Einstein Manifolds,

Self-Dual Weyl Weyl Curvature, &

Conformally Kähler Geometry

Claude LeBrun
Stony Brook University

Interactions in Complex Geometry
Vanderbilt University, December 12, 2021

Most recent references:

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Mathematical Research Letters
28 (2021) 127–144

And

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Einstein Manifolds, Conformal Curvature, and Anti-Holomorphic Involutions

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Annales Mathématiques du Québec
45(2) (2021) 391–405

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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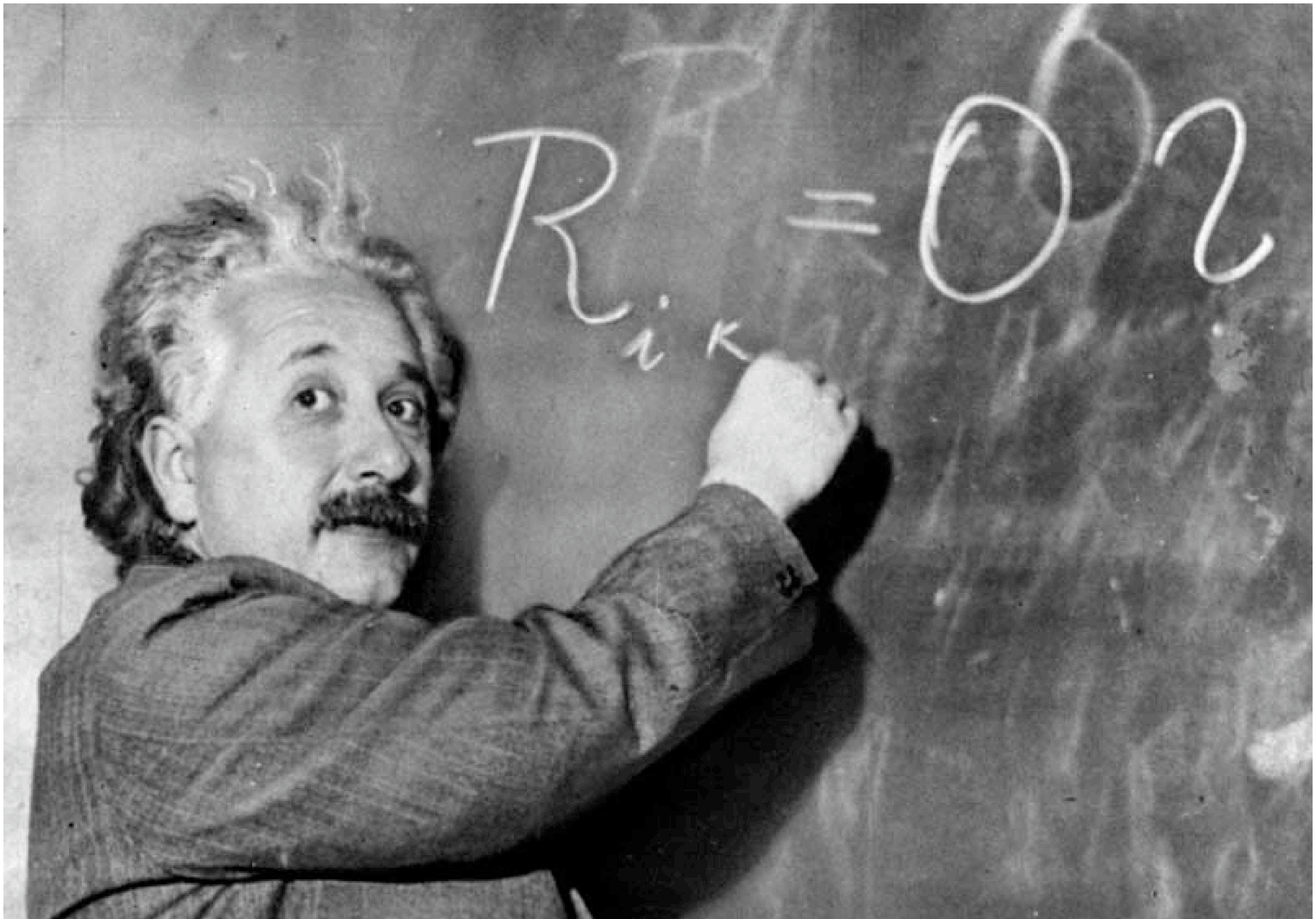

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Has same sign as the *scalar curvature*

$$s = r_j{}^j = \mathcal{R}^{ij}{}_{ij}.$$

Dimension Four is Exceptional

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When $n = 4$, Einstein metrics satisfy a remarkable conformally-invariant condition.

On Riemannian n-manifold (M, g),

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$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}{}_{[c} \delta^{b]}{}_{d]} + \frac{2}{n(n-1)} \, s \, \delta^{a}_{[c} \delta^{b}_{d]}$$

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where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W &= \text{Weyl curvature} \end{aligned}$$

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$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

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Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\Longleftrightarrow W \equiv 0$.

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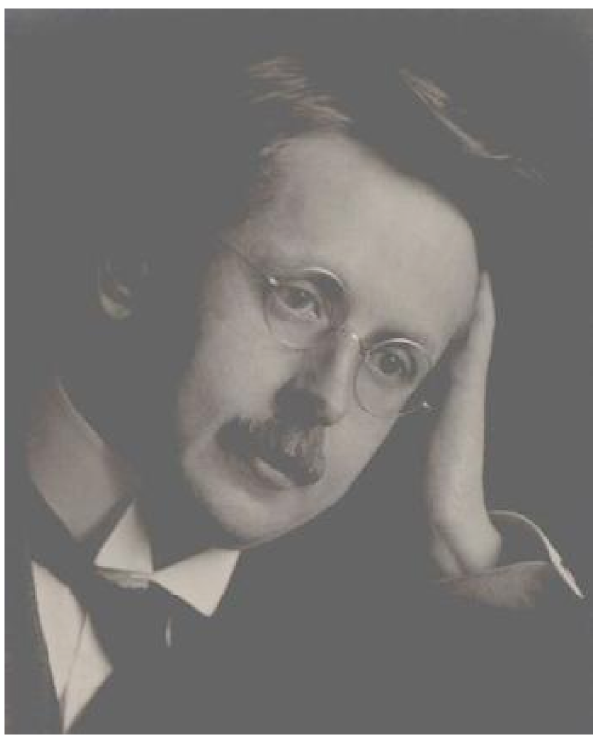

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Measures deviation $[g]$ from conformal flatness.

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When $n = 4$, conf. Einstein $\Rightarrow$ critical for $\mathscr{W}$.

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$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array}\right)$$

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	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

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$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

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Hence

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A case of special interest:

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(M^4, g, J) Kähler.

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$$|W_{+}|^2 = \frac{s^2}{24}$$

Restriction of $\mathscr{W}_+$ to Kähler metrics?

On Kähler metrics,

$$\int |W_+|^2 d\mu = \int \frac{s^2}{24} d\mu$$

so any critical point of restriction must be extremal in sense of Calabi.

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on Kähler g with $[\omega] \in H^2(M)$ fixed.

