MAT 127: Calculus C, Spring 2017
Solutions to Final Exam

Problem 1 (10pts)

Determine whether each of the following sequences or series converges or not. In each case, clearly circle either **YES** *or* **NO**, *but not both. Each correct answer is worth 2 points.*

(a) *the sequence* $a_n = (-1)^n \sin(1/n)$ **YES** **NO**

Since $\sin(1/n) \longrightarrow \sin 0 = 0$, $a_n \longrightarrow 0$ by the Squeeze Theorem for Sequences.

(b) *the sequence* $a_n = n^2\big(1-\cos(1/n)\big)$ **YES** **NO**

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty}\frac{1-\cos(1/n)}{(1/n)^2} = \lim_{x\longrightarrow 0}\frac{1-\cos(x)}{x^2} = \lim_{x\longrightarrow 0}\frac{\sin(x)}{2x} = \lim_{x\longrightarrow 0}\frac{\cos(x)}{2} = \frac{\cos 0}{2} = \frac{1}{2}.$$

The third and fourth equalities use l'Hospital, which is applicable here because $(1-\cos(x)), x^2 \longrightarrow 0$ and $\sin(x), 2x \longrightarrow 0$ as $x \longrightarrow 0$. Alternatively, using $\cos x = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}$,

we find that $a_n = n^2\big(1-(1-\frac{1}{2!n^2}+\frac{1}{4!n^4}+\ldots)\big) = \frac{1}{2} - \frac{1}{24n^2} - \ldots$

where $\ldots$ involve $1/n^4$, $1/n^6$, and so on. As $n \longrightarrow \infty$, a_n thus approaches $\frac{1}{2}$ and so converges.

(c) *the series* $\sum_{n=1}^{\infty}(-1)^n$ **YES** **NO**

Since $(-1)^n$ does not approach 0, the sum diverges by the Test for Divergence.
Alternatively, this is a geometric series with $r=-1$ and thus diverges.

(d) *the series* $\sum_{n=1}^{\infty}(-1)^n\frac{\cos(n)}{n^2+1}$ **YES** **NO**

The series $\sum_{n=1}^{\infty}\frac{|\cos(n)|}{n^2+1}$ converges b/c $\frac{|\cos(n)|}{n^2+1} \le \frac{1}{n^2}$ and the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-Series Test. Thus, the original series converges by the Absolute Convergence Test.

(e) *the series* $\sum_{n=1}^{\infty}\frac{2^n+7^n}{5^n+6^n}$ **YES** **NO**

The series $\sum_{n=1}^{\infty}\frac{7^n}{6^n+6^n} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{7}{6}\right)^n$ diverges b/c it is geometric with $r=7/6>1$.

Since $\frac{2^n+7^n}{5^n+6^n} > \frac{7^n}{6^n+6^n}$, the larger original series also diverges by the Comparison Test.
The Ratio Test or the Limit Comparison Test with $b_n = 7^n/6^n$ can also be used.

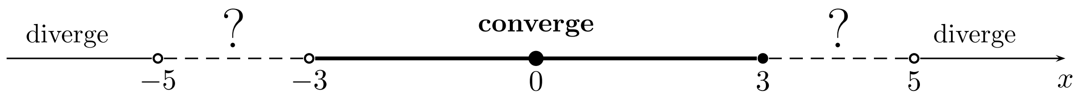
Problem 2 (10pts)

Suppose $c_0, c_1, c_2, \ldots \geq 0$, the series $\sum_{n=0}^{\infty} c_n 3^n$ converges, and $\sum_{n=0}^{\infty} c_n(-5)^n$ diverges. What can be said about the convergence of the infinite series below? In each case, clearly circle exactly one of the three choices. Each correct answer is worth 1 point.

$\sum_{n=0}^{\infty} c_n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} (-1)^n c_n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} (-1)^n c_n 7^n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} c_n 7^n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} n c_n 2^n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} c_n 4^n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} (-1)^n c_n 4^n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} (-1)^n c_n 3^n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} c_n^2$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} (-1)^n c_n^2 4^n$:	**converges**	**diverges**	**impossible to tell**

Explanation on the next page.

By the assumptions, the power series $\sum_{n=0}^{\infty} c_n x^n$ converges if $x=3$ and diverges if $x=-5$.



Thus, the radius of convergence R of this power series is at least 3 and at most 5. The same applies to the power series

$$x\left(\sum_{n=0}^{\infty} c_n x^n\right)' = x\sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} n c_n x^n\,.$$

Since the radii of convergence of these power series are at least 3, the 1st, 2nd, and 5th series on the previous page *converge* (they correspond to $x=1,-1,2$). Since the radius of convergence of the first power series is at most 5, the 3rd and 4th series *diverge* (they correspond to $x=-7,7$). Since $x=4,-4$ fall between the two bounds for the radius of convergence of the first power series, it is *impossible to tell* what happens with the 6th and 7th series.

Since $c_n \ge 0$,

$$\sum_{n=0}^{\infty} \big|(-1)^n c_n 3^n\big| = \sum_{n=0}^{\infty} c_n 3^n\,.$$

Since this series converges by assumption, the 8th series on the previous page also *converges*.

Since the 1st series on the previous page converges, $c_n \longrightarrow 0$ and thus $0 \le c_n \le 1$ for all n large. This implies that $0 \le c_n^2 \le c_n$. Thus, the smaller 9th series also *converges* by the Comparison Test (as on Midterm II 2d).

Since the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$ is at least 3, the series $\sum_{n=0}^{\infty} c_n 2^n$ converges. Thus, $c_n 2^n \longrightarrow 0$ and $0 \le c_n 2^n \le 1$ for all n large. This implies that

$$0 \le c_n^2 4^n = \big(c_n 2^n\big)^2 \le c_n 2^n\,.$$

Thus, the smaller series $\sum_{n=0}^{\infty} c_n^2 4^n$ also converges. By the Absolute Convergence Test, this in turn implies that the last series on the previous page *converges*.

Problem 3 (10pts)

Put your answer to each question in the corresponding box in the simplest possible form. No credit will be awarded if the answer in the box is wrong; partial credit may be awarded if the answer in the box is correct, but not in the simplest possible form.

