MATH 132 Final Exam

Monday December 19, 2011

Name: TIOCS **ID:** SOLNS **Rec:** IX

There are 22 problems in this exam, printed on 10 pages (not including this cover sheet). Make sure that you have them all.

The exam is in two parts: Part 1 consists of questions which should be *quite easy*. Getting part 1 at least 80% correct ensures a C or better on this exam. **Complete Part 1 FIRST**.

Part I:

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											

Part II:

Question:	11	12	13	14	15	16	17	18	19	20	21	22	Total
Points:	15	15	15	15	15	15	15	15	15	15	15	20	185
Score:													

Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate **clearly** what is where if you expect someone to look at it. If you actually read these instructions, write "I can has reading" on the bottom of this page, and you will get two points of extra credit. **Books, calculators, electronic devices, extra papers, and discussions with friends are not permitted.** Leave all answers in exact form (that is, do *not* approximate π, square roots, and so on.)

You must give a correct justification of all answers to receive credit unless otherwise stated.

You have 2 hours and 30 minutes to complete this exam.

I CAN HAS READING

Part 1: Do These First!

Name: _____ Id: _____

10 pts 1. Calculate the indefinite integral: $\int \frac{\cos x\, dx}{\sin^2 x}$.

$u = \sin x$
$du = \cos x\, dx$
$\int u^{-2} du = -u^{-1} + C = \boxed{\frac{1}{-\sin x} + C}$

10 pts 2. Calculate the definite integral $\int_0^1 xe^{-2x}\, dx$. If it does not converge, write "Diverges".

$u = x \quad dv = e^{-2x} dx$
$du = dx \quad v = -\frac{1}{2}e^{-2x}$
$= -\frac{x}{2}e^{-2x} + \frac{1}{2}\int_0^1 e^{-2x} dx = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\Big|_0^1 = \boxed{-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4}}$

10 pts 3. Calculate the definite integral $\int_0^1 \frac{3}{x^5}\, dx$. If it does not converge, write "Diverges".

$\int_0^1 \frac{3}{x^5} dx = -\frac{3}{4}x^{-4}\Big|_0^1 = -\frac{3}{4} - \lim_{x\to 0^+}\left(-\frac{3}{4}x^{-4}\right)$

$\boxed{\text{DIVERGES}}$
cuz $\lim_{x\to 0^+} -\frac{3}{4}x^{-4} = -\infty$

10 pts 4. Calculate the indefinite integral: $\int \frac{dx}{(x+1)(x-1)}$

$= \frac{1}{2}\int \frac{-1}{x+1} + \frac{1}{x-1}\, dx$

$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

$1 = A(x-1) + B(x+1)$
$1 = 2B \Rightarrow B = 1/2$
$1 = -2A \Rightarrow A = -1/2$

$= \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C$

$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$

$x+1 \to x-1$ 22.

Part 1: Do These First!

Name: _____ Id: _____

10 pts 5. Find the sum: $\sum_{n=1}^{\infty} \frac{3}{5^n}$

GEOMETRIC SINCE $n=1$, NOT $n=0$.

$\frac{3}{1-1/5} - 3 = \frac{3}{4/5} - 3 = \frac{15}{4} - 3 = \boxed{\frac{3}{4}}$

10 pts 6. Write a power series for xe^{-2x}.

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^{-2x} = 1 + (-2x) + \frac{2^2x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n x^n (-1)^n}{n!}$

$xe^{-2x} = \boxed{\sum_{n=0}^{\infty} \frac{2^n x^{n+1} (-1)^n}{n!}}$

10 pts 7. Find a function $y(x)$ that solves the differential equation $\frac{dy}{dx} = \frac{x}{\cos y}$ with $y(0) = \frac{\pi}{4}$

$\int \cos y \, dy = \int x \, dx$

$\sin y = \frac{1}{2}x^2 + C$

$y = \arcsin\left(\frac{x^2}{2} + C\right)$

$y(0) = \pi/4 \Rightarrow \arcsin(C) = \pi/4 \Rightarrow C = \frac{1}{\sqrt{2}}$.

$\boxed{y(x) = \arcsin\left(\frac{x^2}{2} + \frac{1}{\sqrt{2}}\right)}$.