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Andrzej Derdziński : For Kähler metrics g,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

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- *$g_t = g + tB$ is Kähler metric for small t.*

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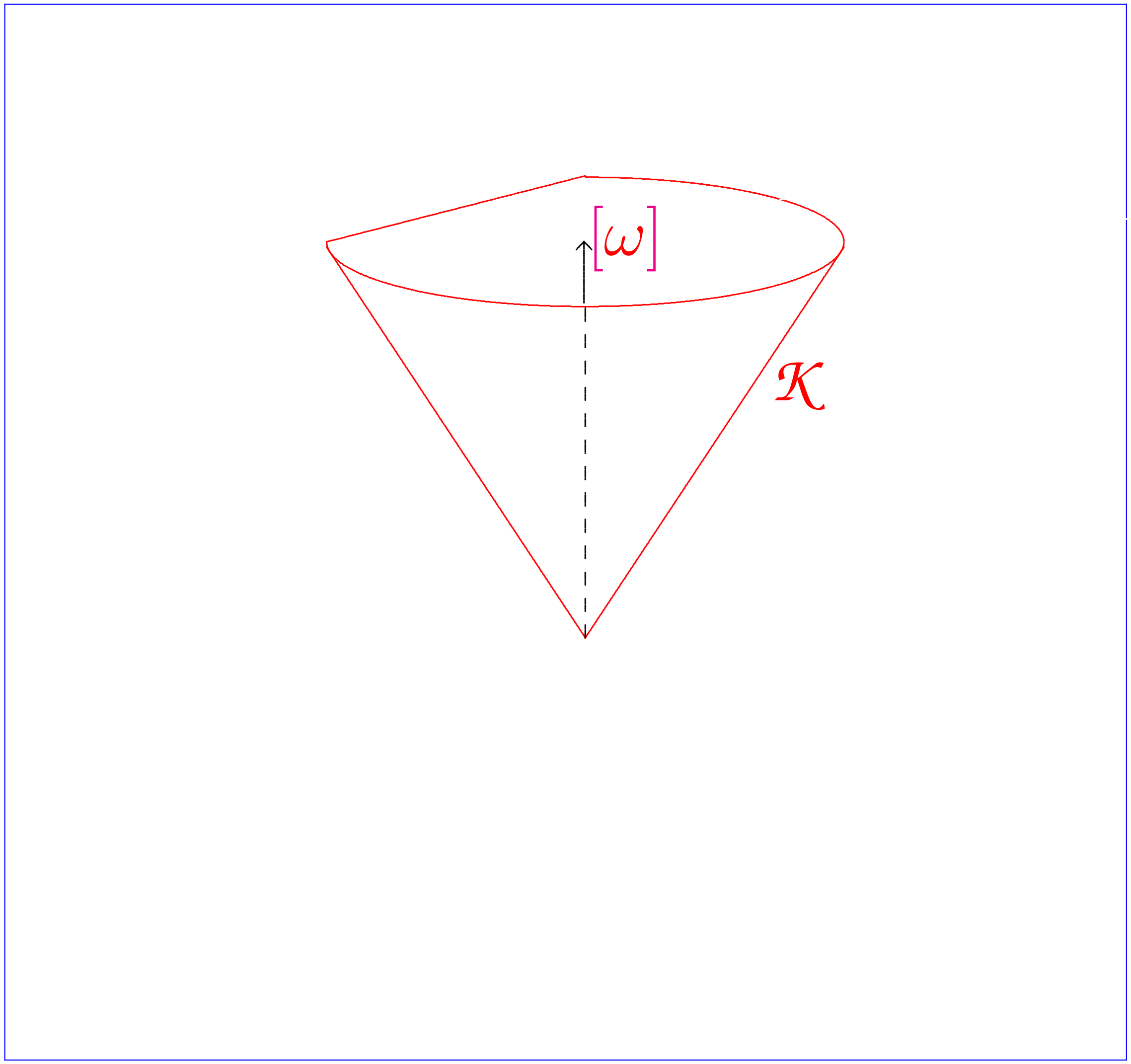
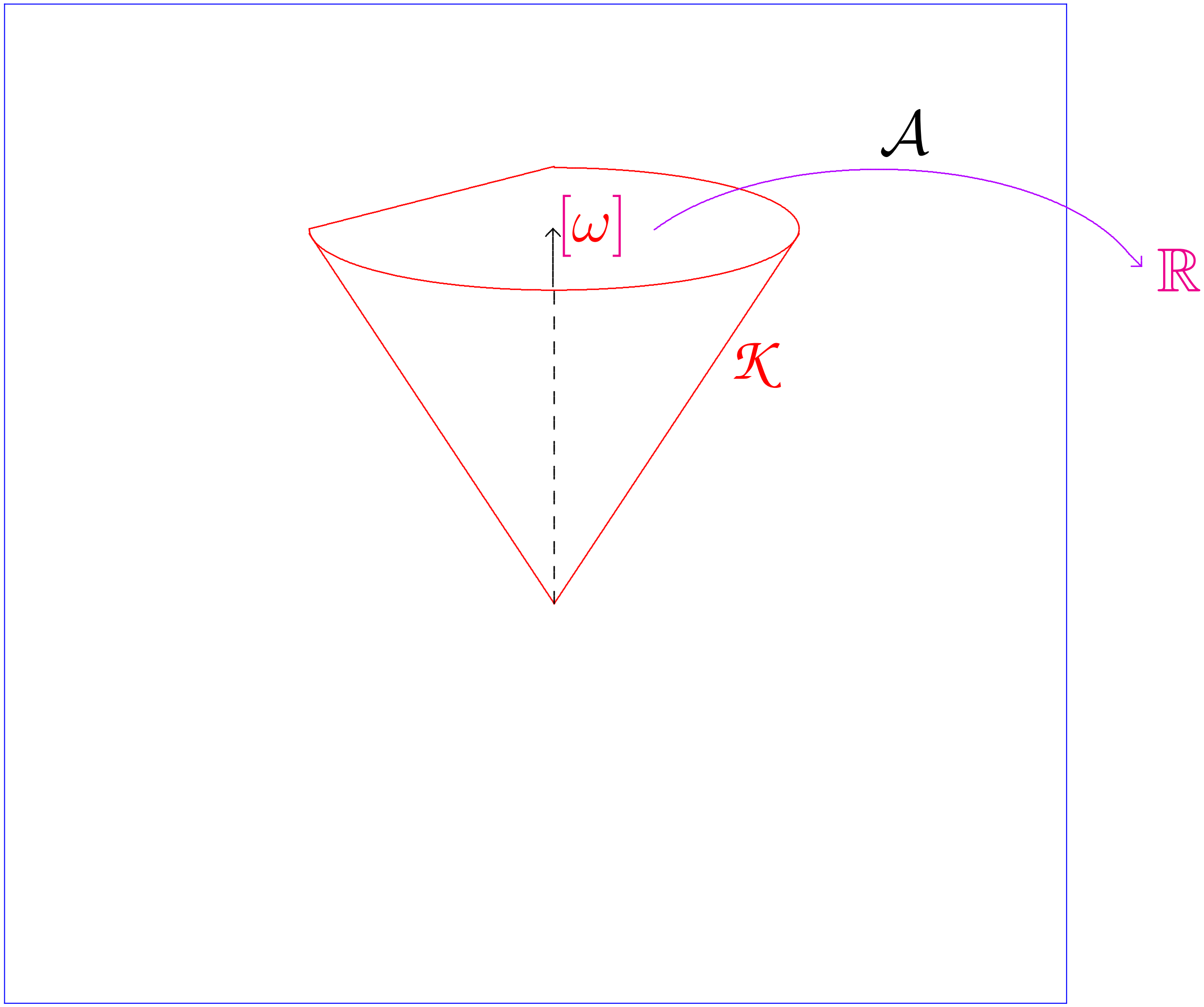
$$\frac{1}{32\pi^2}\int s^2 d\mu_g = \frac{(c_1\cdot[\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2$$
$$=: \mathcal{A}([\omega])$$

where $\mathcal{F}$ is Futaki invariant.



$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$$



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$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.

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$$\frac{1}{32\pi^2}\int s^2 d\mu_g = \frac{(c_1\cdot[\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2$$
$$=: \mathcal{A}([\omega])$$

where $\mathcal{F}$ is Futaki invariant.

$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.

