Four-Manifolds,

Einstein Metrics, &

Differential Topology

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Four-Manifolds,

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Differential Topology, II

Kähler Paradigms
in a Riemannian World

October 20, 2016
University of Pennsylvania

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

$$\text{Einstein} \Rightarrow \qquad = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if Λ^+ is flat on (M, g). The latter happens only if (M, g) finitely covered by a flat T^4 or by a Calabi-Yau $K3$.

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$\therefore$ Topological manifold $|K3|$ has infinitely many smooth structures, but only one of these admits Einstein metrics.

However, don't get too discouraged…

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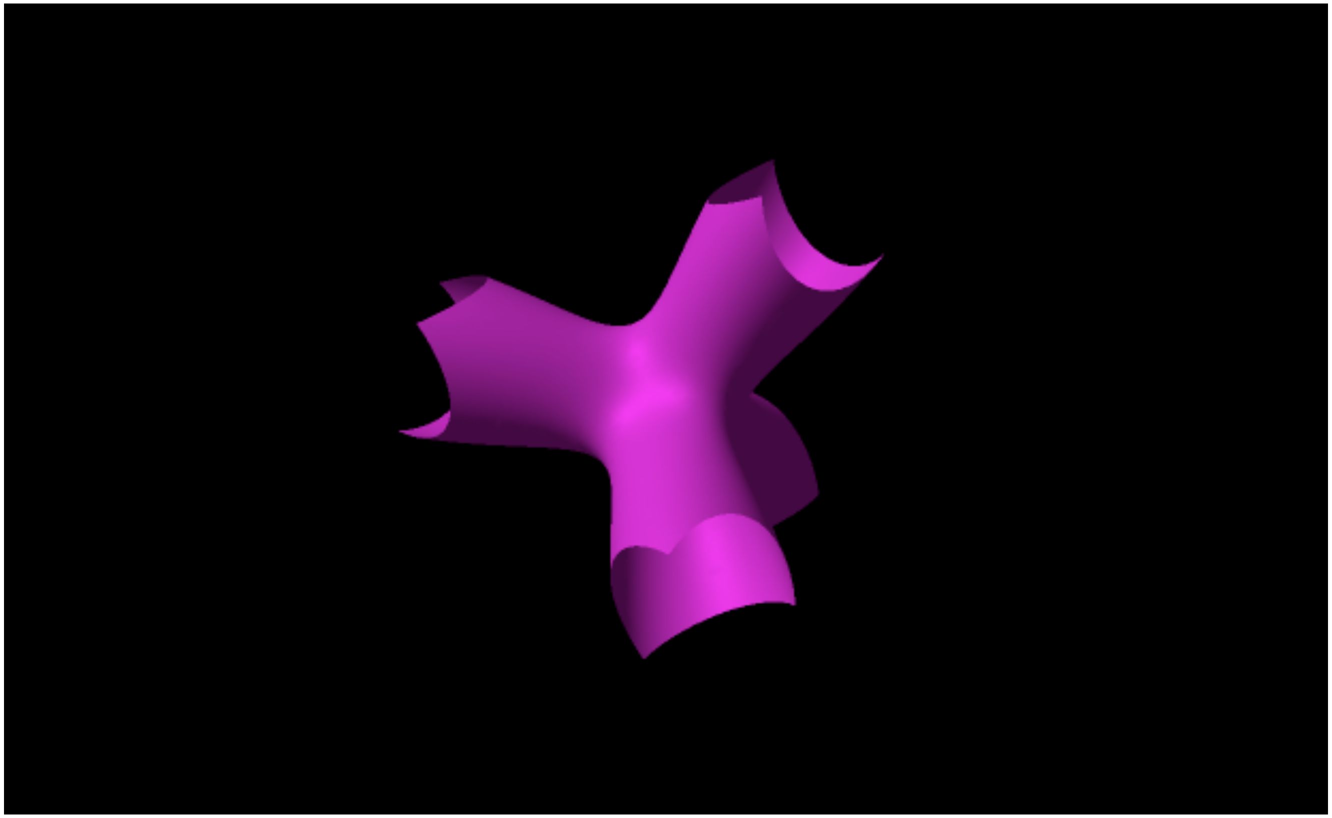