10 pts 8. The series $\sum_{n=1}^{\infty} \frac{1}{3^n\sqrt{n}}$ Converges / Diverges by the what test? (circle one)

Justify:

RATIO: $\lim \frac{3^n\sqrt{n}}{3^{n+1}\sqrt{n+1}} = \frac{1}{3} < 1$

OR COMPARE W/ $\frac{1}{3^n} \geq \frac{1}{3^n\sqrt{n}}$

8. RATIO. OR COMPARISON.

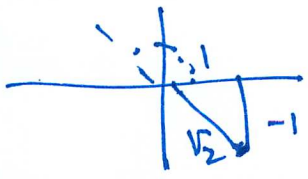
10 pts 9. The series $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$ Converges / Diverges by the what test? (circle one)

Justify:

$\int \frac{dx}{x(\ln x)^2} \quad u = \ln x \quad = \int_2^{\infty} \frac{du}{u^2} = \frac{-1}{u}\Big|_2^{\infty} = \frac{1}{2}$ ✓

9. INTEGRAL

10 pts 10. Write polar coordinates for the point with rectangular coordinates $(1, -1)$ in two different ways, one with $r > 0$ and the other with $r < 0$.



$r = \sqrt{2}$

$(r > 0)\ r = \sqrt{2}\ \theta = -\pi/4$

$(r < 0)\ r = -\sqrt{2}\ \theta = 3\pi/4$

Part 2: Do these after part 1. **Name:** _____ **Id:** _____

15 pts 11. Write the first four nonzero terms of the Taylor series for $f(x) = \ln(x/2)$ centered at $a = 2$.

$f(x) = \ln(x/2)$ $\quad f(2) = \ln 1 = 0$

$f'(x) = \frac{2}{x}\left(\frac{1}{2}\right) = x^{-1}$ $\quad f'(2) = \frac{1}{2}$

$f''(x) = -x^{-2}$ $\quad f''(2) = -\frac{1}{4}$

$f'''(x) = 2x^{-3}$ $\quad f'''(2) = \frac{1}{4}$

$f^{(4)}(x) = -6x^{-4}$ $\quad f^{(4)}(2) = -\frac{3}{8}$

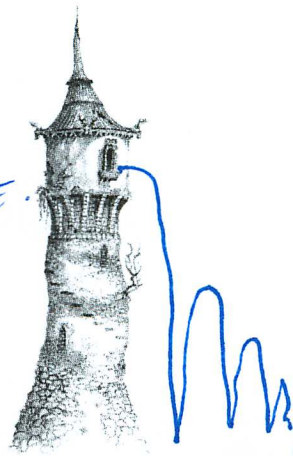
$$f(x) = \frac{1}{2}(x-2) - \frac{1}{4\cdot 2}(x-2)^2 + \frac{1}{4\cdot 3!}(x-2)^3 - \frac{3}{8\cdot 4!}(x-2)^4 + \cdots$$

15 pts 12. Calculate the sum $\frac{\pi}{2} - \frac{\pi^3}{2^3\,3!} + \frac{\pi^5}{2^5\,5!} - \frac{\pi^7}{2^7\,7!} + \cdots$ $= \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{\pi}{2}\right)^{2n-1}/n!$

$= \sin\left(\frac{\pi}{2}\right)$

$= 1$

15 pts 13. Jill is being held prisoner by the evil monkey-king. As a signal to her brother Jack, she drops an enchanted orb out of the window of the tower where she is being held, 100 feet above the ground. Each time the orb strikes the ground, it sends out a beacon of golden light, then bounces and returns to a height two-thirds of its previous maximum height. What is the total distance traveled by the orb if it bounces infinitely many times?



GEOMETRIC. MUST COUNT BOTH BOUNCES.

DOESN'T GO UP FIRST TIME.

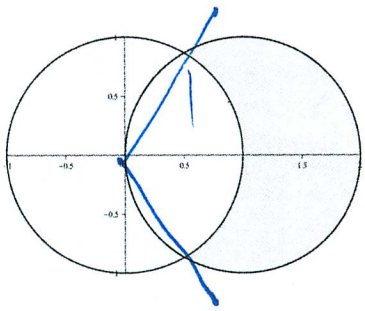
$$200\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n - 100$$

$$= \frac{200}{1-2/3} - 100 = 600 - 100$$

$$= 500'.$$

Part 2: Do these after part 1. **Name:** _____ **Id:** _____

15 pts 14. Find the area of the region that lies inside the circle of radius one given by $r = 2\cos\theta$, but outside the circle $r = 1$.