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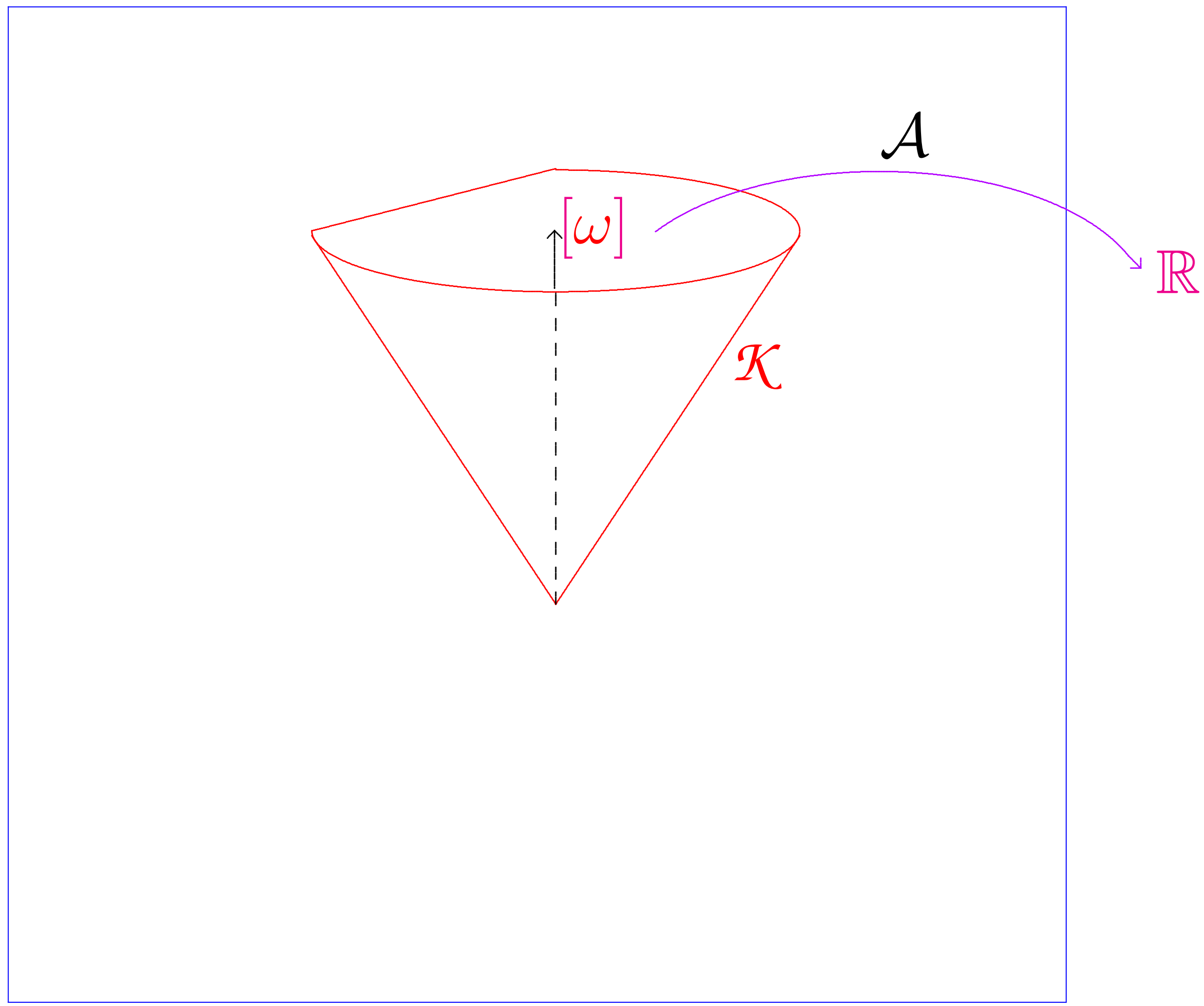

$$\mathcal{K} \subset H^{1,1}(M,\mathbb{R}) \subset H^2(M,\mathbb{R})$$

Restriction of $\mathscr{W}_+$ to Kähler metrics?

On Kähler metrics,

$$\int |W_+|^2 d\mu = \int \frac{s^2}{24} d\mu$$

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Andrzej Derdziński : For Kähler metrics g,

$$B = \frac{1}{12}\left[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\right]$$

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Global implications?

Theorem (L '97)**.**

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Calabi-Eckmann: False in higher dimensions!

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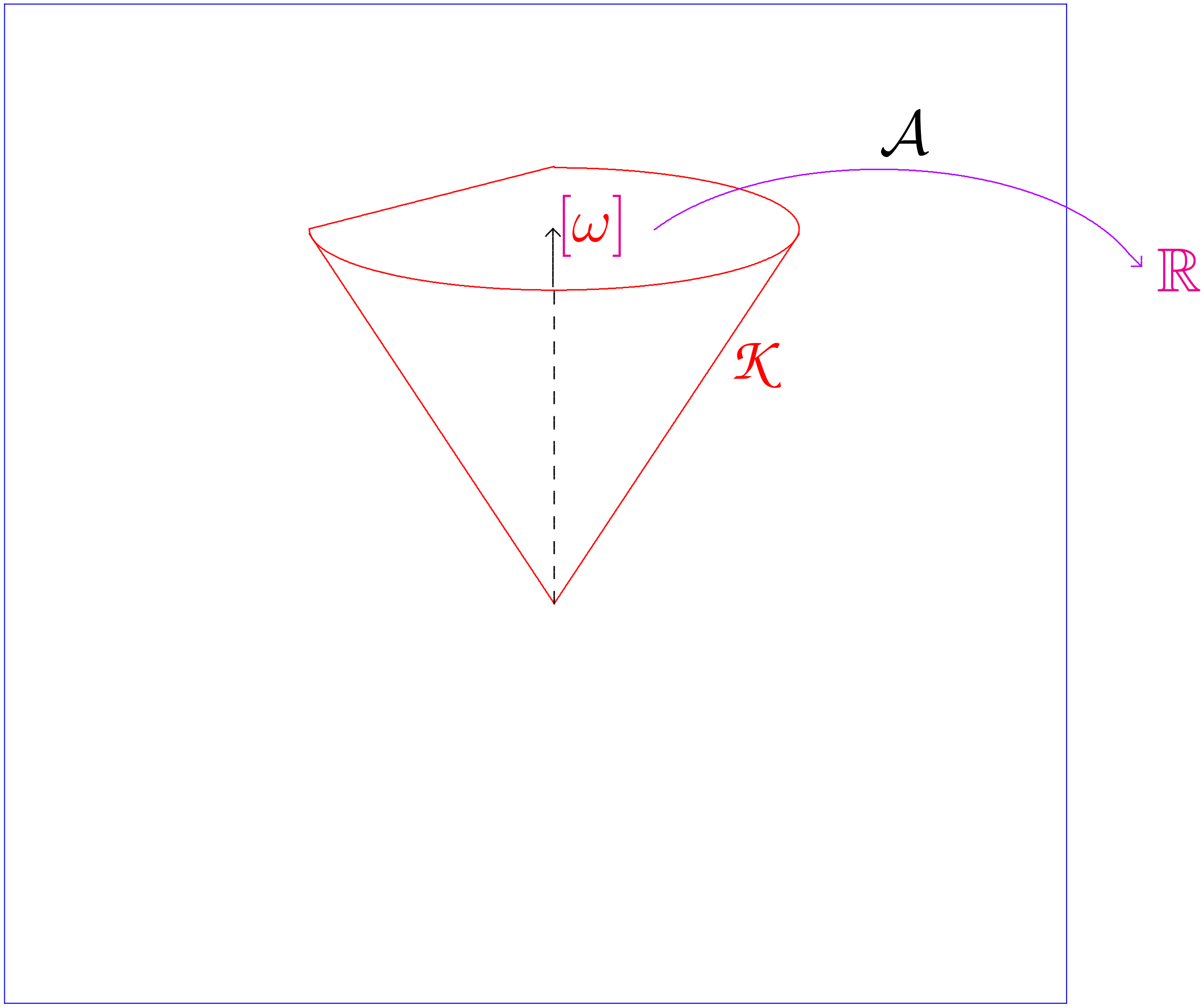

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Implies uniqueness result for h on (M^4, J).