(a; 5pts) *Write the number* $1.\overline{06} = 1.060606\ldots$ *as a simple fraction*

$$\boxed{\frac{35}{33}}$$

$$1.\overline{06} = 1 + .06 + .06 \cdot \frac{1}{100} + .06 \cdot \frac{1}{100^2} + \ldots = 1 + \frac{6/100}{1-1/100} = 1 + \frac{6}{99} = 1 + \frac{2}{33} = \frac{35}{33}$$

Grading: wrong answer 0pts; as above 5pts; $1\frac{2}{33}$ or fraction not simplified 4pts; both issues 3pts

(b; 5pts) *Find the limit of the sequence*

$$\boxed{4}$$

$$\sqrt{12},\ \sqrt{12+\sqrt{12}},\ \sqrt{12+\sqrt{12+\sqrt{12}}},\ \sqrt{12+\sqrt{12+\sqrt{12+\sqrt{12}}}},\ \ldots$$

Assume that this sequence converges.

By the above, $a_{n+1} = \sqrt{12+a_n}$. If this sequence converges to a, then

$$a = \sqrt{12+a} \quad\Longrightarrow\quad a^2 = 12+a \quad\Longrightarrow\quad a^2 - a - 12 = 0 \quad\Longrightarrow\quad (a-4)(a+3) = 0.$$

Since $a \geq 0$ by the first statement, we conclude that $a=4$.

Grading: answer as above 5pts; 0pts otherwise

Problem 4 (15pts)

All questions in this problem refer to the power series

$$f(x) = \sum_{n=1}^{\infty} \frac{2^n x^n}{n}\,.$$

(a; 9pts) *Find the radius and interval of convergence of this power series.*

To find the radius of convergence, use the Ratio Test with $a_n = 2^n x^n/n \neq 0$:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}|x|^{n+1}/(n+1)}{2^n|x|^n/n} = 2|x|\,\frac{n}{n+1} = 2|x|\frac{1}{1+1/n} \longrightarrow 2|x|\frac{1}{1+0} = 2|x|.$$

So the series converges if $2|x|<1$ and diverges if $2|x|>1$. Thus, the radius of convergence is $\boxed{1/2}$ It remains to check convergence for $x = \pm 1/2$, i.e. whether each of the series

$$f(1/2) = \sum_{n=1}^{\infty} \frac{2^n(1/2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{and} \qquad f(-1/2) = \sum_{n=1}^{\infty} \frac{2^n(-1/2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges. The first series diverges by the p-Series Test with $p = 1 \le 1$. The second series converges by the *Alternating Series Test*, since it is alternating (odd terms are negative, even terms are positive), decreasing in absolute value ($1/n > 1/(n+1)$), and approaching 0 ($1/n \longrightarrow 0$). So, the right endpoint is not in the interval of convergence, while the left one is. Thus, the interval of convergence is $\boxed{[-1/2, 1/2)}$

Grading: correct RT setup 3pts; rest of R computation 2pts; at least 3pts for this portion if R is stated correctly; conclusion and justification for each endpoint 1pt each; no more than 5pts for part (a) if R is found to be ∞

(b; 6pts) *Find the Taylor series expansion for the function $f'(x)$ around $x=0$. What are the radius and interval of convergence of this power series?*

The TS for $f'(x)$ around $x=0$ is obtained by differentiating that for f term-by-term:

$$f'(x) = \left(\sum_{n=1}^{\infty} \frac{2^n x^n}{n}\right)' = \sum_{n=1}^{\infty} \frac{2^n n x^{n-1}}{n} = \boxed{\sum_{n=1}^{\infty} 2^n x^{n-1} = \sum_{n=0}^{\infty} 2^{n+1} x^n}$$

Since differentiation does not change the radius of convergence of a power series, the radius of convergence of this power series is still $\boxed{1/2}$ Since differentiation can only drop endpoints from the interval of convergence, this series diverges at $x = 1/2$ by part (a) and we we need to check convergence of the series

$$f'(-1/2) = \sum_{n=1}^{\infty} 2^n\big(-1/2\big)^{n-1} = 2\sum_{n=1}^{\infty} \big(2(-1/2)\big)^{n-1} = 2\sum_{n=1}^{\infty} (-1)^n.$$

This series diverges by the *Test for the Divergence* because the sequence $(-1)^n$ does not approach 0. So the endpoint $x = -1/2$ is dropped from the interval of convergence in (a). Thus, the interval of convergence is $\boxed{(-1/2, 1/2)}$ Alternatively,

$$\sum_{n=0}^{\infty} 2^{n+1} x^n = 2\sum_{n=0}^{\infty} 2^n x^n = 2\sum_{n=0}^{\infty} (2x)^n$$

is a geometric series with $r = 2x$. It converges if and only if $2|x|<1$

Grading: $f'(x)$ as a power series in either form in the box 3pts (at least 1pt if the differentiation is started term-by-term); the radius and each endpoint are 1 point each. If the geometric series approach is used, 1pt off if a geometric series presentation is not specified. If the f'-series is not obtained correctly, but the incorrect expression is used correctly to obtain the radius and interval of convergence, then no carryover penalty.

Problem 5 (20pts)

Find Taylor series expansions of the following functions around the given point. In each case, determine the radius of convergence of the resulting power series and its interval of convergence.

(a; 10pts) $f(x) = x^2$ *around* $x = 2$

In this case, all derivatives can be computed:

$$f^{\langle 0\rangle}(x) = x^2 \implies f^{\langle 0\rangle}(2) = 4, \qquad f^{\langle 1\rangle}(x) = 2x \implies f^{\langle 1\rangle}(2) = 4,$$
$$f^{\langle 2\rangle}(x) = 2 \implies f^{\langle 2\rangle}(2) = 2,$$

and $f^{\langle n\rangle}(x)=0$ if $n\ge 3$. So by the Main Taylor Formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{\langle n\rangle}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{\langle n\rangle}(2)}{n!}(x-2)^n = \frac{4}{0!}(x-2)^0 + \frac{4}{1!}(x-2)^1 + \frac{4}{2!}(x-2)^2$$
$$= \boxed{4 + 4(x-2)+(x-2)^2}$$

Since this series is a sum of finitely many (three) terms, it converges for all x. So the interval of convergence is $\boxed{(-\infty, \infty)}$, while the radius is $\boxed{\infty}$

Remark: you can check the Taylor series expansion by expanding the expression in the long box above and getting x^2.