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Fubini-Study

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$$\pi_1 = 0, \quad \chi = \ell(\ell^2 - 4\ell + 6), \quad \tau = -\frac{1}{3}\ell(\ell^2 - 4), \quad \text{spin} \Leftrightarrow \ell \text{ even.}$$

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(Kodaira embedding theorem)

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These questions will be our main focus...

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Our Focus. *Suppose (M^4, J) is a compact complex surface. When does M^4 admit an Einstein metric g, perhaps completely unrelated to J?*

Kodaira Classification

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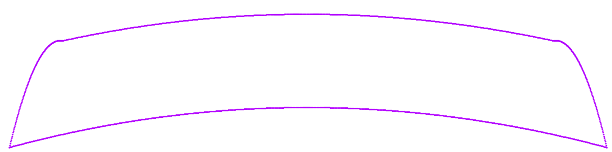

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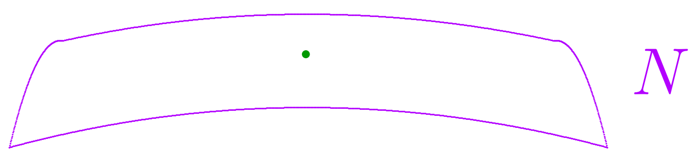

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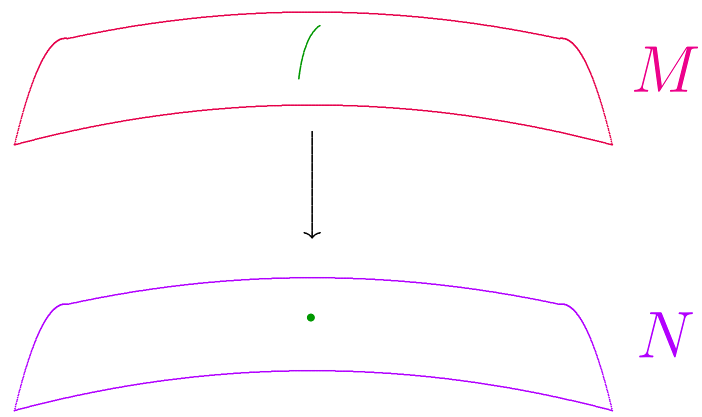

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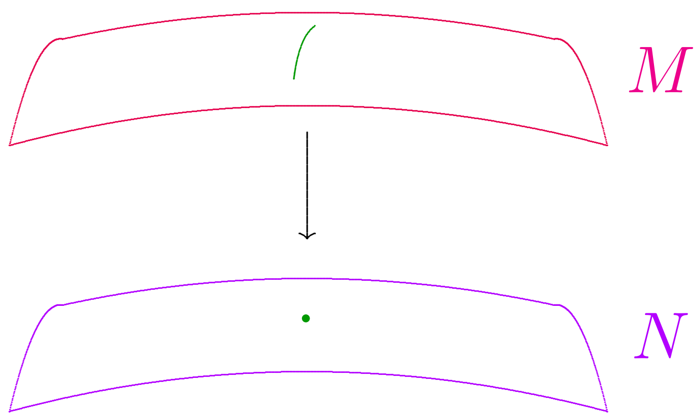