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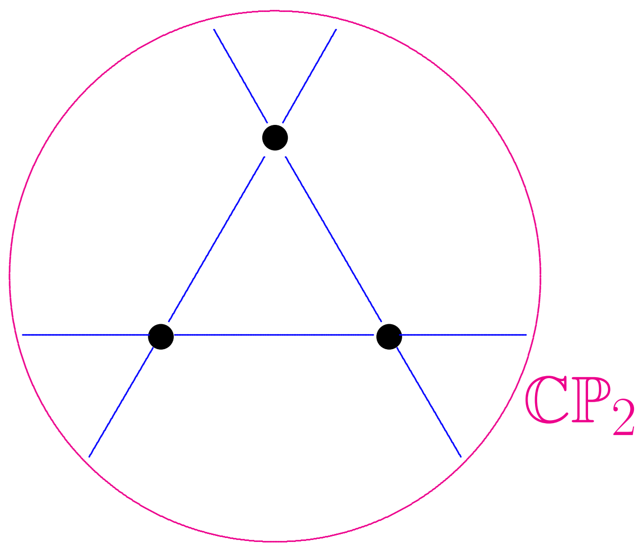

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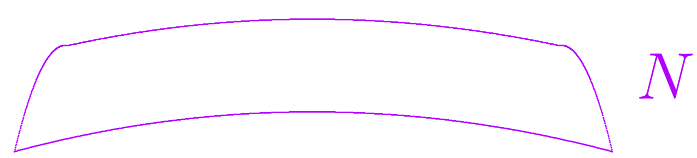

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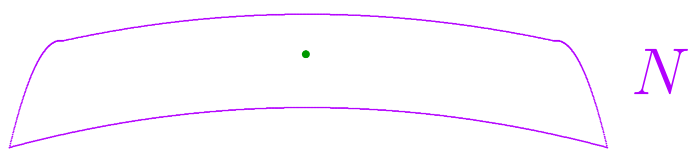

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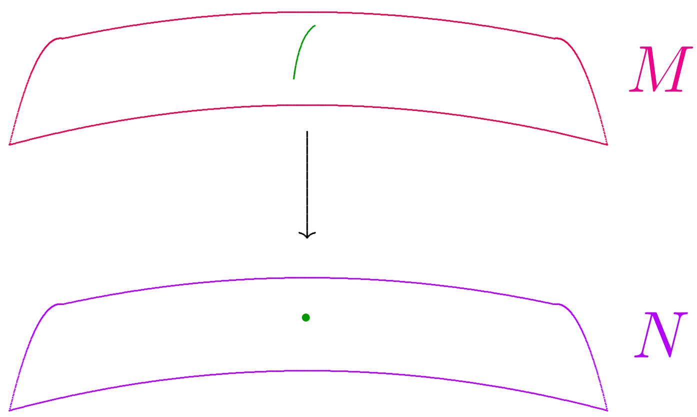

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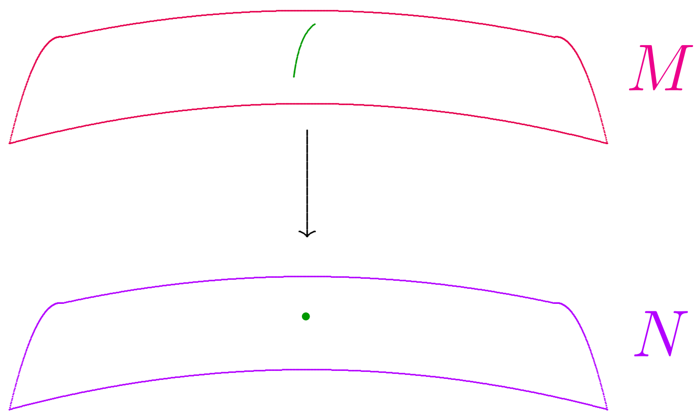

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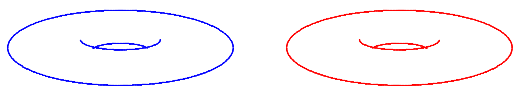

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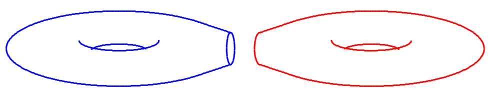

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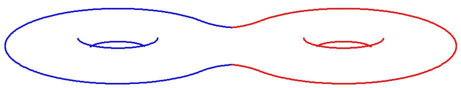

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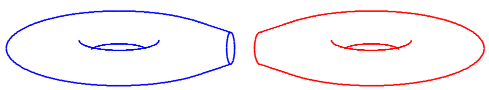

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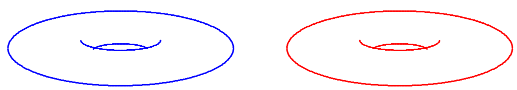

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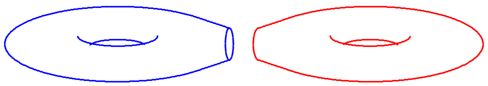

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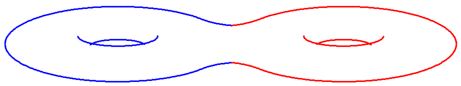

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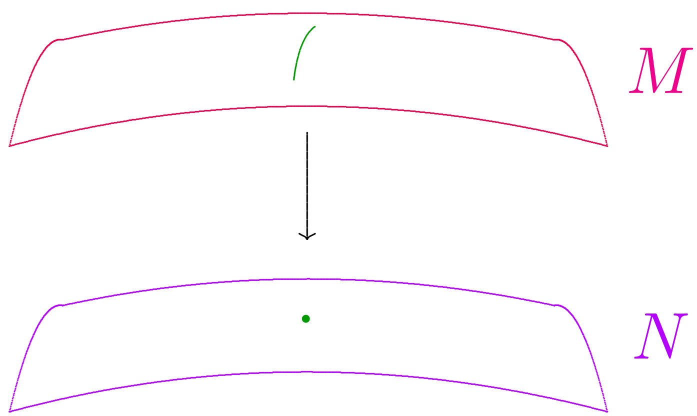

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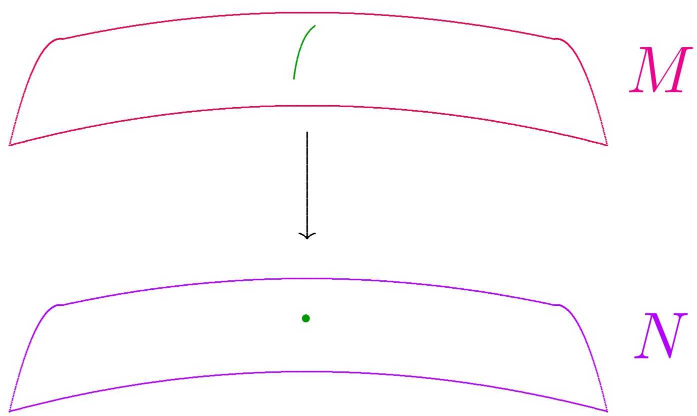