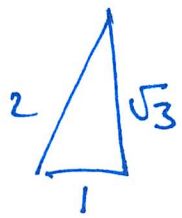
CROSS WHEN $\cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \pm \frac{\pi}{3}$.

$$\int_{-\pi/3}^{\pi/3} \frac{1}{2}(2\cos\theta)^2 - \frac{1}{2}(1)^2 \, d\theta$$

$$= 2\int_{-\pi/3}^{\pi/3} \left(\cos^2\theta - \frac{1}{2}\right) d\theta$$

$$= 2\int_{0}^{\pi/3} (1+\cos 2\theta) d\theta - \left(\frac{\pi}{3}\right)$$



$$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/3} - \pi/3$$

$$= 2\pi/3 + \frac{\sqrt{3}}{2} - \pi/3 = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$$

Part 2: Do these after part 1. **Name:** _____ **Id:** _____

15. Consider the differential equation $y'' + 4y' + 20y = 0$.

10 pts (a) Write the most general form of the solution $y(x)$ which is real-valued for x real.

$r^2 + 4r + 20 = 0$

$r = \frac{-4 \pm \sqrt{16 - 80}}{2} = -2 \pm 4i$

$y(x) = e^{-2x}(A\cos(4x) + B\sin(4x))$

$y'(x) = -2e^{-2x}(A\cos 4x + B\sin 4x) + e^{-2x}(-4A\sin 4x + 4B\cos 4x)$

5 pts (b) Write a formula for the solution $y(x)$ with $y(0) = 1$ and $y'(0) = 14$.

$y(0) = 1 \Rightarrow A = 1$

$y'(0) = 14 \Rightarrow -2(A + 4B = 14$

$4B = 16$

$B = 4.$

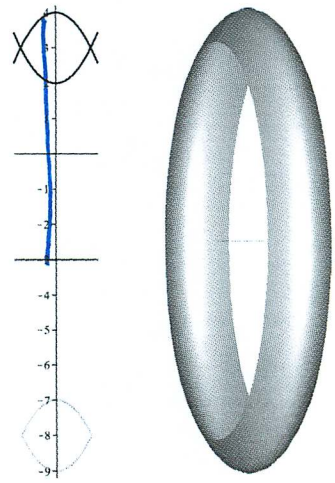
$\Rightarrow y(x) = e^{-2x}(\cos 4x + 4\sin 4x).$

Part 2: Do these after part 1. **Name:** _____ **Id:** _____

15 pts 16. Find the volume of the wedding-band shape obtained by rotating the region between the two curves

$$y = x^2 + 2 \qquad \text{and} \qquad y = 4 - x^2$$

about the horizontal line $y = -3$.



(a) Write an integral which represents the volume.

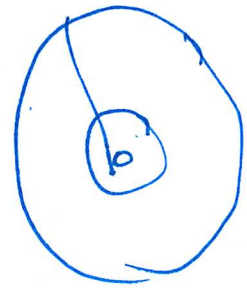
CROSS AT $x^2+2 = 4-x^2$

$2x^2 = 2 \Rightarrow x = \pm 1.$

AREA (SLICE) IS

~~$\pi(x^2+2+3)$~~

$\pi(4-x^2+3)^2 - \pi(x^2+2+3)^2$



$$\pi\int_{-1}^{1} (4-x^2)^2 - (x^2+5)^2\,dx$$

$49 - 14x^2 + x^4 - [x^4 + 10x^2 + 25] = 14 - 24x^2$

(b) Evaluate the integral in (a).

~~$\pi\int_{-1}^{1} 25 - 10x^2 + x^4 - (x^4 + 10x^2 + 25)\,dx$~~

FUK!

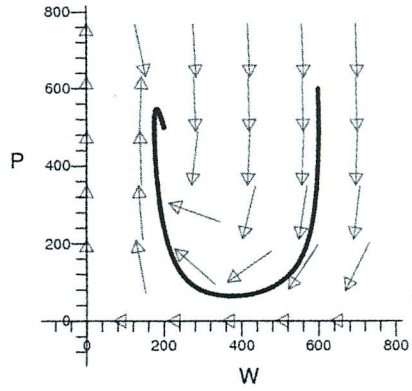
$= \pi\int_{-1}^{1} (14 - 24x^2)\,dx$

$= \pi\left(14x - 8x^3\right)\Big|_{-1}^{1} = \pi(6+6) = 12\pi.$

439

Part 2: Do these after part 1. **Name:** _____ **Id:** _____

15 pts 17. Two populations, the Pacifists and the Warriors, live near one another. The Pacifists are simple rutabaga farmers: if left to themselves, their population would be well modelled by a logistic growth model. However, the nearby Warriors survive by making regular raids on the Pacifists. The two populations are modelled by the predator-prey system below, where t is in years, $W(t)$ is the population of the Warriors after t years, and $P(t)$ is the population of the Pacifists. The phase portrait for this system is shown at right.



