\cdots+x^n+\cdots$$

by replacing x with $\ln x$. We know that the geometric series converges for $-1<x<1$, and diverges for all other values of x. Therefore the series in the problem converges when $-1<\ln x<1$, which translates into $e^{-1}<x<e$.

15 pts 18. Sketch the curves $y = \cos x$ and $y = 1 - \cos x$ for $0 \le x \le \pi$, and observe that the region between the curves has two separate parts. Find the area of this region.

Solution: Here is a sketch of the two graphs $y = \cos x$ and $y = 1 - \cos x$:



To find the point where they intersect, we solve $\cos x = 1 - \cos x$ to get $x = \frac{\pi}{3}$. The area of the region is therefore

$$A = \int_0^{\frac{\pi}{3}} \Big(\cos x - (1 - \cos x)\Big)\, dx + \int_{\frac{\pi}{3}}^{\pi} \Big((1 - \cos x) - \cos x\Big)\, dx$$
$$= \int_0^{\frac{\pi}{3}} \Big(2\cos x - 1\Big)\, dx + \int_{\frac{\pi}{3}}^{\pi} \Big(1 - 2\cos x\Big)\, dx.$$

Because the antiderivative of $2\cos x - 1$ is $2\sin x - x$, we then get

$$A = \big(2\sin x - x\big)\Big|_0^{\frac{\pi}{3}} + \big(x - 2\sin x\big)\Big|_{\frac{\pi}{3}}^{\pi} = \left(\sqrt{3} - \frac{\pi}{3}\right) + \pi - \left(\frac{\pi}{3} - \sqrt{3}\right) = \frac{\pi}{3} + 2\sqrt{3}.$$

15 pts 19. Suppose I walk north for 1 mile, then east for 1/2 mile, then south for 1/4 mile, then west for 1/8 mile, then north for 1/16 mile, and so on. Where do I end up?

Solution: Let's first consider the north-south direction. Since I walk north for 1 mile, south for 1/4 mile, north for 1/16 mile, south for 1/64 mile, and so on, I end up

$$1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots = \frac{1}{1 + \frac{1}{4}} = \frac{4}{5}$$

miles north of where I started, using the formula for the sum of a geometric series. In the east-west direction, I walk east for 1/2 mile, west for 1/8 mile, east for 1/32 mile, west for 1/128 mile, and so on. So I end up

$$\frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \frac{1}{128} + \cdots = \frac{1}{2} \cdot \left(1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots\right) = \frac{1}{2} \cdot \frac{4}{5} = \frac{2}{5}$$

miles east of where I started.

15 pts 20. Calculate the indefinite integral $\int \arctan(x)\,dx$.

Solution: The trick is to use integration by parts with $u = \arctan(x)$ and $dv = dx$. Then $du = \frac{dx}{1+x^2}$ and $v = x$, and so we get

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{1+x^2}\,dx = x\arctan(x) - \frac{1}{2}\ln\left|1+x^2\right| + C.$$

15 pts 21. In the chemical reaction $A+B \to C$, one molecule of substance A reacts with one molecule of substance B to form one molecule of substance C. According to the law of mass action, this reaction happens at a rate that is proportional to the concentrations of A and B. Suppose that we start from a certain concentration of A and B. Find a differential equation with suitable initial condition that describes the concentration of C.

Solution: Let $C(t)$ be the concentration of substance C after t minutes; similarly for $A(t)$ and $B(t)$. Using the law of mass action, we get

$$\frac{dC}{dt} = kAB,$$

where k is a constant. If a and b denote the initial concentrations of A and B, then

$$A(t) = a - C(t) \qquad \text{and} \qquad B(t) = b - C(t),$$

because each time one molecule of C is produced, one molecule of A and one molecule of B are used up in the reaction. The differential equation is therefore

$$\frac{dC}{dt} = k(a-C)(b-C),$$

and the appropriate initial condition is $C(0) = 0$.