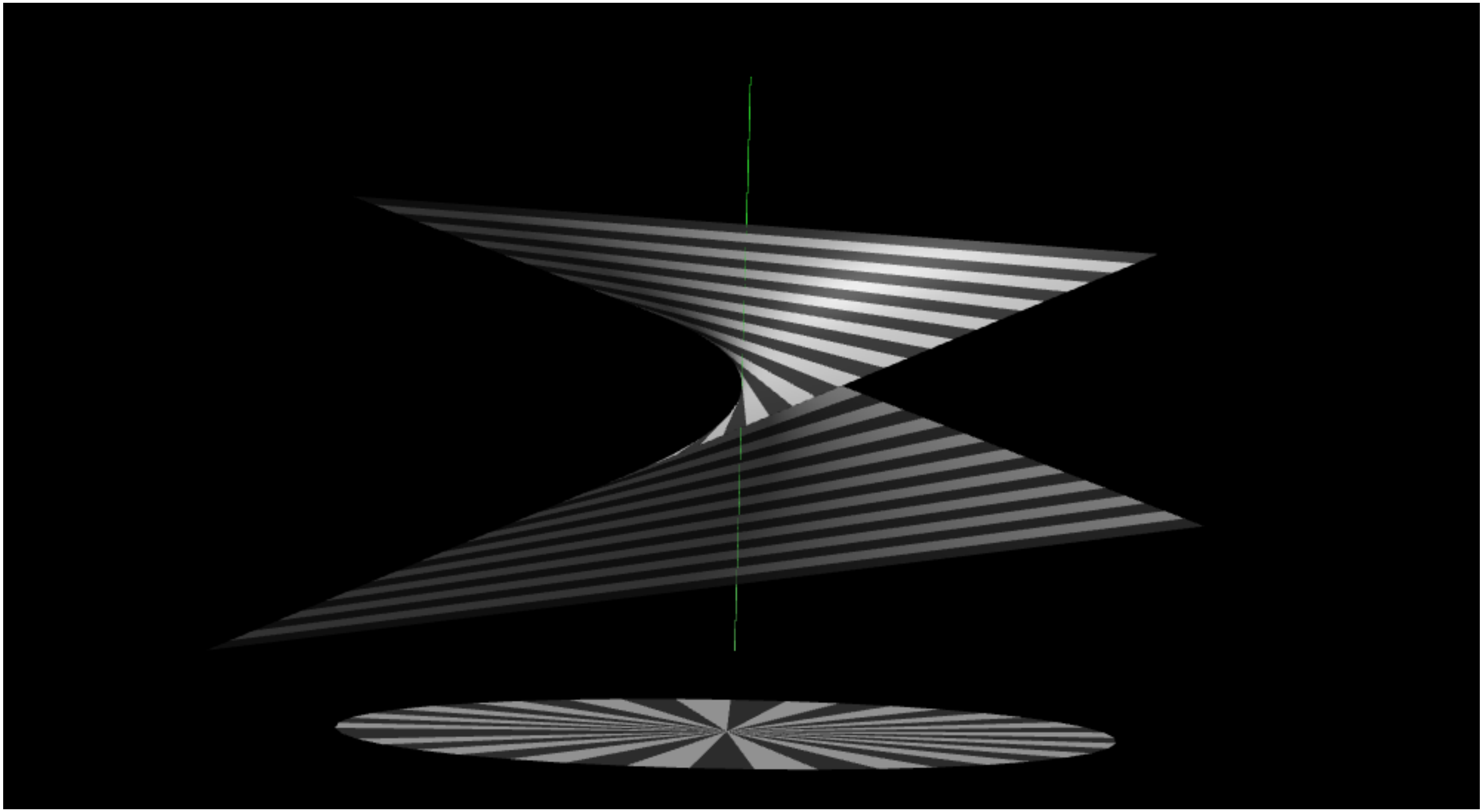

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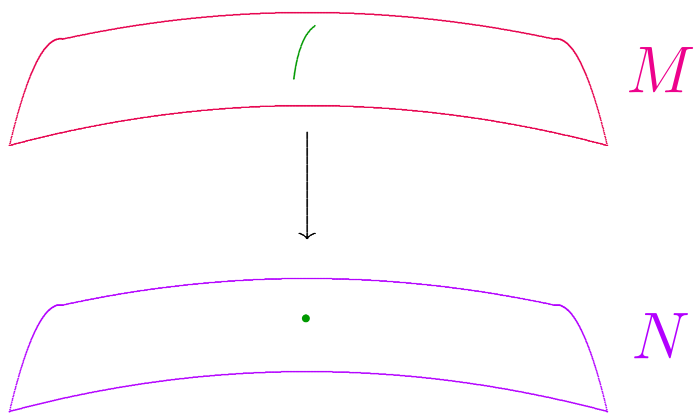