Grading: statement of general Taylor formula 1pt, with $a=2$ 3pts (*not* in addition to 1pt); vanishing of higher derivatives 1pt and computation of the remaining derivatives 2pts (no separate statement is required if general Taylor formula with $a=2$ is stated and correctly applied in this case); final answer 1pt; interval of convergence, radius of convergence, and explanation 1pt each (with 3pt loss if the interval of convergence is not correct); at least 2pts off if final answer is not a polynomial in $(x-2)$.

(b; 10pts) $f(x) = \dfrac{2x^2}{8 - x^3}$ *around* $x = 0$

Since $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ and this power series converges if $|x| < 1$,

$$\frac{2x^2}{8-x^3} = \frac{2x^2}{8} \cdot \frac{1}{1-(x^3/8)} = \frac{x^2}{4}\sum_{n=0}^{\infty}\left(\frac{x^3}{8}\right)^n = \frac{x^2}{4}\sum_{n=0}^{\infty}\frac{(x^3)^n}{8^n} = \boxed{\sum_{n=0}^{\infty}\frac{x^{3n+2}}{4\cdot 8^n} = \sum_{n=0}^{\infty}\frac{x^{3n+2}}{2^{3n+2}}}$$

and this series converges whenever

$$|x^3/8| < 1 \iff -8 < x^3 < 8 \iff -2 < x < 2.$$

So the interval of convergence is $\boxed{(-2, 2)}$ and the radius is $\boxed{2}$

Grading: use of correct standard power series 2pts (no separate statement is required if properly used in the given case); substitution and multiplication statement 1pt each; 3pts for simplifying to a power series in x; interval of convergence, radius of convergence, and explanation 1pt each (with 3pt loss if the interval of convergence is not correct, except endpoints error 1pt off).

(c; bonus 10pts) $f(x) = \dfrac{1}{1 - x^2 - 2x^4}$ *around* $x = 0$

First, factor out the denominator:

$$\frac{1}{1 - x^2 - 2x^4} = \frac{1}{(1-2x^2)(1+x^2)}\,.$$

Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ if $|x| < 1$,

$$\frac{1}{1-2x^2} = \sum_{n=0}^{\infty} \left(2x^2\right)^n = \sum_{n=0}^{\infty} 2^n x^{2n} \qquad \text{if } \ |2x^2| < 1$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad \text{if } \ |-x^2| < 1.$$

Using partial fractions thus gives

$$\begin{aligned}
\frac{1}{1-x^2-2x^4} &= \frac{-1}{(2x^2-1)(x^2+1)} = \frac{-1/2}{(x^2-1/2)(x^2+1)} = \frac{-1/2}{(+1)-(-1/2)}\left(\frac{1}{x^2-1/2} - \frac{1}{x^2+1}\right) \\
&= \frac{-1}{3}\left(\frac{1}{x^2-1/2} - \frac{1}{x^2+1}\right) = \frac{1}{3}\left(\frac{1}{1/2-x^2} + \frac{1}{1+x^2}\right) = \frac{1}{3}\left(\frac{2}{1-2x^2} + \frac{1}{1+x^2}\right) \\
&= \frac{1}{3}\left(2\sum_{n=0}^{\infty} 2^n x^{2n} + \sum_{n=0}^{\infty} (-1)^n x^{2n}\right) = \boxed{\sum_{n=0}^{\infty} \frac{2^{n+1}+(-1)^n}{3} x^{2n}}
\end{aligned}$$

Since the $1/(1-2x^2)$ series converges whenever $x^2 < 1/2$, while the $1/(1+x^2)$ series converges whenever $x^2 < 1$, their sum converges whenever $x^2 < 1/2$. So the interval of convergence is $\boxed{(-1/\sqrt{2}, 1/\sqrt{2})}$ and the radius is $\boxed{1/\sqrt{2}}$

Alternatively, multiplication of power series can be used:

$$\begin{aligned}
\frac{1}{1 - x^2 - 2x^4} &= \frac{1}{1-2x^2} \cdot \frac{1}{1+x^2} = \left(\sum_{n=0}^{\infty} 2^n x^{2n}\right) \cdot \left(\sum_{n=0}^{\infty} (-1)^n x^{2n}\right) \\
&= \Big(1 + 2x^2 + 2^2(x^2)^2 + \dots\Big)\Big(1 - x^2 + (x^2)^2 + \dots\Big) \\
&= \sum_{n=0}^{\infty} \Big(1 \cdot (-1)^n + 2 \cdot (-1)^{n-1} + \dots + 2^n \cdot (-1)^0\Big) x^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} - 2^{n+1}}{(-1) - 2} x^{2n} = \boxed{\sum_{n=0}^{\infty} \frac{2^{n+1}+(-1)^n}{3} x^{2n}}
\end{aligned}$$

Since the last power series is the sum of a power series convergent for $x^2 < 1/2$ and a power series convergent for $x^2 < 1$, it converges for $x^2 < 1/2$.

Grading: product decomposition of the fraction, power series for $1/(1-2x^2)$, and power series for $1/(1+x^2)$ 1pt each; 4pts for either the partial fraction or multiplication of power series computation to the answer; interval of convergence, radius of convergence, and explanation 1pt each (with 3pt loss if the interval of convergence is not correct, except endpoints error 1pt off).

Problem 6 (10pts)

Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (\ln 5)^n}{n!}$$

converges and find its sum.

First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=1}^{\infty} \frac{(-1)^n (\ln 5)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-\ln 5)^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{n!}\bigg|_{x=-\ln 5}.$$

Since the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x and its sum equals e^x, the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (\ln 5)^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \frac{x^0}{0!} \right)\bigg|_{x=-\ln 5}$$

converges and equals to the evaluation of $\mathrm{e}^x - 1$ at $x = -\ln 5$, i.e.

$$\mathrm{e}^{-\ln 5} - 1 = \left(\mathrm{e}^{\ln 5}\right)^{-1} - 1 = 5^{-1} - 1 = \frac{1}{5} - 1 = \boxed{-\frac{4}{5}}$$

You can also justify convergence using the Ratio Test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|\ln 5|^{n+1}/(n+1)!}{|\ln 5|^n/n!} = \frac{(\ln 5)^{n+1}}{(\ln 5)^n} \cdot \frac{n!}{(n+1)!} = (\ln 5) \cdot \frac{1}{(n+1)} \longrightarrow 0.$$

Since $0 < 1$, the series converges by the Ratio Test.