$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{1000}\right) - \frac{PW}{200}$$

$$\frac{dW}{dt} = -\frac{W}{4} + \frac{PW}{2000}$$

(a) Are there any equilibrium solutions? If so, find **all** of them. If not, write "none", and **justify your answer.**

$P' = 2P\left(1 - \frac{P}{1000} - \frac{W}{400}\right) \neq 0$

$W' = W\left(-\frac{1}{4} + \frac{P}{2000}\right)$

$\Rightarrow W' = 0$ IF $W = 0$ OR $P = 500$

$P' = 0$ IF $P = 0$ OR $1 - \frac{P}{1000} - \frac{W}{400} = 0$ IF

IF $W=0$, $P = 1000$ / 0

IF $P=500$, $W = 200$.

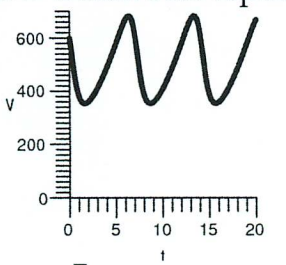
(0,0)

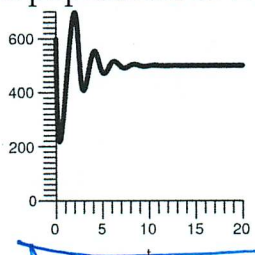
(0, 1000)

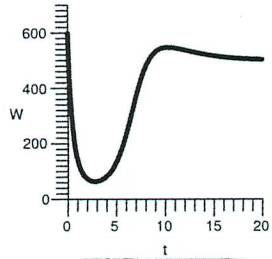
(200, 500)

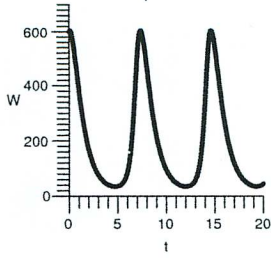
200

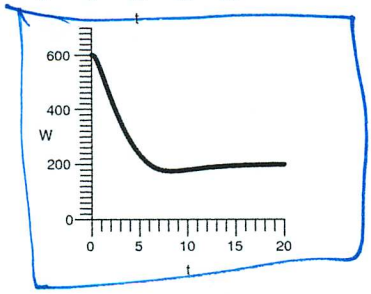
(b) If the populations start out with 600 Pacifists and 600 Warriors, circle the graph below which best represents the population of Warriors.









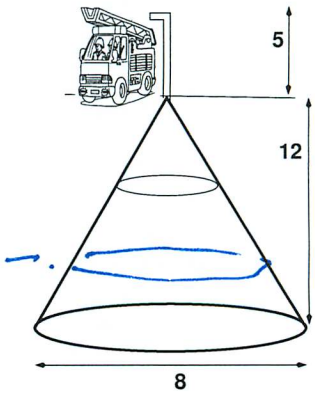


444

Part 2: Do these after part 1. **Name:** _____ **Id:** _____

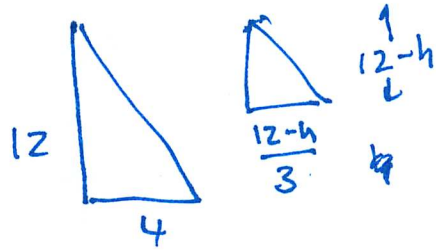
15 pts 18. Fireman Fred has an underground tank partially full of Fluorotelomer Fire-Fighting Foam. The tank is conical, with the vertex at the top of the tank. The height of the tank is 12 feet, with a diameter of 8 feet, and is filled to a height of 9 feet. Fred wants to pump the foam out of the tank and into his truck, which fills at a height 5 feet above groundlevel. The foam has a density of 1 pound per cubic foot and has a delightful minty scent.

Write an integral which represents the amount of work required for Fred to pump all of the foam out of the tank and into his truck. (You do not need to calculate the integral).