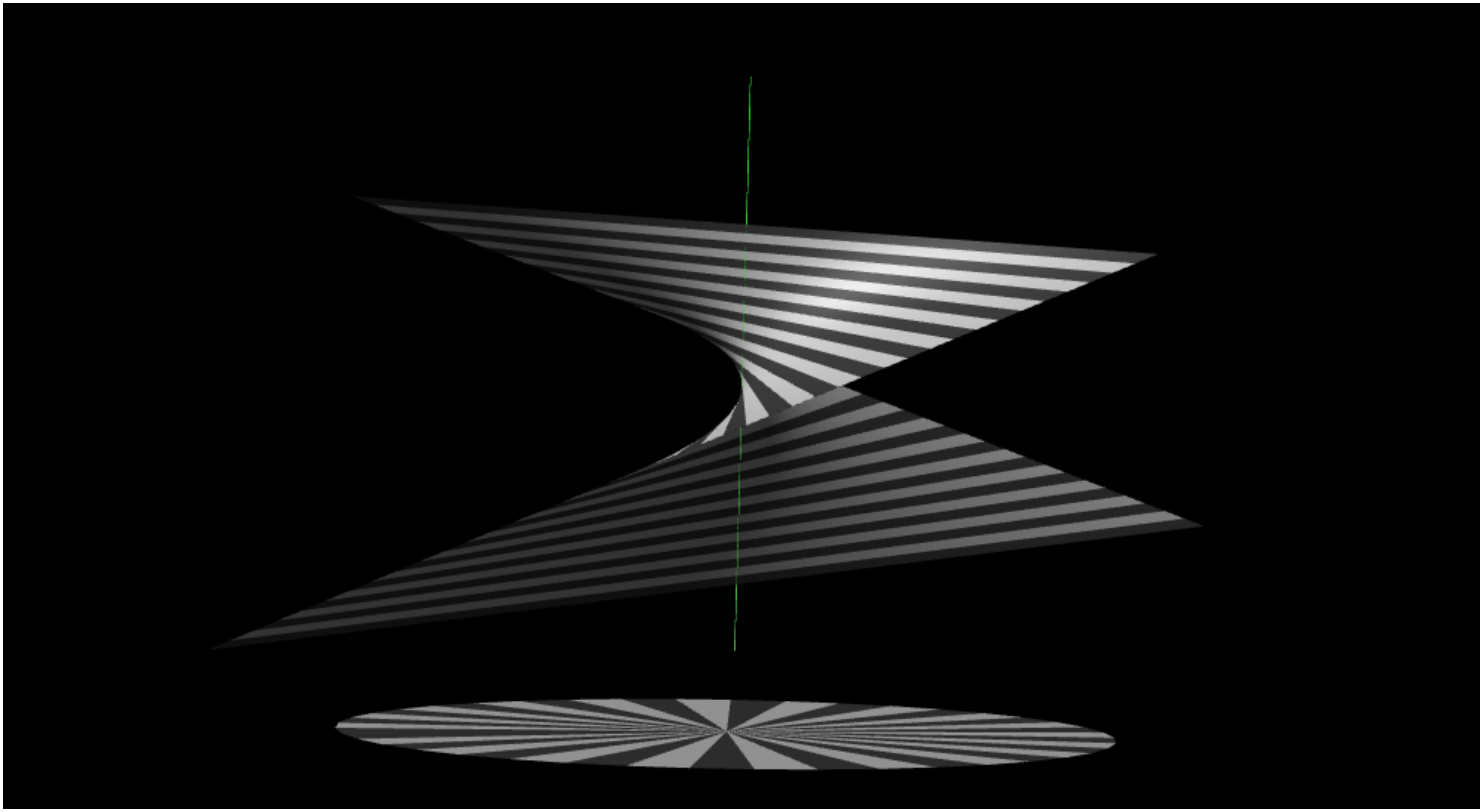

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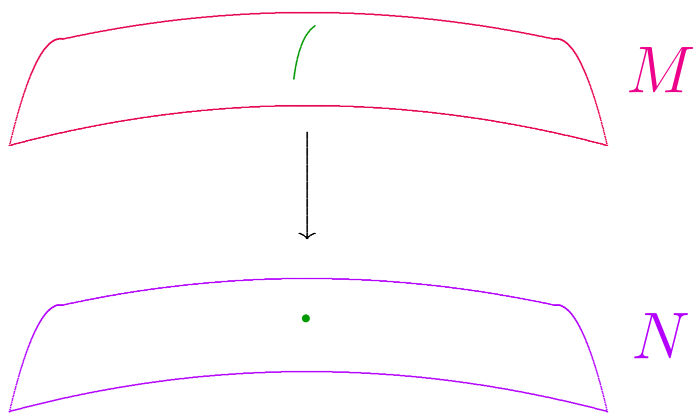

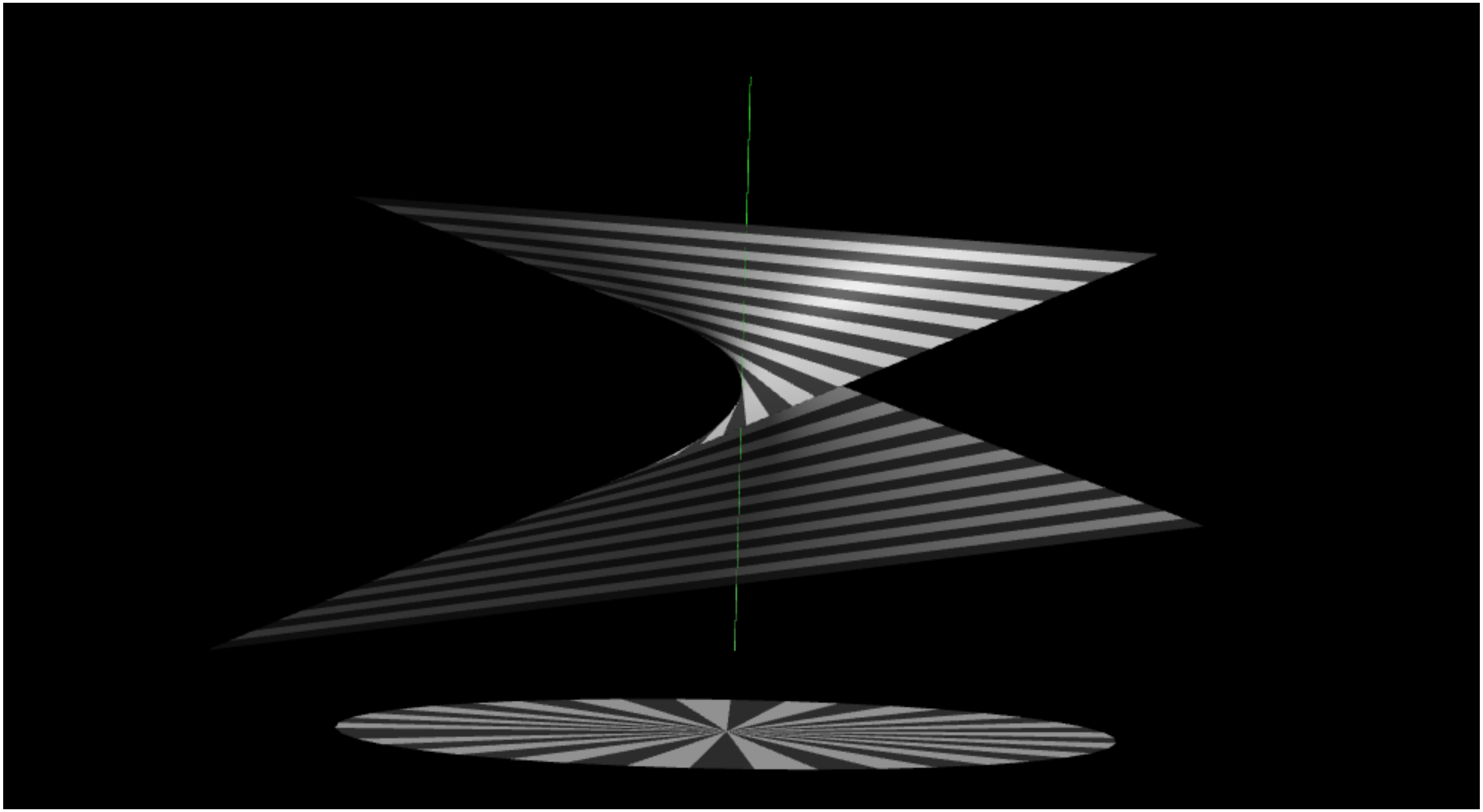

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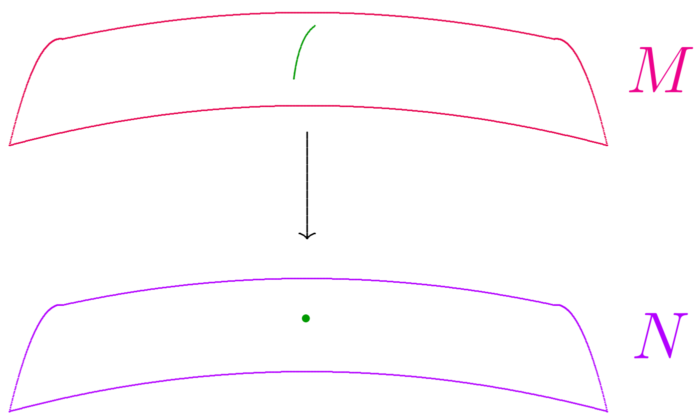

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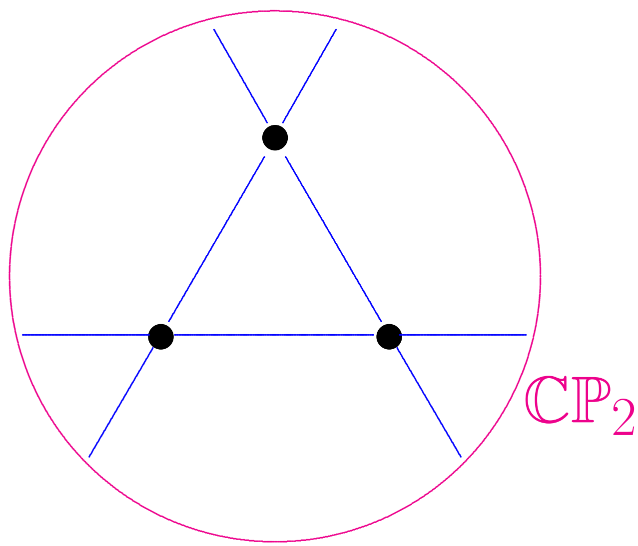

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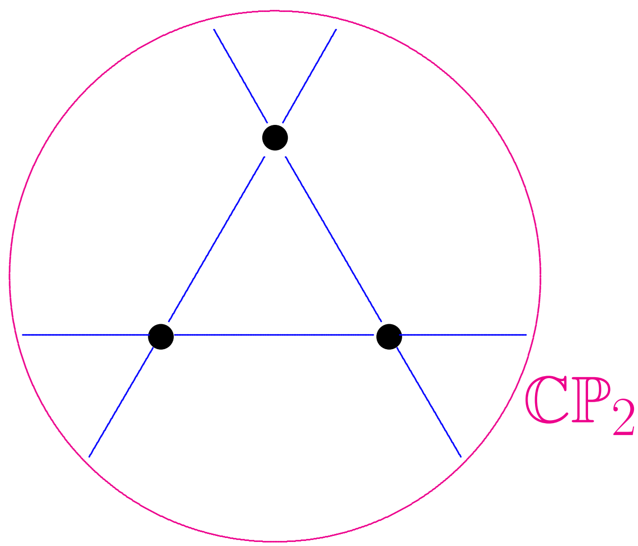