Grading: correct power series 3pts; evaluation point 3pts; sum of power series, $\mathrm{e}^x - 1$, 1pt; answer 1pt; justification of convergence 2pts; 2pts reduction if no lower-limit adjustment; 3pts reduction if left as $\mathrm{e}^{-\ln 5}$ or similar; if *only* the convergence part is done, up to 5pts.

Problem 7 (10pts)

All questions in this problem refer to the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\cdot(2n)!\cdot 2^n}$$

(a; 3pts) *Explain why this series converges.*

This series converges because it is alternating (odd terms are negative, even terms are positive),

$$\frac{1}{(n+1)(2n)!2^n} \longrightarrow 0 \text{ as } n \longrightarrow 0, \qquad \text{and} \qquad \frac{1}{(n+1)(2n)!2^n} > \frac{1}{((n+1)+1)(2(n+1))!2^{n+1}}.$$

It also converges by the Ratio Test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1/(((n+1)+1)(2(n+1))!2^{n+1})}{1/((n+1)(2n)!2^n)} = \frac{n+1}{n+2}\cdot\frac{(2n)!}{(2n+2)!}\cdot\frac{2^n}{2^{n+1}} = \frac{1+1/n}{1+2/n}\cdot\frac{1}{n+1}\cdot\frac{1}{2} \longrightarrow 1\cdot 0\cdot\frac{1}{2} = 0.$$

Since $0<1$, the series converges by the Ratio Test.

Grading: assumption checks are required for full credit, names of tests are not

(b; 4pts) *What is the minimal number of terms required to approximate the sum of this series with error less than* $1/1000$*? Justify your answer.*

Since the 3 assumptions for the Alternating Series Test hold,

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\cdot(2n)!\cdot 2^n} - \sum_{n=1}^{n=m} \frac{(-1)^n}{(n+1)\cdot(2n)!\cdot 2^n}\right| = \left|\sum_{n=m+1}^{\infty} \frac{(-1)^n}{(n+1)\cdot(2n)!\cdot 2^n}\right|$$
$$< |a_{m+1}| = \frac{1}{((m+1)+1)(2(m+1))!2^{m+1}}.$$

We need to choose the smallest m so that $(m+2)(2m+2)!2^{m+1} \geq 1000$. Plugging in $m = 1, 2$, we find that the smallest value that works is $\boxed{2}$

Note: According to the book's recipe, you need to take $m=2$ as done above because this is the smallest value of m for which the book's upper-bound on the remainder of the infinite series is no greater than the required precision (with $m=1$, the upper-bound is $1/(3\cdot 24\cdot 4)=1/288 > 1/1000$). However, the remainder is smaller than the upper bound, so that a smaller m could still work. In fact, $m=1$ would not work in this case because a lower bound for the remainder for A.S.T. is given by the sum of the first *two* dropped terms. In this case, this bound is

$$\frac{1}{((m+1)+1)(2(m+1))!2^{m+1}} - \frac{1}{((m+2)+1)(2(m+2))!2^{m+2}} = |a_{m+1}+a_{m+2}| < \left|\sum_{n=m+1}^{\infty} \frac{(-1)^{n-1}}{(n+1)(2n)!2^n}\right|.$$

If $m=1$, this lower bound is $1/(3\cdot 24\cdot 4) - 1/(4\cdot 720\cdot 8) = 79/(288\cdot 80) > 1/1000$.

Grading: *Alternating Series Estimation Theorem* statement for the given case 2pts; conclusion that $m=2$ 2pts; bonus 5pts (all or nothing) for full justification that $m=2$ is actually sharp (as in *Note* above)

(c; 3pts) *Based on your answer in part (b), estimate the sum of this series with error less than* $1/1000$*; leave your answer as a simple fraction* p/q *for some integers* p *and* q *with no common factor. Is your estimate an under- or over-estimate for the sum? Explain why. (If you do not know how to do (b), state a guess for the answer in (b)).*
Based on part (b), the required estimate is

$$\sum_{n=1}^{n=m}\frac{(-1)^n}{(n+1)\cdot(2n)!\cdot 2^n}=\sum_{n=1}^{n=2}\frac{(-1)^n}{(n+1)\cdot(2n)!\cdot 2^n}=\frac{(-1)^1}{2\cdot 2!\cdot 2}+\frac{(-1)^2}{3\cdot 4!\cdot 4}=\frac{-3\cdot 3\cdot 4+1}{3\cdot 24\cdot 4}=\boxed{-\frac{35}{288}}$$

This is an $\boxed{\text{over-estimate}}$ for the infinite sum, because the last term used is positive.

Grading (based on m used): computation of finite sum 2pts (1pt off if fraction not simplified); under/over-estimate (depending on m) with justification 1pt; if using $m=1$, only the first term should be taken, resulting in $-1/8$, which is an under-estimate because the last term used is negative; for $m=3$, the sum is $-2801/23040$, which is also an under-estimate

(d; bonus 10pts) *Find the sum of the infinite series exactly.*

We need to relate this infinite series to some power series evaluated at some point. Because of $(-1)^n$ in the numerator and $(2n)!$ in the denominator, we'll relate it to infinite series to the power series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}=\cos x \qquad \text{and} \qquad \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=\sin x$$

in two ways.

The extra factor $n+1=\frac{1}{2}\cdot 2(n+1)$ suggests integration of x^{2n+1} and $x\cos x$:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(2n)!}x^{2n+2}=2\int_0^x u\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}u^{2n}\mathrm{d}u=2\int_0^x u\cos u\,\mathrm{d}u=2\big(x\sin x+\cos x-1\big);$$

the last equality is obtained by integrating by parts. This gives

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)(2n)!2^n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(2n)!}\left(\frac{1}{\sqrt{2}}\right)^{2n}-\frac{(-1)^0}{(0+1)(2\cdot 0)!}\left(\frac{1}{\sqrt{2}}\right)^{2\cdot 0}=2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(2n)!}x^{2n+2}\Big|_{x=\frac{1}{\sqrt{2}}}-1$$

$$=4\big(x\sin x+\cos x-1\big)\big|_{x=\frac{1}{\sqrt{2}}}-1=\boxed{2\sqrt{2}\sin(1/\sqrt{2})+4\cos(1/\sqrt{2})-5}$$

