MAT 203 Spring 2014 Midterm 1

Instructions: The exam is closed book, closed notes, calculators are not allowed, and all cellphones and other electronic devices must be turned off for the duration of the exam. You will have approximately 50 minutes for this exam. The point value of each problem is written next to the problem – use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown. You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., then please raise your hand.
1. Given \( \mathbf{u} = 6i - 5j + 2k \) and \( \mathbf{v} = -4i + 2j + 3k \),

(a) (5 points): Calculate \( \mathbf{u} \times \mathbf{v} \).

By definition,
\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ 6 & -5 & 2 \\ -4 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -5 & 2 \\ 2 & 3 \end{vmatrix} i - \begin{vmatrix} 6 & 2 \\ -4 & 3 \end{vmatrix} j + \begin{vmatrix} 6 & 2 \\ -4 & 2 \end{vmatrix} k = -19i - 26j - 8k.
\]

(b) (5 points): Calculate \( (\mathbf{u} + \mathbf{v}) \times \mathbf{u} \)

Here we can use some properties of the vector product:
\[
(\mathbf{u} + \mathbf{v}) \times \mathbf{u} = \mathbf{u} \times \mathbf{u} + \mathbf{v} \times \mathbf{u} \quad \text{(distributivity)}
\]
\[
= \mathbf{v} \times \mathbf{u} \quad \text{(since \( \mathbf{u} \times \mathbf{u} = 0 \))}
\]
\[
= -\mathbf{u} \times \mathbf{v} \quad \text{(anticommutativity)}
\]

therefore, from (a) we conclude that \( (\mathbf{u} + \mathbf{v}) \times \mathbf{u} = 19i + 26j + 8k \)

(c) (10 points): Find a unit vector that is orthogonal to both \( \mathbf{w}_1 = \langle 2, -10, 8 \rangle \) and \( \mathbf{w}_2 = \langle 4, 6, -8 \rangle \)

A vector orthogonal to a given pair can be obtained by means of the cross product. Thus consider
\[
\mathbf{w}_1 \times \mathbf{w}_2 = \begin{vmatrix} i & j & k \\ 2 & -10 & 8 \\ 4 & 6 & -8 \end{vmatrix} = \begin{vmatrix} -10 & 8 \\ 6 & -8 \end{vmatrix} i - \begin{vmatrix} 2 & 8 \\ 4 & -8 \end{vmatrix} j + \begin{vmatrix} 2 & -10 \\ 4 & 6 \end{vmatrix} k = 32i + 48j + 52k.
\]

Since \( ||\mathbf{w}_1 \times \mathbf{w}_2|| = \sqrt{(32)^2 + (48)^2 + (52)^2} = 4\sqrt{377} \), a unit vector perpendicular to \( \mathbf{w}_1 \) and \( \mathbf{w}_2 \) would be \( \frac{1}{4\sqrt{377}} \langle 8, 12, 13 \rangle \).
2.- (a) (10 points): Use the triple scalar product to find the volume of the parallelepiped having adjacent edges \(u = 2i + j\), \(v = 2j + k\) and \(w = -j + 2k\).

Recall that the triple scalar product of \(u\), \(v\) and \(w\) is defined as
\[
u \cdot (v \times w) = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 2 \cdot \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix} = 10.
\]

In general, the triple scalar product measures oriented volume, but in this case, since the triple \(u\), \(v\), \(w\) forms a right-handed triple, the volume of the parallelepiped spanned by these vectors is 10.

(b) (20 points): Find the equation of the plane containing the lines given by
\[
\frac{x - 1}{-2} = y = z + 1 \quad \text{and} \quad \frac{x + 1}{-2} = y - 1 = z - 2.
\]

In principle, it is false that two random lines in space will lie on a plane, however, in this case, the lines have parametric equations
\[
\langle x, y, z \rangle = \langle 1, 0, -1 \rangle + t\langle -2, 1, 1 \rangle \quad \text{and} \quad \langle x, y, z \rangle = \langle -1, 1, 2 \rangle + t\langle -2, 1, 1 \rangle
\]
in particular, these lines are parallel. One tentative, simple option would have been to find the normal vector to the plane by taking the cross product of the direction vectors of the two lines, but this would be \(0\). Instead, we can find 3 points in the prospective plane. Setting \(t = 0\), we find \((1, 0, -1)\) and \((-1, 1, 2)\). A third point is for instance, with \(t = 1\) in the first line, \((-1, 1, 0)\). Thus
\[
\langle 1, 0, -1 \rangle - \langle -1, 1, 0 \rangle = \langle 2, -1, -1 \rangle \quad \text{and} \quad \langle -1, 1, 2 \rangle - \langle -1, 1, 0 \rangle = \langle 0, 0, 2 \rangle
\]
are vectors parallel to the plane. Therefore
\[
\begin{vmatrix} i & j & k \\ 2 & -1 & -1 \\ 0 & 0 & 2 \end{vmatrix} = -2i - 4j
\]
is a vector normal to the plane, and
\[-2(x + 1) - 4(y - 1) = 0, \quad \text{or} \quad x + 2y = 1
\]
is the equation for the plane.
3.-(20 points): Convert the rectangular equation to an equation in (a) cylindrical coordinates and (b) spherical coordinates.

\[ x^2 - y^2 = 2z. \]