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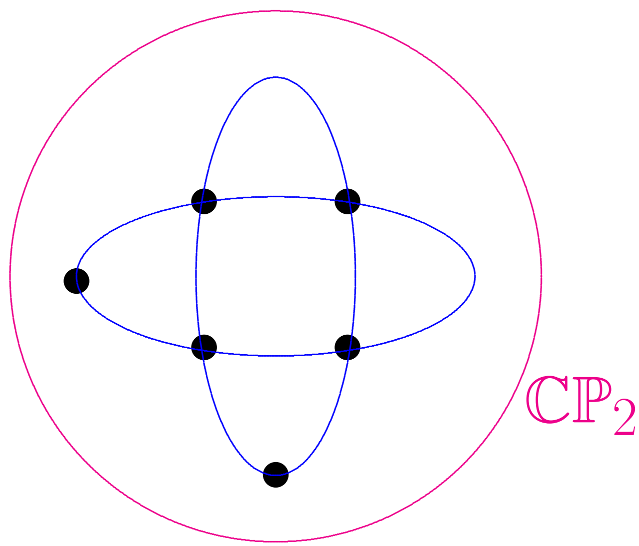

No 3 on a line, no 6 on conic,

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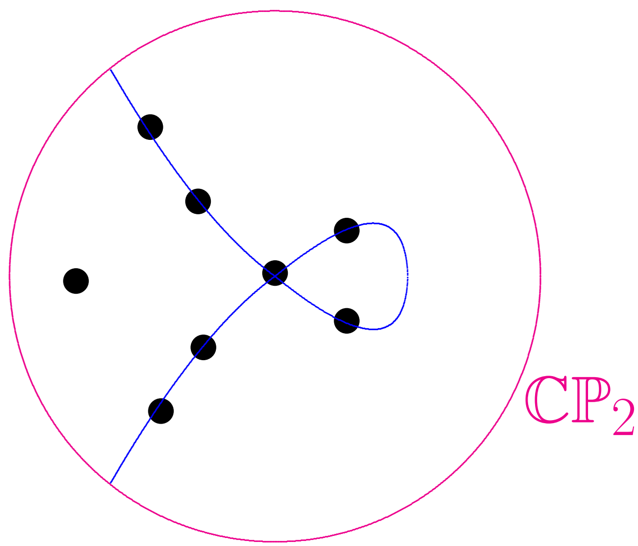

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible*

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Existence: Page

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One reason this seems satisfying. . .

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Exactly one connected component of moduli space!

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$$W^+ = \text{trace-free part of } \begin{bmatrix} 0 & & \\ & 0 & \\ & & \frac{s}{4} \end{bmatrix}$$

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for these metrics & conformal rescalings:

$$g \rightsquigarrow h = f^2 g \quad \Longrightarrow \quad \det(W^+) \rightsquigarrow f^{-6} \det(W^+).$$

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either on M or double cover $\widetilde{M}$.

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Claim: (M, h) compact Einstein $\Longrightarrow$ J integrable.

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Theorem A. *Let* (M, h) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

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Simply connected hypothesis is essential!

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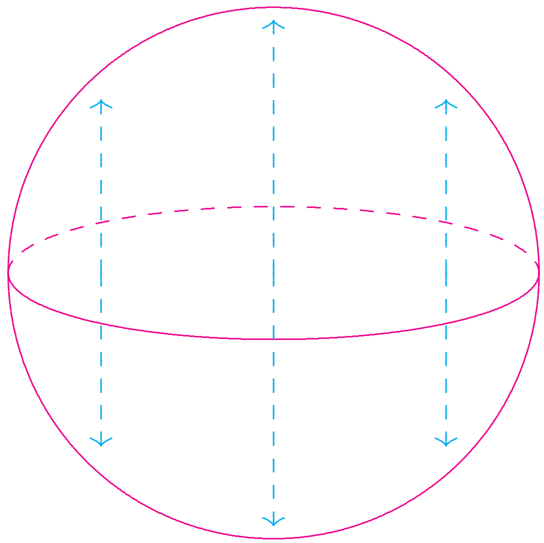
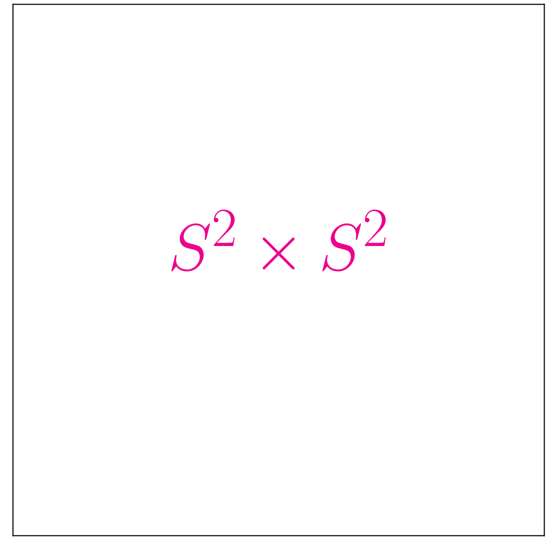
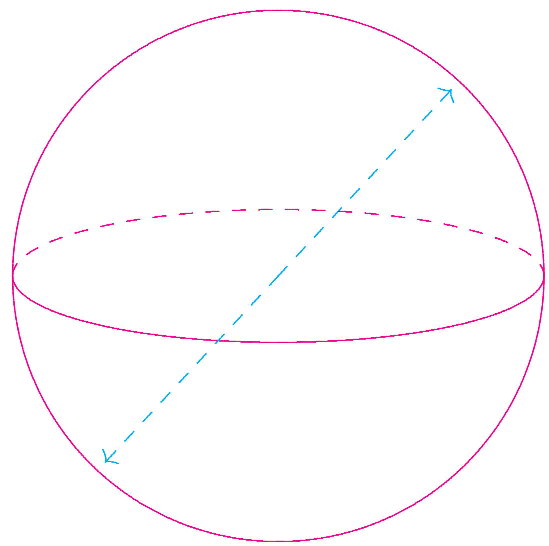




Oriented spin 4-manifold

$\mathscr{P} := (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{r} \rangle$