Alternatively, we can complete the denominator to $(2n+2)!$:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)(2n)!2^n}=\sum_{n=1}^{\infty}\frac{(-1)^n 2(2n+1)}{(2n+2)(2n+1)(2n)!2^n}=\sum_{n=1}^{\infty}\frac{(-1)^n(2n+2-1)}{(2n+2)(2n+1)!2^{n-1}}$$

$$=\left(2\sqrt{2}\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}+4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!}\right)\Bigg|_{x=\frac{1}{\sqrt{2}}}=\left(2\sqrt{2}\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}+4\sum_{n=2}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\right)\Bigg|_{x=\frac{1}{\sqrt{2}}}$$

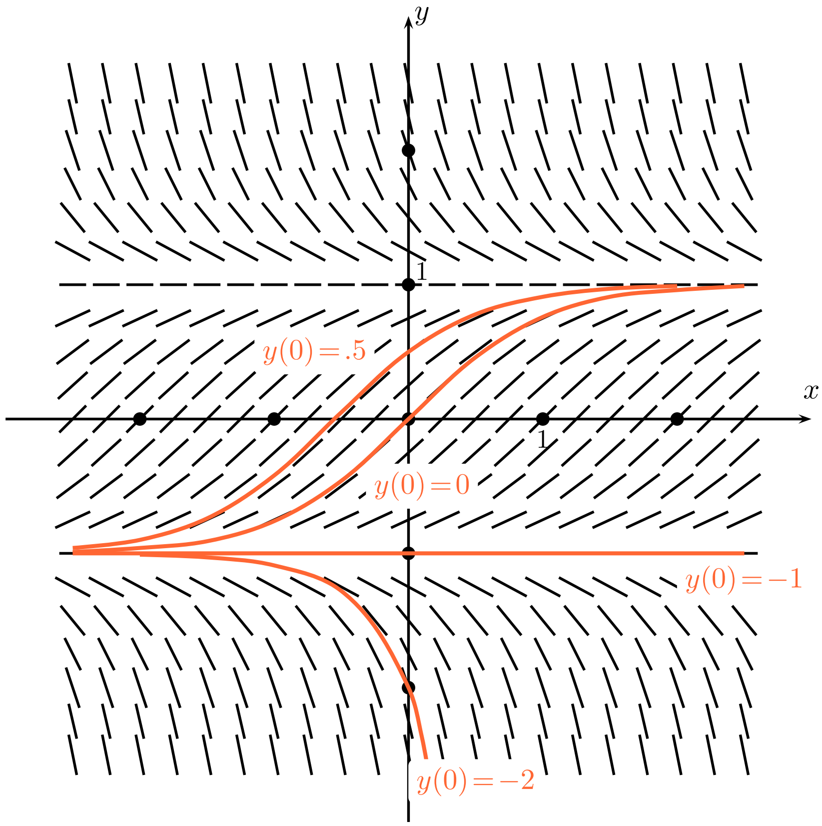
$$=\left(2\sqrt{2}\left(\sin x-\frac{(-1)^0x^{2\cdot 0+1}}{(2\cdot 0+1)!}\right)+4\left(\cos x-\frac{(-1)^0x^{2\cdot 0}}{(2\cdot 0)!}-\frac{(-1)^1x^{2\cdot 1}}{(2\cdot 1)!}\right)\right)\Bigg|_{x=\frac{1}{\sqrt{2}}}$$

$$=2\sqrt{2}\big(\sin(1/\sqrt{2})-1/\sqrt{2}\big)+4\big(\cos(1/\sqrt{2})-1+1/4\big)=\boxed{2\sqrt{2}\sin(1/\sqrt{2})+4\cos(1/\sqrt{2})-5}$$

Grading: 2pts each for adjustments in infinite series, integration of $x\cos x$ or completion to $(2n+2)!$, adjustment of lower limits, sum of power series, and final answer (any reasonable form ok)

Problem 8 (10pts)

The direction field for a differential equation is shown below.



(a; 8pts) *On the direction field, sketch and clearly label the graphs of the four solutions with the initial conditions* $y(0) = 0$, $y(0) = .5$, $y(0) = -1$, *and* $y(0) = -2$ *(each of these four conditions determines a solution to the differential equation).*

The four curves must pass through the points $(0,0)$, $(0,.5)$, $(0,-1)$, and $(0,-2)$, respectively. They must be tangent to the slope lines at those points and roughly approximate the slopes everywhere else. They should never intersect. The solution curves with $y(0) = 0$ and $y(0) = .5$ ascend toward the line $y=1$ as $x \longrightarrow \infty$ and descend toward the line $y=-1$ as $x \longrightarrow -\infty$. The solution curve with $y(0)=-1$ is the horizontal line $y=-1$. The solution curve $y(0)=-2$ ascends toward the line $y=-1$ as $x \longrightarrow -\infty$ and descends rapidly as x increases.

Grading: for each curve, passing through specified point and roughly tangent there 1pt; general shape ok 1pt; the 3rd curve must be the correct line; curves clearly intersect (as opposed to being asymptotic) 3pts off; one curve not labeled 1pt off, 2-4 curves not labeled 2pts off.

(b; 2pts) *The direction field above is for one of the following differential equations for* $y = y(x)$:

$$\text{(i) } y' = 1 - x^4\,, \qquad \text{(ii) } y' = 1 - y^4\,, \qquad \text{(iii) } y' = 1 - x^4 - y^4\,.$$

Which of these three equations does the direction field correspond? Circle your answer above. No explanation is required.

The slopes do not depend on x, so the answer cannot be (i) or (iii).

Problem 9 (10pts)

Let $f=f(x)$ and $g=g(x)$ be two solutions of the differential equation

$$y''+2y'+y=\mathrm{e}^{-x}, \qquad y=y(x).$$

Show that $h=f-g$ is a solution of the differential equation

$$y''+2y'+y=0, \qquad y=y(x).$$

Show your work and/or explain your reasoning.

Since $f=f(x)$ and $g=g(x)$ are solutions of the first differential equation,

$$f''+2f'+f=\mathrm{e}^{-x} \qquad \text{and} \qquad g''+2g'+g=\mathrm{e}^{-x}\,.$$

Since $h'=f'-g'$ and $h''=f''-g''$,

$$h''+2h'+h=\big(f''-g''\big)+2\big(f'-g'\big)+\big(f-g\big)=\big(f''+2f'+f\big)-\big(g''+2g'+g\big)=\mathrm{e}^{-x}-\mathrm{e}^{-x}=0.$$

Thus, the function h satisfies the second differential equation in the statement.

