MATH 127 Solutions to Midterm 2

15 pts 1. State whether each of the following sequences is convergent or divergent, and briefly justify your answer. If the sequence is convergent, compute its limit.

(a) $\left\{ \dfrac{3n^2 + n - 1}{7n^2 + 3n + 6} \right\}$

Solution: The sequence is convergent, and the limit is $\frac{3}{7}$. We can see this by rewriting the fraction as

$$\frac{3n^2 + n - 1}{7n^2 + 3n + 6} = \frac{n^2 \cdot (3 + \frac{1}{n} - \frac{1}{n^2})}{n^2 \cdot (7 + \frac{3}{n} + \frac{6}{n^2})} = \frac{3 + \frac{1}{n} - \frac{1}{n^2}}{7 + \frac{3}{n} + \frac{6}{n^2}}.$$

This shows that

$$\lim_{n\to\infty} \frac{3n^2 + n - 1}{7n^2 + 3n + 6} = \lim_{n\to\infty} \frac{3 + \frac{1}{n} - \frac{1}{n^2}}{7 + \frac{3}{n} + \frac{6}{n^2}} = \frac{3 + 0 - 0}{7 + 0 + 0} = \frac{3}{7}.$$

(b) $\left\{ \ln \dfrac{n+1}{n} \right\}$

Solution: The sequence is convergent, and the limit is 0. The reason is that

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1,$$

and therefore

$$\lim_{n\to\infty} \ln \frac{n+1}{n} = \ln 1 = 0.$$

(c) $\{ne^n\}$

Solution: The sequence is divergent (to $+\infty$); this is clear because both n and e^n grow without bounds as $n \to \infty$.

15 pts 2. A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = -a_n^2 + 4a_n - 1$ for $n \geq 1$.

(a) Compute the first six terms of the sequence.

Solution: The formula for a_{n+1} tells us how to get each term from the previous one:

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= -1 + 4 - 1 = 2 \\
a_3 &= -4 + 8 - 1 = 3 \\
a_4 &= -9 + 12 - 1 = 2 \\
a_5 &= -4 + 8 - 1 = 3 \\
a_6 &= -9 + 12 - 1 = 2
\end{aligned}$$

(b) Is the sequence convergent or divergent? Briefly justify your answer. If the sequence is convergent, compute its limit.

Solution: The sequence goes back and forth between 2 and 3, and is therefore divergent. In fact, the pattern that appears from the first six terms is that $a_n = 2$ whenever n is even, and $a_n = 3$ whenever n is odd (with the exception of $a_1 = 1$). The formula for a_{n+1} bears this out. Indeed, if $a_{2k} = 2$, then the next term is

$$a_{2k+1} = -4 + 8 - 1 = 3,$$

and the next term after that is

$$a_{2k+2} = -9 + 12 - 1 = 2.$$

15 pts

3. Consider the sequence defined by

$$a_n = \frac{n}{n^2+1} \qquad \text{for } n \geq 1.$$

Show that this sequence is decreasing.

Solution: We have to explain why $a_1 \geq a_2 \geq a_3 \geq \cdots$. This can be done in two ways.

The first way is to show directly that $a_n \geq a_{n+1}$ for every $n \geq 1$, or in other words, that

$$\frac{n}{n^2+1} \geq \frac{n+1}{(n+1)^2+1}.$$

Cross-multiplying turns this into

$$n \cdot \big((n+1)^2 + 1\big) \geq (n^2+1) \cdot (n+1)$$

and if we expand the parentheses, we get

$$n^3 + 2n^2 + 2n \geq n^3 + n^2 + n + 1,$$

or equivalently, $n^2 + n \geq 1$. This last inequality is clearly true when $n \geq 1$, and so we can say that $a_n \geq a_{n+1}$ for every $n \geq 1$.

The second way is to show that the function $f(x) = \dfrac{x}{x^2+1}$ is decreasing for $x \geq 1$. For that, we compute the derivative

$$f'(x) = \frac{(x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

For $x \geq 1$, we have $1 - x^2 \leq 0$, and therefore $f'(x) \leq 0$. This proves that the function $f(x)$ is decreasing; so in particular

$$f(1) \geq f(2) \geq f(3) \geq \cdots$$

Because $a_n = f(n)$, this does the job.

15 pts

4. Use the integral test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+3}}$$

is convergent or divergent.

Solution: The relevant function is $f(x) = \dfrac{1}{\sqrt{2x+3}}$. It is clearly positive and decreasing, and so we are allowed to use the integral test. The improper integral

$$\int_1^{\infty} \frac{1}{\sqrt{2x+3}}\, dx = \sqrt{2x+3}\Big|_1^{\infty} = +\infty$$

is infinite, and by the integral test, the series is therefore divergent.

20 pts

5. Determine whether the following series are convergent or divergent. Briefly justify your answer in each case.

(a) $16 + 12 + 9 + \dfrac{27}{4} + \dfrac{81}{16} + \cdots$

Solution: The series is convergent. The reason is that it is a geometric series with initial term 16 and ratio $\frac{12}{16} = \frac{3}{4}$, and $\frac{3}{4} < 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n^3+1}$

Solution: The series is convergent. We have

$$\frac{n}{4n^3+1} \leq \frac{n}{4n^3} = \frac{1}{4n^2}$$

for every $n \geq 1$. The bigger series

$$\sum_{n=1}^{\infty} \frac{1}{4n^2}$$

converges (by the p-series test), and therefore our original series also converges (by the comparison test).

(c) $\displaystyle\sum_{n=1}^{\infty} \ln(n)$

Solution: The series is divergent. In fact, the terms $\ln(n)$ are not going to zero, and so we can simply apply the divergence test.

(d) $1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \cdots$

Solution: The series is divergent. To see this, we write the series in the form

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}.$$

Now $\frac{1}{2n-1} \geq \frac{1}{2n}$, and the smaller series

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

is a harmonic series, and therefore divergent. By the comparison test, our original series is also divergent.

20 pts 6. A ball is being dropped from a height of 100 feet above the ground. Each time the ball strikes the ground, it bounces and returns to a height equal to two-thirds of its previous maximum height. What is the total distance travelled by the ball if it bounces infinitely many times?

Solution: The ball falls 100 feet; bounces back to a height of $\frac{2}{3} \cdot 100$ feet and falls the same distance; bounces back to a height of $\frac{2}{3} \cdot \frac{2}{3} \cdot 100$ feet and falls the same distance; etc. The total distance D travelled by the ball is therefore equal to the infinite series

$$D = 100 + 2 \cdot 100 \cdot \frac{2}{3} + 2 \cdot 100 \cdot \left(\frac{2}{3}\right)^2 + 2 \cdot 100 \cdot \left(\frac{2}{3}\right)^3 + \cdots$$

Leaving the first term alone, but factoring out $2 \cdot 100 \cdot \frac{2}{3}$ from all the other terms, we can rewrite this as

$$D = 100 + 2 \cdot 100 \cdot \frac{2}{3} \cdot \left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right)$$

This is a geometric series with ratio $\frac{2}{3}$, and so the answer is

$$D = 100 + 2 \cdot 100 \cdot \frac{2}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 100 + 2 \cdot 100 \cdot \frac{2}{3} \cdot 3 = 100 + 400 = 500.$$

In other words, the ball travels a total distance of 500 feet.