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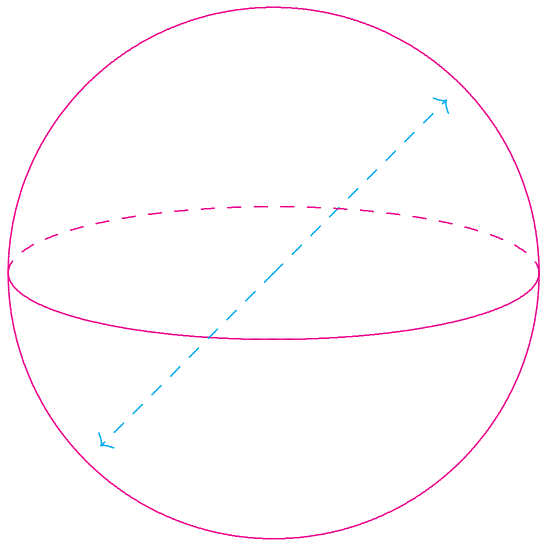
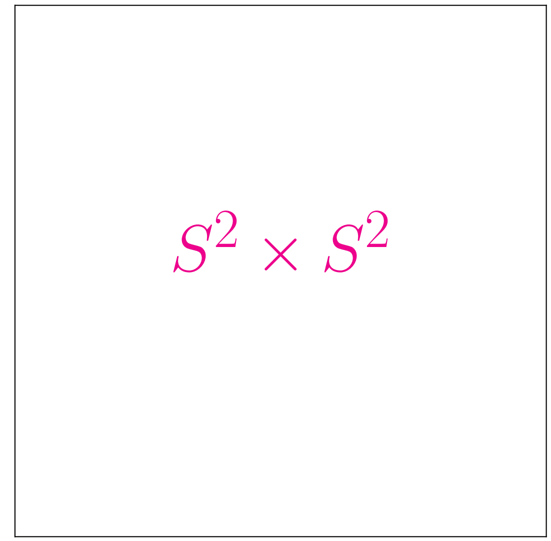
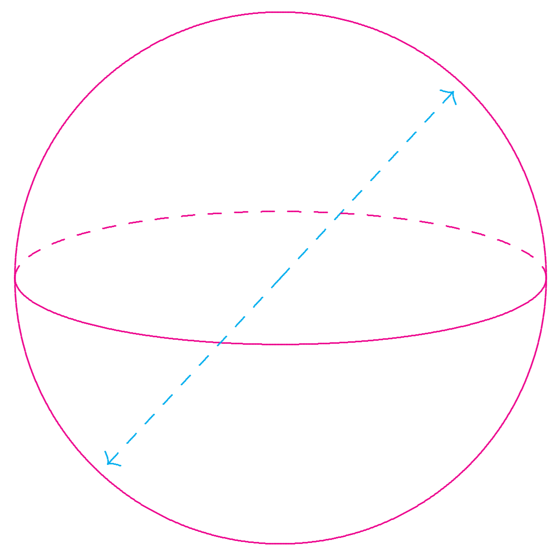




Non-spin 4-manifold

$\mathscr{Q} := (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{a} \rangle$



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Why is $\mathscr{E}_{\det}(M) \subset \mathscr{E}(M)$ open and closed?

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Why is $\mathscr{E}_{\det}(M) \subset \mathscr{E}(M)$ open and closed?

Open: $\det(W^+) > 0$.

Closed: $\det(W^+) = \frac{1}{3\sqrt{6}}|W^+|^3$ and $s \geq 0$.

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Theorem D. *Let* (M, h) *be a compact oriented Einstein* 4*-manifold. If*

$$\det(W^+) > -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

everywhere on M*, then actually* $\det(W^+) > 0$*. Consequently, all the results described remain true if we merely impose this ostensibly weaker hypothesis.*

Some indication of the proof:

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For clarity, let's just assume $\det(W^+) > 0\ldots$

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as proxy for Einstein equation.

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for $fW^+ \in \mathrm{End}(\Lambda^+)$.

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with $\omega \otimes \omega$, and integrate by parts. This yields:

$$0 = \int_M \left[\langle W^+, \nabla^*\nabla(\omega\otimes\omega)\rangle + \frac{s}{2}W^+(\omega,\omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2\right] f \, d\mu$$

Let $\alpha \geq \beta \geq \gamma$ be eigenvalues of W^+:

$$W^+ = \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$$

$$\alpha + \beta + \gamma = 0$$

$$\alpha > 0, \quad \gamma < 0, \quad \text{if } W^+ \neq 0$$

$$\det(W^+) = \alpha\beta\gamma$$

$$\det(W^+) > 0 \implies \alpha \text{ has multiplicity } 1.$$

So $\alpha = \alpha_h : M \to \mathbb{R}^+$ a smooth function.

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Eigenvalues of W^+ carry a conformal weight:

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$$\begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix} = \begin{bmatrix} f^2\alpha & & \\ & f^2\beta & \\ & & f^2\gamma \end{bmatrix}$$

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$$\alpha = \alpha^{1/3} = f^{-1}$$

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Now choose $\omega \in \Gamma\Lambda^+$ so that

$$W_g^+(\omega) = \alpha\, \omega, \quad |\omega|_g \equiv \sqrt{2},$$

after at worst passing to double cover $\hat{M} \to M$.

$$0 = \int_{\hat{M}} \Big[\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle \\ + \frac{s}{2} W^+(\omega, \omega) - 6 |W^+(\omega)|^2 + 2 |W^+|^2 |\omega|^2 \Big] f \, d\mu$$

$$0 = \int_M \Big[\langle W^+, \nabla^*\nabla(\omega \otimes \omega)\rangle + \frac{s}{2} W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2 \Big] f \; d\mu$$

$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) - 2W^+(\omega, \nabla^e \nabla_e \omega) + \frac{s}{2} W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2 \Big] f \, d\mu$$

$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) - 2\alpha\langle \omega, \nabla^e \nabla_e \omega\rangle$$

$$+\frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 2|W^+|^2|\omega|^2\Big] f \, d\mu$$

because

$$W_g{}^+(\omega) = \alpha\omega$$

$$0 = \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha \langle \omega, \nabla^* \nabla \omega \rangle$$
$$+\frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 2|W^+|^2|\omega|^2 \Big] f \; d\mu$$

$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha\langle \omega, \nabla^*\nabla\omega\rangle$$
$$+\frac{s}{2}\alpha|\omega|^2 - 6\alpha^2|\omega|^2 + 3\alpha^2|\omega|^2\Big] f \; d\mu$$

because

$$|{W_g}^+|^2 \geq \frac{3}{2}\alpha^2$$

$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha\langle \omega, \nabla^*\nabla\omega\rangle + \frac{s}{2}\alpha|\omega|^2 - 3\alpha^2|\omega|^2 \Big] f \, d\mu$$

$$|\omega|^2_g = 2 \quad \Longrightarrow \quad (\nabla_e \omega) \perp \omega$$

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$$|\omega|^2_g = 2 \quad \Longrightarrow \quad (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad W^+ \sim \begin{bmatrix} + & & \\ & - & \\ & & - \end{bmatrix}$$

$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha \langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2}\alpha |\omega|^2 - 3\alpha^2 |\omega|^2 \Big] f \, d\mu$$

$$|\omega|_g^2 = 2 \implies (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \implies W^+(\nabla_e \omega, \nabla^e \omega) \leq 0$$

$$0 \geq \int_M \Big[\quad 2\alpha\langle\omega, \nabla^*\nabla\omega\rangle$$

$$+\frac{s}{2}\alpha|\omega|^2 - 3\alpha^2|\omega|^2\Big] f \; d\mu$$

$$|\omega|_g^2 = 2 \quad \Longrightarrow \quad (\nabla_e\omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad -W^+(\nabla_e\omega, \nabla^e\omega) \geq 0$$

$$0 \geq \int_M \left[2\alpha \langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} \alpha |\omega|^2 - 3\alpha^2 |\omega|^2 \right] f \, d\mu$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^*\nabla\omega \rangle + \frac{s}{2}|\omega|^2 - 3\alpha|\omega|^2 \right] (\alpha f) \; d\mu$$

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But

$$\alpha f \equiv 1$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2}|\omega|^2 - 3|\omega|^2 \alpha \right] d\mu$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle - 3W^+(\omega, \omega) + \frac{s}{2}|\omega|^2 \right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \left(\nabla^*\nabla - 2W^+ + \frac{s}{3}\right)\omega\rangle\right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \quad (d+d^*)^2 \quad \omega\rangle\right] d\mu$$

Because

$$(d+d^*)^2 = \nabla^*\nabla - 2W^+ + \frac{s}{3}$$

on $\Gamma\Lambda^+$.

$$0 \geq \frac{1}{2} \int_M |\nabla \omega|^2 \, d\mu + 3 \int_M |d\omega|^2 \, d\mu$$

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So $\nabla\omega \equiv 0$, and g is Kähler!

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QED

Thanks for the invitation!

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