Grading: The first pair of statements above and their explanations are worth 3+2 pts. The statements could appear just as parts of the third equation above, instead of separately, but there still needs to be an explanatory comment to receive the full 5pts. The remainder is 5pts.

(bonus 15pts) *Find the general solution of the first differential equation above and use it to show that $h=f-g$ is a solution of the second differential equation.*

Multiply both sides of the first differential equation by e^x to obtain

$$\big(\mathrm{e}^x y\big)'' = \mathrm{e}^x y'' + 2\mathrm{e}^x y' + \mathrm{e}^x y = 1.$$

Integrating this equation twice, we obtain

$$\big(\mathrm{e}^x y\big)' = x + C_1, \qquad \mathrm{e}^x y = \frac{1}{2}x^2 + C_1 x + C_2, \qquad \boxed{y(x) = \frac{1}{2}x^2\mathrm{e}^{-x} + C_1 x\mathrm{e}^{-x} + C_2\mathrm{e}^{-x}}$$

This uses an approach similar to Problem D.

Alternatively, the associated quadratic polynomial r^2+2r+1 has a double root $r=-1$. Thus, the general solution of the associated homogeneous differential equation (the second equation in the statement) is $C_1 x\mathrm{e}^{-x} + C_2\mathrm{e}^{-x}$. For a particular solution of the first differential equation, we try $y_p = ax^2\mathrm{e}^{-x}$. Then

$$y_p'' + 2y_p' + y_p = a\big(2\mathrm{e}^{-x} - 4x\mathrm{e}^{-x} + x^2(-1)^2\mathrm{e}^{-x}\big) + 2a\big(2x\mathrm{e}^{-x} - x^2\mathrm{e}^{-x}\big) + ax^2\mathrm{e}^{-x} = 2ae^{-x}\,.$$

Thus, y_p satisfies the first differential equation if $a=\frac{1}{2}$. Combining this y_p with the general solution of the associated homogeneous differential equation, we obtain the general solution of the first differential equation in the statement as in the box above. This is the method of *Undetermined Coefficients*, which is covered in MAT 303, but which some of you have used to solve Problem D.

Since the general solution of the first differential equation is given by the box above,

$$f(x) = \frac{1}{2}x^2\mathrm{e}^{-x} + A_f x\mathrm{e}^{-x} + B_f\mathrm{e}^{-x} \qquad \text{and} \qquad g(x) = \frac{1}{2}x^2\mathrm{e}^{-x} + A_g x\mathrm{e}^{-x} + B_g\mathrm{e}^{-x}$$

for some constants A_f, B_f, A_g, B_g and

$$h = f - g = (A_f - A_g)x\mathrm{e}^{-x} + (B_f - B_g)\mathrm{e}^{-x}\,.$$

Since the general solution of the second differential equation in the statement is $C_1 x\mathrm{e}^{-x} + C_2\mathrm{e}^{-x}$, h is a particular solution of this equation.

Grading: 10pts for the general solution with reduction for nominal mistakes; 5pts for the last part with full justification

Problem 10 (15pts)

Find the general real solution to each of the following differential equations. Put your answer to each question in the corresponding box in the simplest possible form. Each correct answer in the simplest possible form is worth 5pts. Partial credit may be awarded for not completely correct answers.

(a) $y'' + 2y' + y = 0, \quad y = y(x)$

$$\boxed{C_1 e^{-x} + C_2 x e^{-x}}$$

(b) $y'' + 2y' = 0, \quad y = y(x)$

$$\boxed{C_1 + C_2 e^{-2x}}$$

(c) $y'' + y = 0, \quad y = y(x)$

$$\boxed{C_1 \cos x + C_2 \sin x}$$

Grading (each part): two-dim solution space at least 2pts overall; if not, at most 3pts; 1pt for 1 correct basis element; 2pts for 2 correct basis elements with wrong sign; 1pt reduction for not simplifying; 2pts reduction for leaving in a complex form.

The associated polynomial equation for the DE in (a) is $r^2+2r+1 = (r+1)^2$. The two roots $r_1 = r_2 = -1$ are the same. Thus, the general real solution is as stated in (a).

The associated polynomial equation for the DE in (b) is $r^2+2r = r(r+2)$. The two roots $r_1 = 0$ and $r_2 = -2$ are real and distinct. Thus, the solution is as stated in (b).

The associated polynomial equation for the DE in (c) is r^2+1. The two roots $r_1, r_2 = \pm \mathrm{i}$ are complex with $p = 0$ and $q = 1$. Thus, the general real solution is

$$y(x) = C_1 e^{0x} \cos(1x) + C_2 e^{0x} \sin(1x) = C_1 \cos x + C_2 \sin x\,.$$

Problem 11 (10pts)

A two-species interaction is modeled by the following system of differential equations

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = x - \frac{1}{10}x^2 + \frac{1}{100}xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{20}y - \frac{1}{100}y^2 + \frac{1}{20}xy \end{cases} \qquad (x,y) = (x(t), y(t)),$$

where t denotes time.

(a; 2pts) *Which of the following best describes the interaction modeled by this system?*

(i) predator-prey (ii) competition for same resources (iii) cooperation for mutual benefit

Circle your answer above.

Because of the coefficient of $\frac{1}{100}$ in front of xy in the first equation, the x-species is helped by the presence of the y-species (the growth rate of the former is increased if the population of the latter is nonzero). Because of the coefficient of $\frac{1}{20}$ in front of xy in the second equation, the y-species is helped by the presence of the x-species.