SLICE @ h.

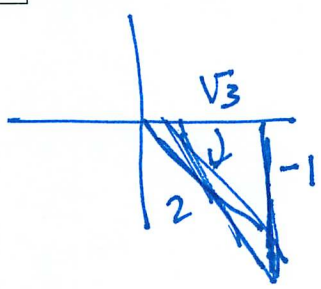
$r = \frac{12-h}{3}$, $V = \pi\left(\frac{12-h}{3}\right)^2$



DISTANCE = $17-h$.

$$\int_0^9 \pi\left(\frac{12-h}{3}\right)^2 (17-h)\,dh.$$

15 pts 19. Simplify the complex number $(\sqrt{3}-i)^{47}$, writing it in the form $a+bi$ with a and b real.



$\theta = \frac{\pi}{6}$

$$-\frac{47\pi}{6} = -\frac{48\pi + \pi}{6} \approx +\frac{\pi}{6}$$

$$= 2^{47}\sqrt{3} + 2^{47}i$$

Part 2: Do these after part 1. **Name:** _____ **Id:** _____

15 pts 20. Let $f(x) = \sqrt{x}$. Find a value c between 4 and 9 so that $f(c)$ is equal to the average of $f(x)$ in $[4, 9]$. You should leave your answer in rough form; it is not necessary to simplify fully.

AVG VAL OF $\sqrt{x}$ ON $[4,9]$ =

$$\frac{1}{5}\int_4^9 x^{1/2}dx = \frac{1}{5}\cdot\frac{2}{3}x^{3/2}\Big|_4^9 = \frac{2}{15}(27-8)$$

$$\sqrt{c} = \frac{38}{15}.$$

$$c = \left(\frac{38}{15}\right)^2$$

15 pts 21. State all values of x for which the series $\sum_{n=0}^{\infty} \frac{(2x-3)^n}{n \ln n}$ converges. Don't forget to check the endpoints.

RATIO: $\lim_{n\to\infty}\left|\frac{(2x-3)^{n+1}}{(2x-3)^n}\cdot\frac{n\ln n}{(n+1)\ln(n+1)}\right| = 2x-3$

$\Rightarrow -1 < 2x-3 < 1 \Rightarrow 2 < 2x < 4$

$1 < x < 2$

IF $x=1$, $\sum \frac{(-1)^n}{n\ln n}$ CONV, ALT SERIES.

IF $x=2$, $\sum \frac{1}{n\ln n}$ DIVERGES, INTEGRAL TEST.

$\therefore 1 \le x < 2$

Name: _____ Id: _____

20 pts 22. Compute the following integrals.

(a) $\int \arctan(1/x)\,dx$

$u = \arctan\left(\frac{1}{x}\right) \quad dv = dx$

$du = \frac{1}{1+\frac{1}{x^2}}\left(\frac{-1}{x^2}\right)dx \quad v = x\,dx$

$\frac{x^2}{1+x^2}\left(\frac{-1}{x^2}\right)dx$

$= x\arctan\left(\frac{1}{x}\right) + \int \frac{x\,dx}{1+x^2}$

$= x\arctan\left(\frac{1}{x}\right) + \frac{1}{2}\ln(1+x^2) + C$

(b) $\int \sqrt{16-5x^2}\,dx = \int \sqrt{1-\left(\frac{\sqrt{5}x}{4}\right)^2}\,dx$

$\frac{\sqrt{5}x}{4} = \sin\theta, \quad \frac{\sqrt{5}}{4}dx = \cos\theta\,d\theta$

$dx = \frac{4}{\sqrt{5}}\cos\theta\,d\theta$

$= \frac{4}{\sqrt{5}}\int \cos^2\theta\,d\theta$

$= \frac{4}{\sqrt{5}}\left(\frac{1}{2}\int 1+\cos 2\theta\,d\theta\right)$

$= \frac{4}{\sqrt{5}}\left(\theta + \frac{1}{2}\sin 2\theta\right)$

$= \frac{4}{\sqrt{5}}\left(\theta + \sin\theta\cos\theta + C\right)$

$= \frac{4}{\sqrt{5}}\left(\arcsin\left(\frac{\sqrt{5}x}{4}\right) + \frac{\sqrt{5}x}{4}\sqrt{1-\left(\frac{\sqrt{5}x}{4}\right)^2}\right) + C.$

EWWW.