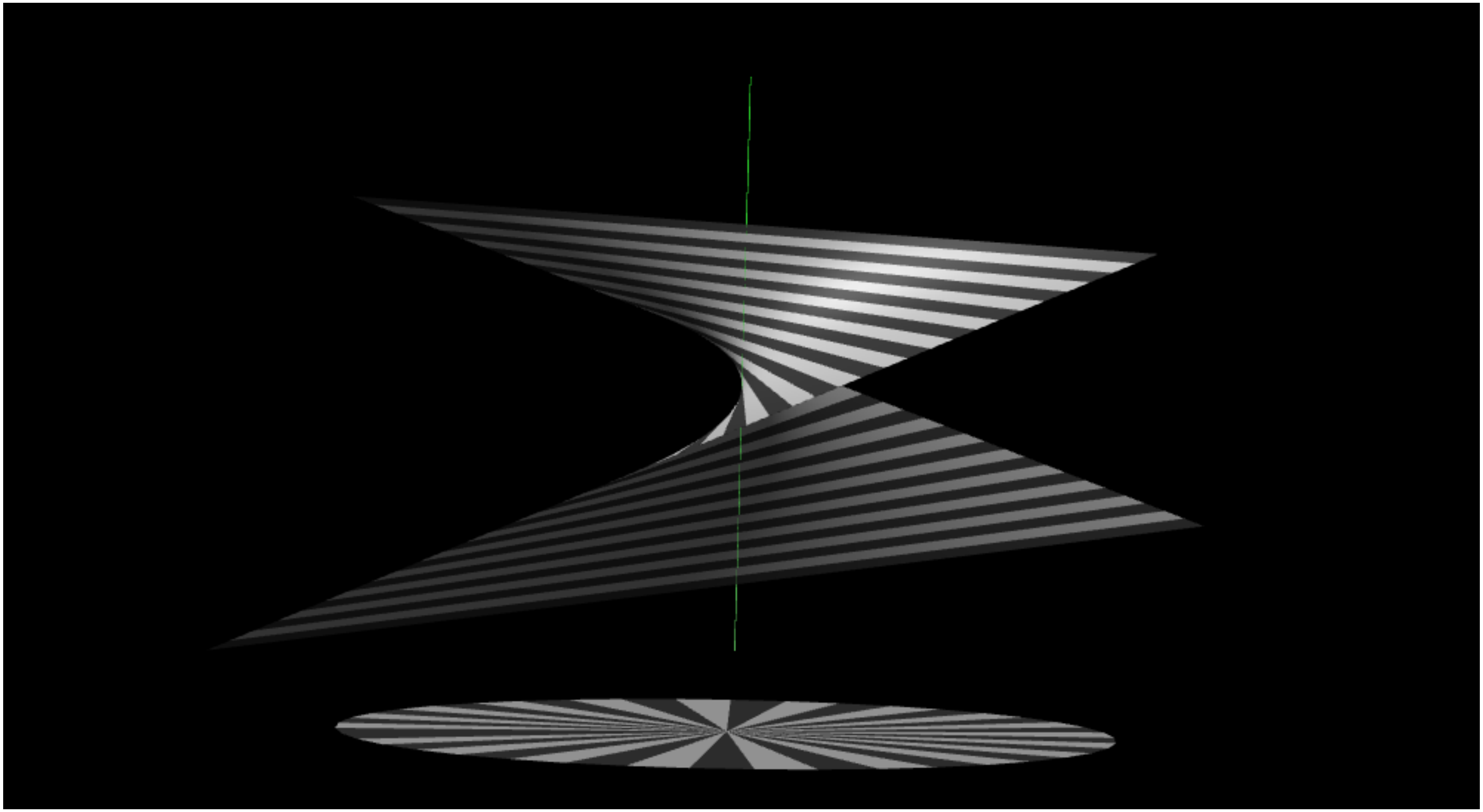

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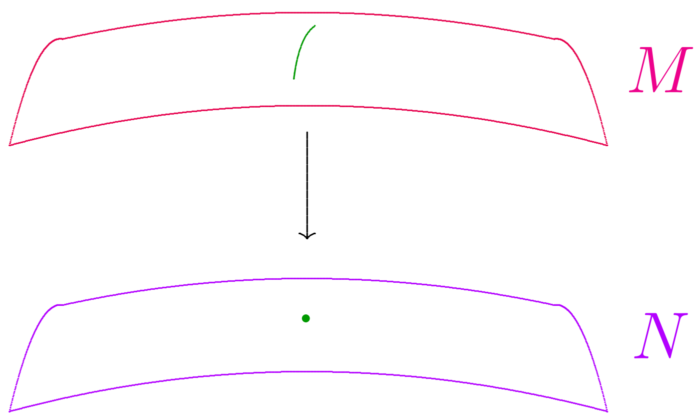