(b; 8pts) *This system has 4 equilibrium (constant) solutions; find all of them. Put one equilibrium solution in each box below.*

$(0,0)$	$(0,5)$	$(10,0)$	$(21,110)$

The constant solutions are described by $(x'(t), y'(t)) = 0$. For the above system, this gives

$$\begin{cases} 0 = \frac{x}{100}(100 - 10x + y) \\ 0 = \frac{y}{100}(5 - y + 5x) \end{cases} \iff \begin{cases} x = 0 \;\text{ or }\; 10x - y = 100 \\ y = 0 \;\text{ or }\; y - 5x = 5 \end{cases}$$

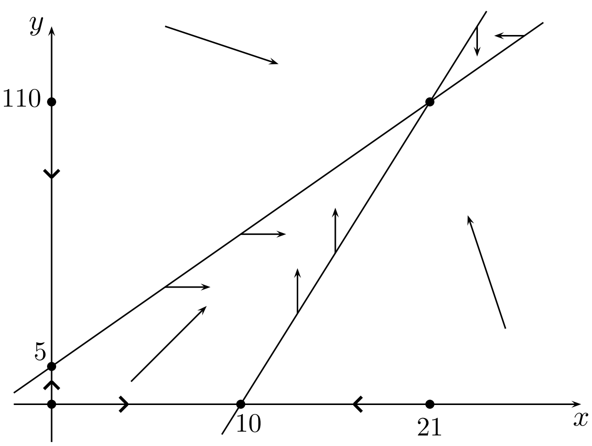
Thus, the constant solutions (x,y) are the solutions of the following systems:

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \quad \begin{cases} x = 0 \\ y - 5x = 5 \end{cases} \quad \begin{cases} 10x - y = 100 \\ y = 0 \end{cases} \quad \begin{cases} 10x - y = 100 \\ y - 5x = 5 \end{cases}$$

The first three systems immediately give the solutions in the first three boxes above. Adding up the two equations in the last system, we obtain $5x = 105$. This gives $x = 21$ and $y = 110$.

Grading: 2pts per box containing correct solution (c; bonus 10pts) *Let* $(x,y)=(x(t),y(t))$ *be a solution of the system above such that* $x(0),y(0)>0$. *Find* $\lim\limits_{t\longrightarrow\infty} x(t)$.

We consider in what direction $(x(t),y(t))$ moves in the xy-plane depending on its position. This is similar to the direction field for the $\mathrm{d}y/\mathrm{d}x$-equation obtained by dividing the second equation in the system by the first (see Figures 1,2 on p542), but with arrows on the little slope segments indicating the direction of movement as t increases. In fact, we do not need the precise slopes, just the general direction (e.g. up and to the right). For example, on the x-axis (where $y=0$), $\mathrm{d}y/\mathrm{d}t=0$ according to the second equation in the system. Thus, along the x-axis, the direction of movement is horizontal. It is to the right for $x\in(0,10)$ and to the left for $x\in(10,\infty)$, according to the first equation in the system. Similarly, on the y-axis (where $x=0$), the direction of movement is vertical, up for $y\in(0,5)$ and down for $y\in(5,\infty)$. According to the first equation in the system, $\mathrm{d}x/\mathrm{d}t=0$ (the movement is vertical) at every point of the line $10x-y=100$ (with x-intercept of 10). According to the second equation, $\mathrm{d}y/\mathrm{d}t=0$ (the movement is horizontal) at every point of the line $y-5x=5$ (with y-intercept of 5). The general direction of the flow does not change anywhere else. Thus, it must be up on the segment of the first line below its intersection with the second line and down above this intersection (comparing with the direction on the y-axis). It must also be to the right on the segment of the second line below its intersection with the first line and to the left above this intersection. There is no movement at the intersection point of these lines; this is the equilibrium point $(21,110)$. The movement in the regions formed by the above two lines and the coordinate axes is according to the nearby arrows on these lines; see the figure below.



A point $(x(t),y(t))$ in the bottom left or top right portion of the first quadrant formed by the two lines thus cannot cross either of the two lines to leave this region. It must thus approach the intersection point $(21,110)$. A point $(x(t),y(t))$ in the bottom right or top left portion of the first quadrant formed by the two lines must either approach $(21,110)$ or cross into either the bottom left or top right portion of the first quadrant. In all cases, $(x(t),y(t))$ thus approaches $(21,110)$ and so $\boxed{\lim\limits_{t\longrightarrow\infty} x(t)=21}$

Grading: no partial credit; explanation must be complete on the substance in order to receive the 10 bonus points.

You can learn more about analyzing systems of differential equations in MAT 303, if you pass MAT 127 (or in MAT 308 if you get an A in MAT 127 and find it not sufficiently challenging).

Problem 12A (20pts)

Only the higher of your scores on Problems 12A and 12B will count toward the total for the exam

A tank contains 100 liters of salt solution with 400 grams of salt dissolved in it. A salt solution containing 6g of salt per liter enters the tank at a rate of 5 liters per minute. The solution is kept thoroughly mixed and drains at a rate of 5L/min (so the volume in the tank stays constant). Let $y(t)$ be the amount of salt in the tank, measured in grams, after t minutes.

(a; 8pts) *Explain why the function $y=y(t)$ solves the initial-value problem*

$$y' = 30 - \frac{y}{20}, \quad y = y(t), \qquad y(0) = 400.$$

Since the initial amount of salt in the tank is 400 grams, $y(0) = 400$. Furthermore, $y'(t) = y'_{\text{in}}(t) - y'_{\text{out}}(t)$, where

$$\begin{aligned} y'_{\text{in}}(t) &= (\text{flow rate of salt})_{\text{in}} = (\text{flow rate of solution})_{\text{in}} \cdot (\text{salt concentration})_{\text{in}} = 5 \cdot 6 = 30; \\ y'_{\text{out}}(t) &= (\text{flow rate of salt})_{\text{out}} = (\text{flow rate of solution})_{\text{out}} \cdot (\text{salt concentration})_{\text{out}}. \end{aligned}$$

Since the salt in the tank is thoroughly mixed and the volume is kept constant at 100 gallons, the outgoing salt concentration is the same as the salt concentration in the tank and

$$(\text{salt concentration})_{\text{out}} = \frac{\text{amount salt in tank}}{\text{volume in tank}} = \frac{y(t)}{100}, \qquad y'_{\text{out}}(t) = 5 \cdot \frac{y(t)}{100} = \frac{y(t)}{20}.$$

It follows that $y(t)$ is a solution to the differential equation $y' = 30 - y/20$.
Grading: mention of initial condition 1pt; computation of y'_{in} 2pts and of y'_{out} 3pts; remainder 2pts; mostly formulas is ok

(b; 8pts) *Find the solution $y=y(t)$ to the initial-value problem stated in (a).*

