Name: ____________________

ID#: ____________________

Final Exam

MAT 127 Spring 2005

Directions: There are 8 questions. You have until 1:30 PM (150 minutes). For credit, you must show all your work, using the backs of the pages if necessary. You may not use a calculator.

1. ____/10 2. ____/10 3. ____/10 4. ____/10 5. ____/15 6. ____/15 7. ____/15 8. ____/15

Total Score. ____/100

1. A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y^2 - y - 6.$$

(a) What are the constant solutions of the equation?

(b) For what values of y is y increasing?

(c) For what values of y is y decreasing?

(a) A SOLUTION IS CONSTANT IF $\frac{dy}{dt} = 0$, SO WE SOLVE

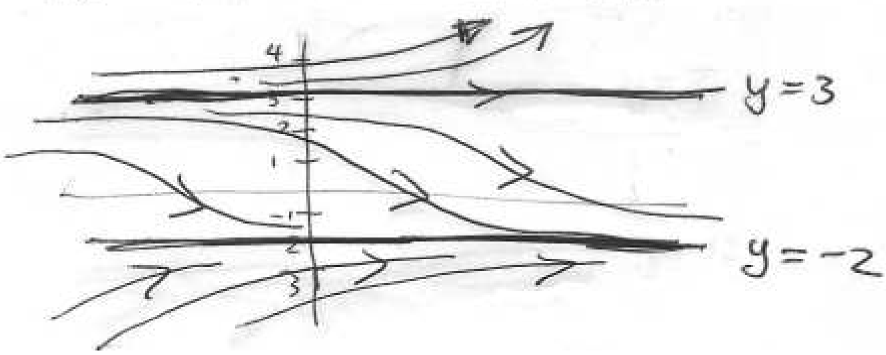
$$\frac{dy}{dt} = 0 = y^2 - y - 6 = (y-3)(y+2).$$

THUS THE CONSTANT SOLUTIONS ARE $\boxed{y(t) = 3 \quad \text{AND} \quad y(t) = -2.}$

(b) $y(t)$ IS INCREASING WHEN $\frac{dy}{dt} > 0$, WHICH HAPPENS WHEN $\boxed{y < -2 \quad \text{OR} \quad y > 3.}$

(c) $\frac{dy}{dt} < 0$ FOR $\boxed{-2 < y < 3}$, SO $y(t)$ DECREASES THERE

THE SOLUTIONS LOOK LIKE THIS!



2. Solve the following initial value problems. (Hint: they are both seperable.)

(a) $\frac{dx}{dt} = -2x^2 t$, $x(0) = 1/3$

(b) $\frac{dy}{dt} = y\cos t$, $y(\pi) = 1$

(a) $\frac{dx}{dt} = -2x^2 t$

$\int \frac{dx}{x^2} = -\int 2t\,dt$

$-\frac{1}{x} = -t^2 + C$

$x(t) = \frac{1}{t^2 + C}$

SINCE $x(0) = 1/3$, $C = 3$.

SO

$$\boxed{x(t) = \frac{1}{t^2+3}}$$

(b) $\frac{dy}{dt} = y\cos t$

$\int \frac{dy}{y} = \int \cos t\,dt$

$\ln|y| = \sin t + C$

EXPONENTIATING, $|y| = e^{\sin t + C}$

SO $y = Ae^{\sin t}$ (WHERE $A = \pm e^C$)

SINCE $y(\pi) = 1$,

$1 = Ae^{\sin \pi} = A$.

$\therefore$ $\boxed{y(t) = e^{\sin t}}$

3. Assume a contagious disease spreads at a rate proportional to the number of infected people. Initially there are 10 people infected and after 1 month there are 100 people infected.

(a) Find an expression for the number of infected people after t months.

(b) When will there be 1000 infected people?

LET $P(t)$ BE THE NUMBER OF PEOPLE INFECTED AFTER t MONTHS. SINCE THE RATE OF INCREASE OF $P(t)$ IS PROPORTIONAL TO $P(t)$, WE HAVE

$$P'(t) = kP(t), \quad P(0) = 10, \quad P(1) = 100$$

SO $P(t) = Ae^{kt}$

SINCE $P(0) = 10$, $A = 10$.

SINCE $P(1) = 100 = 10e^k$

$10 = e^k$,

SO $k = \ln 10$

(a) $\therefore$ $\boxed{P(t) = 10e^{t\ln 10}}$

(OR $P(t) = 10^{t+1}$)

(b) SINCE $P(t) = 10^{t+1}$

1000 PEOPLE WILL BE INFECTED WHEN

$1000 = 10^{t+1}$

THAT IS, AFTER

$t = \boxed{2 \text{ MONTHS}}$

4. Compute the limits of the following convergent sequences:

(a) $\left\{\frac{\sin 5n}{2n}\right\}$

(b) $\left\{\frac{n}{(\ln n)^2}\right\}$

(a) $\lim_{n\to\infty} \frac{\sin 5n}{2n}$. SINCE $-1 \leq \sin 5n \leq 1$,

WE HAVE

$$\lim_{n\to\infty} \frac{-1}{2n} \leq \lim_{n\to\infty} \frac{\sin 5n}{2n} \leq \lim_{n\to\infty} \frac{1}{2n}$$

SO

$$0 \leq \lim_{n\to\infty} \frac{\sin 5n}{2n} \leq 0$$

So $\boxed{\text{THE LIMIT IS } 0}$ USING THE SQUEEZE THM

(b) $\lim_{n\to\infty} \frac{n}{(\ln n)^2} = \lim_{n\to\infty} \frac{1}{2\ln n/n}$ USING L'HOPITAL'S RULE.