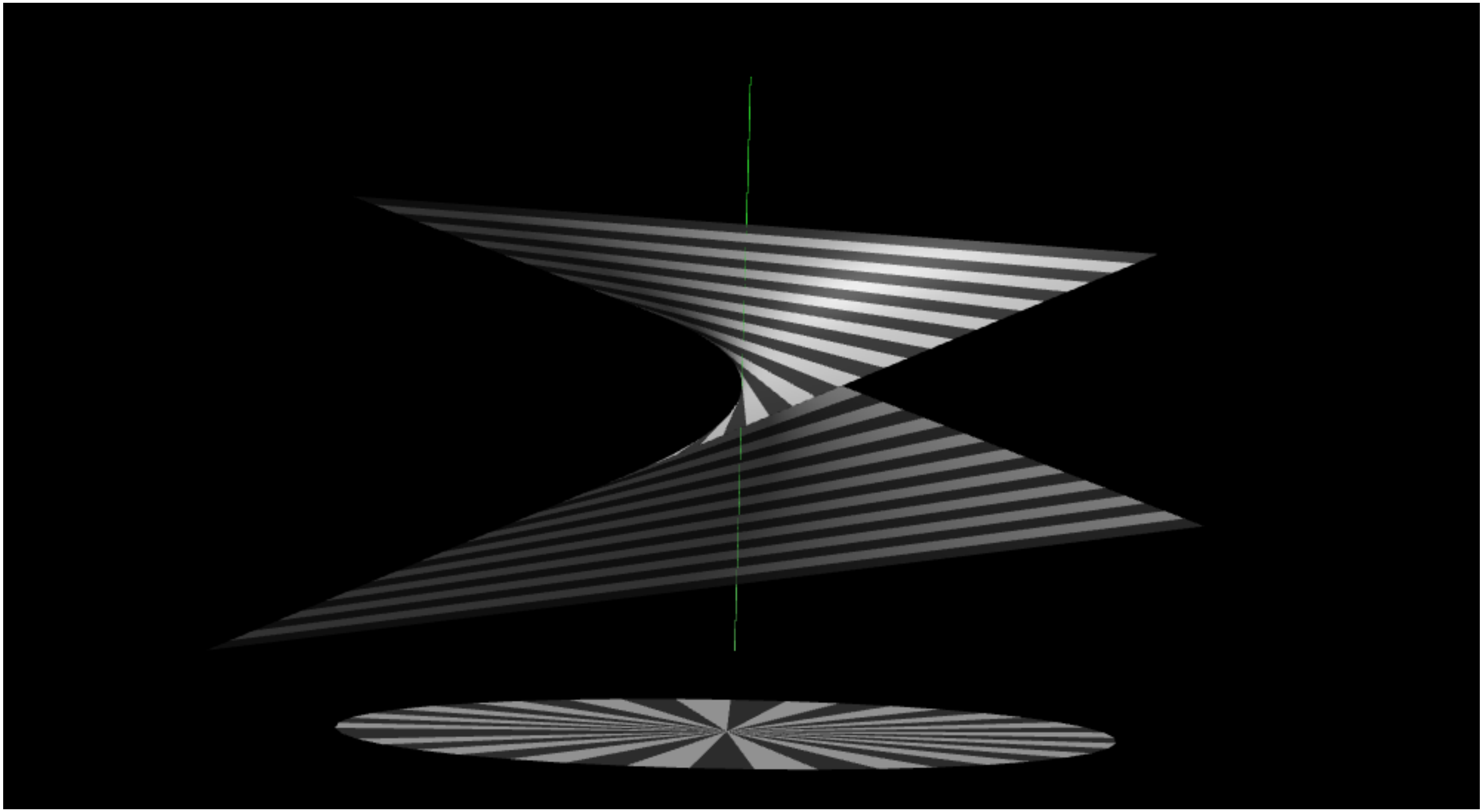

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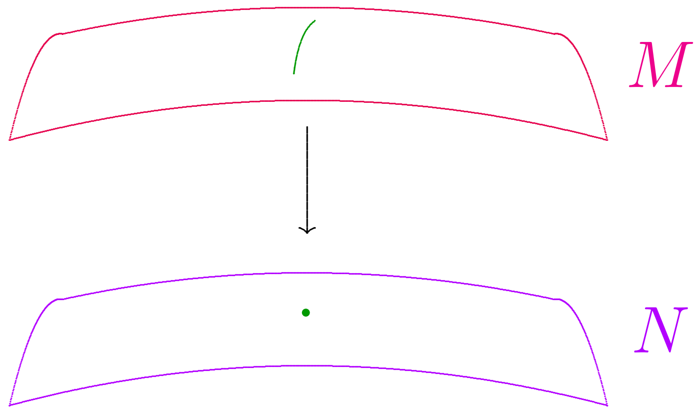