(a) In cylindrical coordinates,

\[ x^2 - y^2 = (r \cos \theta)^2 - (r \sin \theta)^2 = r^2 (\cos^2 \theta - \sin^2 \theta) = r^2 \cos (2\theta), \]

thus the equation would be

\[ r^2 \cos (2\theta) = 2z. \]

(b) In spherical coordinates,

\[ x^2 - y^2 = (\rho \sin \phi \cos \theta)^2 - (\rho \sin \phi \sin \theta)^2 = \rho^2 \sin^2 \phi (\cos^2 \theta - \sin^2 \theta) = \rho^2 \sin^2 \phi \cos (2\theta), \]

thus, since \( z = \rho \cos \phi \), we get

\[ \rho^2 \sin^2 \phi \cos (2\theta) = 2\rho \cos \phi \]

or

\[ \rho \sin^2 \phi \cos (2\theta) = 2 \cos \phi. \]
4.- Consider the vector-valued function

\[ \mathbf{r}(t) = e^t \cos(t) \mathbf{i} + e^t \sin(t) \mathbf{j} \]

(a) \(15 \text{ points}\) Find the unit tangent vector \( \mathbf{T}(t) \) and the principal unit normal vector \( \mathbf{N}(t) \) at \( t = \frac{\pi}{2} \).

Notice that the trace of this vector-valued function is planar. Since \( \mathbf{r}'(t) = e^t (\cos(t) - \sin(t)) \mathbf{i} + e^t (\sin(t) + \cos(t)) \mathbf{j} \), then

\[
||\mathbf{r}'(t)|| = e^t \sqrt{(\cos(t) - \sin(t))^2 + (\sin(t) + \cos(t))^2}
= e^t \sqrt{2 (\cos^2(t) + \sin^2(t))} = \sqrt{2} e^t,
\]

therefore \( \mathbf{T}(t) = \frac{1}{\sqrt{2}} ((\cos(t) - \sin(t)) \mathbf{i} + (\sin(t) + \cos(t)) \mathbf{j}). \)

Similarly, \( \mathbf{T}'(t) = -\frac{1}{\sqrt{2}} ((\sin(t) + \cos(t)) \mathbf{i} + (\sin(t) - \cos(t)) \mathbf{j}). \) Thus

\[
||\mathbf{T}'(t)|| = \frac{1}{\sqrt{2}} \sqrt{(\sin(t) + \cos(t))^2 + (\sin(t) - \cos(t))^2} = 1.
\]

Therefore \( \mathbf{N}(t) = \mathbf{T}'(t) \). At \( t = \frac{\pi}{2} \) we find

\[ \mathbf{T}(\pi/2) = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle, \quad \mathbf{N}(\pi/2) = \langle -1/\sqrt{2}, -1/\sqrt{2} \rangle. \]

(b) \(15 \text{ points}\) Find the tangential and normal componens of acceleration \( a_T \) and \( a_N \) at \( t = \frac{\pi}{2} \).

A direct computation shows that \( \mathbf{a}(t) = \mathbf{r}''(t) = -2e^t \sin(t) \mathbf{i} + 2e^t \cos(t) \mathbf{j} \). Without any further computations, we can observe that

\[ \mathbf{a}(t) = \sqrt{2} e^t \mathbf{T}(t) + \sqrt{2} e^t \mathbf{N}(t) \]

Therefore, \( a_T(t) = a_N(t) = \sqrt{2} e^t \). In particular, \( a_T(\pi/2) = a_N(\pi/2) = \sqrt{2} e^{\pi/2}. \)
5.- Extra credit (15 points): Using Newton’s Second Law of Motion, $F = ma$, and Newton’s Law of Gravitation

$$F = -\frac{GmM}{r^3}r$$

show that $\mathbf{a}$ and $\mathbf{r}$ are parallel, and that $\mathbf{r}(t) \times \mathbf{r}'(t) = \mathbf{L}/m$ is a constant vector. So, $\mathbf{r}(t)$ moves in a fixed plane, orthogonal to $\mathbf{L}$ (In physics, $\mathbf{L}$ is know as angular momentum. As this problem shows, an intrinsic property of any central force problem is that motion always occurs in a plane. Probably the first documented manifestation of the conservation of angular momentum is Kepler’s second law of planetary motion).

(a) As a consequence of Newton’s second law, we find that $\mathbf{a} = -\frac{GM}{r^3}\mathbf{r} = f(r)\cdot \mathbf{r}$, where $f(r)$ is a scalar function. Even though this function is not constant, for every value of $r$ we find that the vectors $\mathbf{r}$ and $\mathbf{a}$ are parallel.

(b) Let $\mathbf{L}/m = \mathbf{r} \times \mathbf{r}'$ (notice that here $m$ is just a constant, representing the mass of a moving particle). Then

$$\frac{d(\mathbf{L}/m)}{dt} = \frac{1}{m} \frac{d\mathbf{L}}{dt} = \frac{1}{m} \frac{d}{dt} (\mathbf{r} \times \mathbf{r}') = \frac{1}{m} (\mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'')$$

$$= \frac{1}{m} (\mathbf{r} \times \mathbf{a})$$

$$= \frac{f(r)}{m} (\mathbf{r} \times \mathbf{r}) = 0.$$

Therefore, $\mathbf{L}$ is constant.