$= \lim_{n\to\infty} \frac{n}{2\ln n} = \lim_{n\to\infty} \frac{1}{2/n}$ BY L'HOPITAL'S AGAIN

$= \lim_{n\to\infty} \frac{n}{2} = \infty$.

SO THE SEQUENCE DIVERGES TO $+\infty$.

5. Determine whether the series is convergent or divergent. State which test you're using.

(a) $\sum_{n=1}^{\infty} \frac{3n}{n^3+4}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$

(a) WE'LL USE LIMIT COMPARISON TO COMPARE

$\sum_1 \frac{3n}{n^3+4}$ TO $\sum_1 \frac{1}{n^2}$.

$$\lim_{n \to \infty} \frac{3n/n^3+4}{1/n^2} = \lim_{n \to \infty} \frac{3n^3}{n^3+4} = 3.$$

SINCE THE LIMIT ISN'T 0 AND IT ISN'T ∞, THE TWO SERIES CONVERGE OR DIVERGE TOGETHER.

SINCE $\sum_1 \frac{1}{n^2}$ CONVERGES (P-SERIES, P=2)

SO DOES $\sum_1 \frac{3n}{n^3+4}$.

(b) $\sum_1 \frac{(-1)^n}{\sqrt{n+2}}$ IS AN ALTERNATING SERIES, SO WE USE THE ALT. SERIES TEST.

SINCE $\lim_{n \to \infty} \frac{1}{\sqrt{n+2}} = 0$, THE SERIES

$\sum_1 \frac{(-1)^n}{\sqrt{n+2}}$ CONVERGES BY ALT, SERIES TEST.

6. Compute the following sums.

(a) $\sum_{n=0}^{\infty} \frac{3^{2n+1}}{10^n}$

(b) $\sum_{n=1}^{\infty} [\sin(1/n) - \sin(1/(n+1))]$

(a) $\sum_{n=0}^{\infty} \frac{3^{2n+1}}{10^n} = \sum_{n=0}^{\infty} \frac{3 \cdot 9^n}{10^n}$, A GEOMETRIC SERIES w/ $r = \frac{9}{10}$.

SO THE SUM IS $\frac{3}{1 - 9/10} = 30$

(b) LET'S WRITE OUT SOME TERMS:

$\left[\sin(1) - \sin\left(\frac{1}{2}\right)\right] + \left[\sin\left(\frac{1}{2}\right) - \sin\left(\frac{1}{3}\right)\right] + \left[\sin\left(\frac{1}{3}\right) - \sin\left(\frac{1}{4}\right)\right] + \cdots$

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$= \boxed{\sin(1)}$.

7. Compute and write out the following series. If applicable, you can use the table of Maclaurin series provided with your exam:

(a) The Maclaurin series for $f(x) = x^3 e^{x^3}$.

(b) The Taylor series, centered around $a = 1$, for $f(x) = x^3$.

(a) $x^3 e^{x^3} = x^3 \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{3n+3}}{n!} = x^3 + x^6 + \frac{x^9}{2} + \frac{x^{12}}{3!} + \frac{x^{15}}{4!} + \cdots$

(b)

n	$f^{(n)}(x)$	$f^{(n)}(1)/n!$
0	x^3	1
1	$3x^2$	3
2	$6x$	$\frac{6}{2} = 3$
3	6	$\frac{6}{3!} = 1$
4, ...	0	0

SO THE SERIES IS

$1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$

8. Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}.$$

(a) Write the "center" of this power series.

(b) Find the open interval of absolute convergence.

(c) Determine whether the series converges or diverges at each of the interval's endpoints.

(a) THE INTERVAL OF CONVERGENCE IS ABOUT $\boxed{X=2,}$ SO THAT'S THE "CENTER".

(b) APPLYING THE RATIO TEST:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(x-2)^{n+1}/(n+1)3^{n+1}}{(x-2)^n/n3^n} = \lim_{n\to\infty} \frac{n3^n}{(n+1)3^{n+1}} \cdot |x-2|$$

$$|x-2| \lim_{n\to\infty} \frac{n}{(n+1)3} = \frac{|x-2|}{3}.$$

THIS WILL BE <1 WHEN $\frac{|x-2|}{3} < 1$,

IE $|x-2| < 3$

IE $-3 < x-2 < 3$ IE $\boxed{-1 < x < 5}$

IE ~~$-1 < x < 5$~~

(c) NOW WE CHECK WHAT HAPPENS FOR $x = 5$ AND $x = -1$.

IF $x=5$, THE SERIES IS $\sum_1^{\infty} \frac{3^n}{n3^n} = \sum_1^{\infty} \frac{1}{n}$, WHICH DIVERGES. (HARMONIC SERIES)

IF $x=-1$, WE HAVE $\sum_1^{\infty} \frac{(-3)^n}{n3^n} = \sum_1^{\infty} \frac{(-1)^n}{n}$ WHICH CONVERGES. (ALT SERIES)

SO THE INTERVAL OF CONV. IS $[-1,5)$