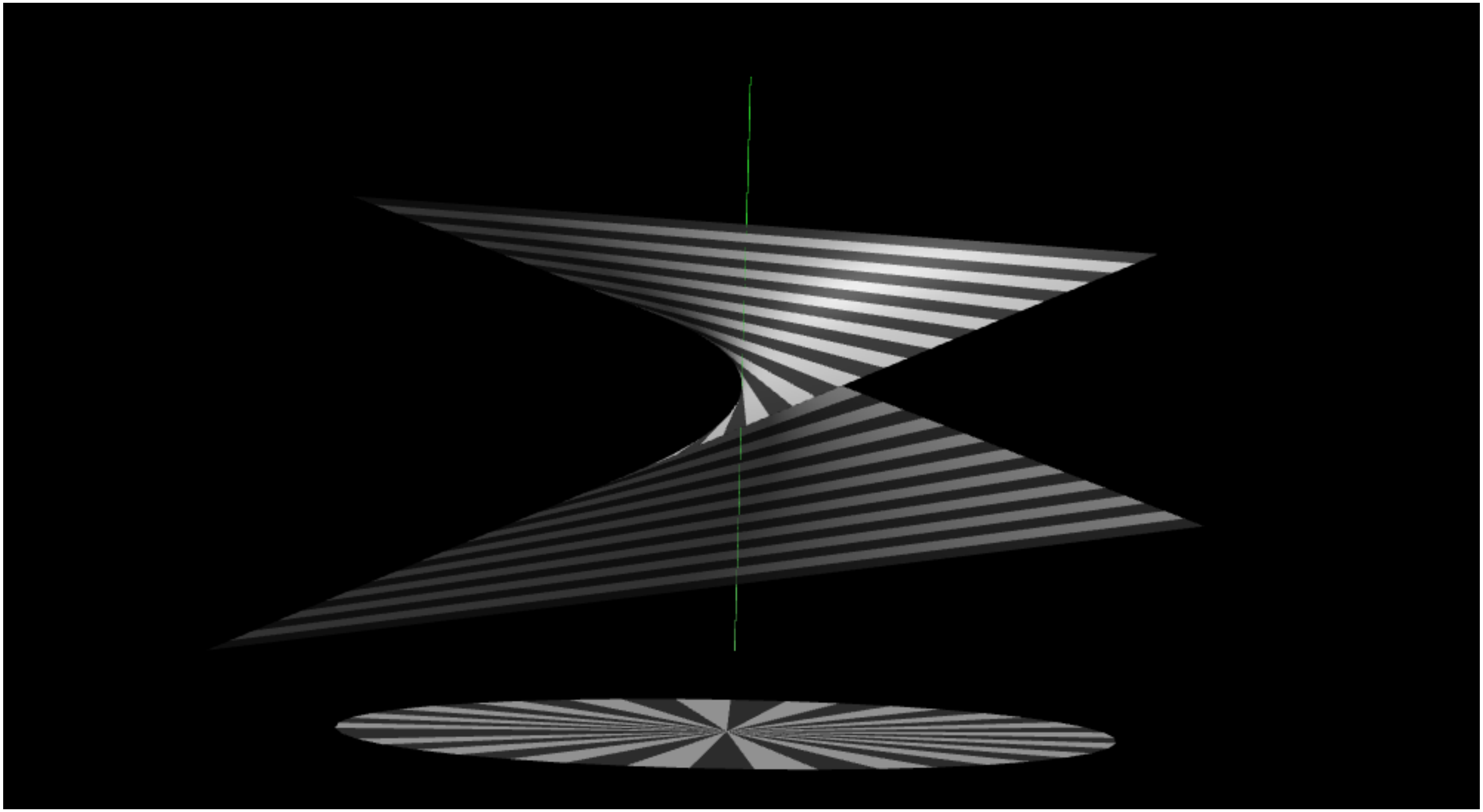

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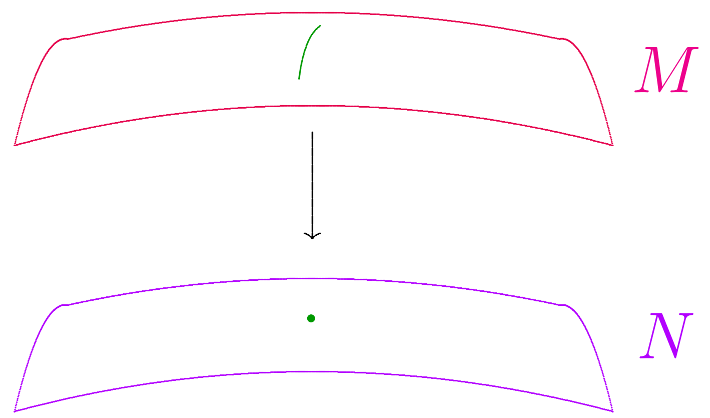

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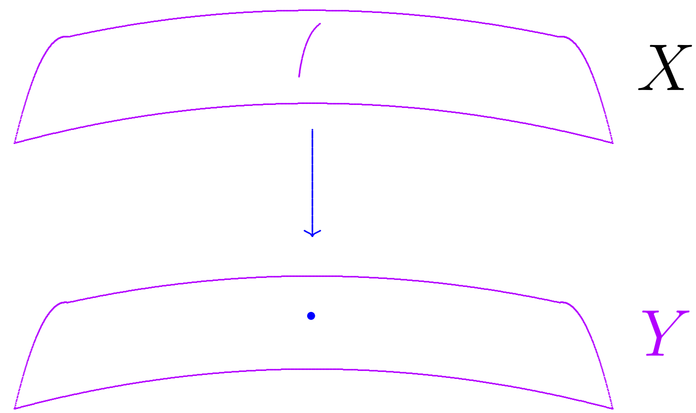