First find the general solution to the differential equation. Since it is separable, writing $y' = \mathrm{d}y/\mathrm{d}t$, moving everything involving y to LHS and everything involving t to RHS, and integrating, we obtain

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{600-y}{20} \quad &\Longleftrightarrow \quad \frac{\mathrm{d}y}{600-y} = \frac{\mathrm{d}t}{20} \quad \Longleftrightarrow \quad \int \frac{\mathrm{d}y}{600-y} = \int \frac{\mathrm{d}t}{20} \quad \Longleftrightarrow \quad -\ln|600-y| = \frac{t}{20} + C \\ &\Longleftrightarrow \quad \ln|600-y| = -\frac{t}{20} + C \quad \Longleftrightarrow \quad \mathrm{e}^{\ln|600-y|} = \mathrm{e}^{-t/20+C} = \mathrm{e}^{C}\mathrm{e}^{-t/20} \\ &\Longleftrightarrow \quad |600-y| = A\mathrm{e}^{-t/20} \quad \Longleftrightarrow \quad 600-y = \pm A\mathrm{e}^{-t/20} \quad \Longleftrightarrow \quad y(t) = 600 + C\mathrm{e}^{-t/20}. \end{aligned}$$

Plugging in the initial condition $(t, y) = (0, 400)$, we obtain

$$400 = 600 + C\mathrm{e}^{-0/20} = 600 + C \qquad \Longleftrightarrow \qquad C = -200.$$

So $\boxed{y(t) = 600 - 200\mathrm{e}^{-t/20}}$
Grading: separating variables 2pts; integration 1pt; remainder of computation to general solution 2pts; determining C 2pts; final answer 1pt

(c; 4pts) *How long will it take for the amount of salt in the tank to reach 500 grams?*

We need to find t so that $y(t) = 500$:

$$\begin{aligned} y(t) = 600 - 200\mathrm{e}^{-t/20} = 500 \quad \Longleftrightarrow \quad \mathrm{e}^{-t/20} = \frac{1}{2} \quad &\Longleftrightarrow \quad -\frac{t}{20} = \ln(1/2) = -\ln 2 \\ &\Longleftrightarrow \quad \boxed{t = 20(\ln 2) \ \ \text{mins}} \end{aligned}$$

Grading: setup 1pt; numerical solution 2pts; units 1pt; 2pts off for $-\ln(1/2)$

Problem 12B (20pts)

Only the higher of your scores on Problems 12A and 12B will count toward the total for the exam

(a; 8pts) *Show that the orthogonal trajectories to the family of curves $y^4 = kx^3$ are described by the differential equation*

$$y' = -\frac{4}{3}\frac{x}{y}, \qquad y = y(x).$$

Differentiate $y^4 = kx^3$ with respect to x, using chain rule and remembering that k is a constant:

$$4y^3y' = 3kx^2 \qquad \Longleftrightarrow \qquad y' = \frac{3kx^2}{4y^3}.$$

From the original equation, we find that $k = y^4/x^3$ and so our curves have slope

$$y' = k\frac{3x^2}{4y^3} = \frac{y^4}{x^3} \cdot \frac{3x^2}{4y^3} = \frac{3y}{4x}$$

at (x, y). The slopes of the orthogonal curves are the negative reciprocal of this; so they satisfy

$$y' = -\frac{1}{3y/4x} = -\frac{4x}{3y}.$$

Grading: computation of slopes of the initial curves 4pts; the negative reciprocal statement 3pts; conclusion 1pt

(b; 6pts) *Find the general solution to the differential equation stated in (a).*

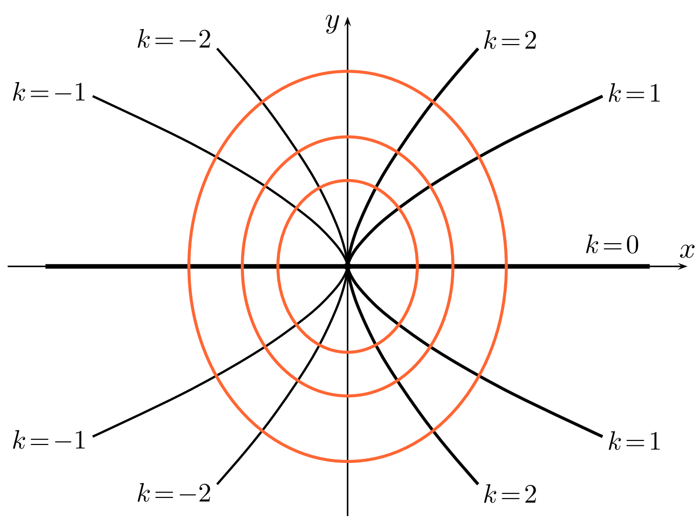
This equation is separable, so after writing $y' = \mathrm{d}y/\mathrm{d}x$, we can move everything involving y to LHS and everything involving x to RHS and then integrate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x}{3y} \iff 3y\,\mathrm{d}y = -4x\,\mathrm{d}x \iff \int 3y\,\mathrm{d}y = -\int 4x\,\mathrm{d}x \iff y^2 = -\frac{4}{3}x^2 + C$$
$$\iff \boxed{4x^2 + 3y^2 = C}$$

Grading: splitting the variables 2pts; integration 2pts; simplification 2pts (answer with both square roots is ok) (c; 6pts) *Sketch at least 3 representatives of the original family of curves and at least 3 orthogonal trajectories on the same diagram; indicate clearly which is which.*

Draw the above curves for different values of k and C:

- $y^4 = 0x^3$ is the x-axis; $y^4 = 1x^3$ is the graph of the function $x = y^{4/3}$, which is tangent to the y-axis at (0,0); $y^4 = (-1)x^3$ is the graph of the function $x = -y^{4/3}$ and is the reflection of the above graph about the y-axis;
- the equation $4x^2 + 3y^2 = C$ has no solutions (x, y) if $C < 0$; if $C = 0$, this "curve" is just the origin $(0, 0)$; the curve $4x^2 + 3y^2 = 12$ is the ellipse passing through the points $(\pm\sqrt{3}, 0)$ and $(0, \pm 2)$, and so it is stretched vertically; the ellipse $4x^2 + 3y^2 = 12r^2$ is the same ellipse scaled by the factor $|r|$.



Grading: 1pt for each relevant curve; 2pts off if no indication is given which curves are which (k, C values not required); 1pt off if the axes are not labeled