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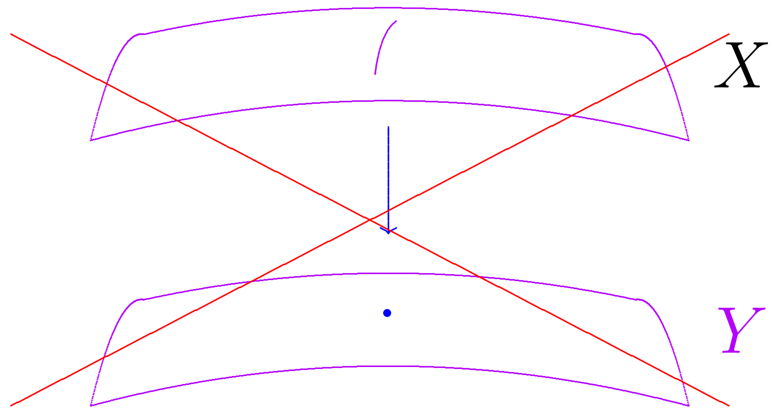

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Symplectic structure:

2-form ω with $d\omega = 0$ and $\omega \wedge \omega > 0$.

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- $M \approx \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$*, where* $0 \leq k \leq 8$*; or*
- M *is is finitely covered by* T^4*; or*
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Let's think more about Riemannian 4-manifolds...

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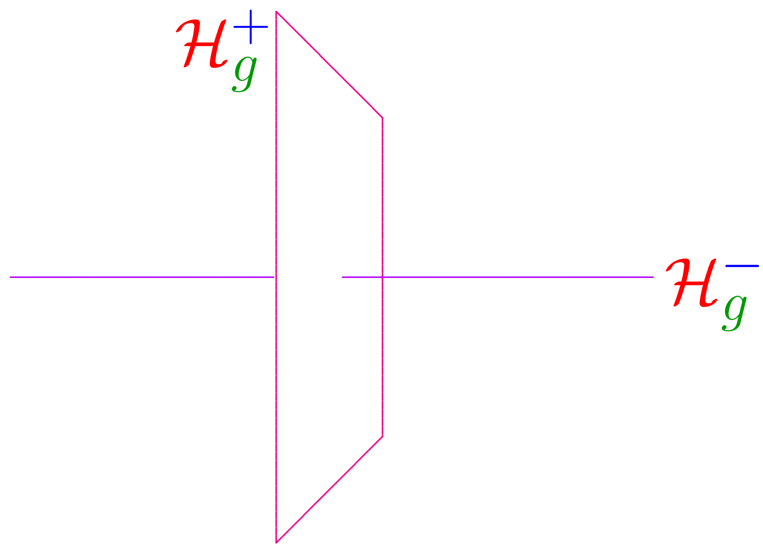
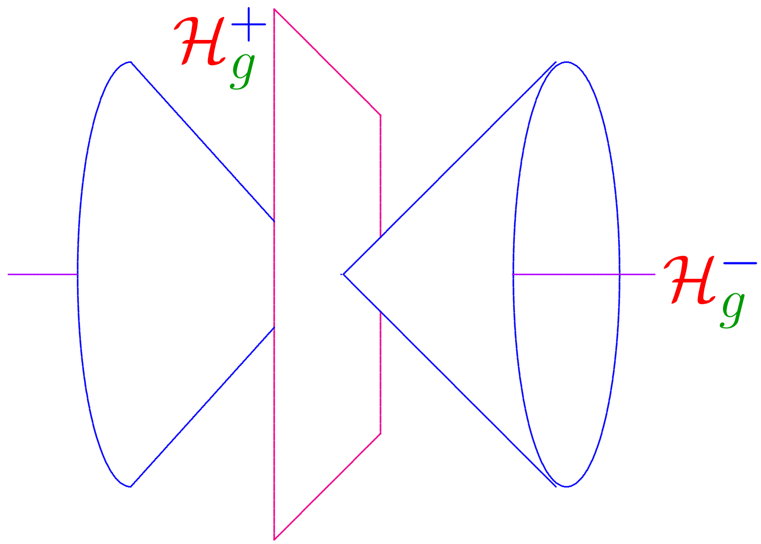


$$H^2(M, \mathbb{R})$$



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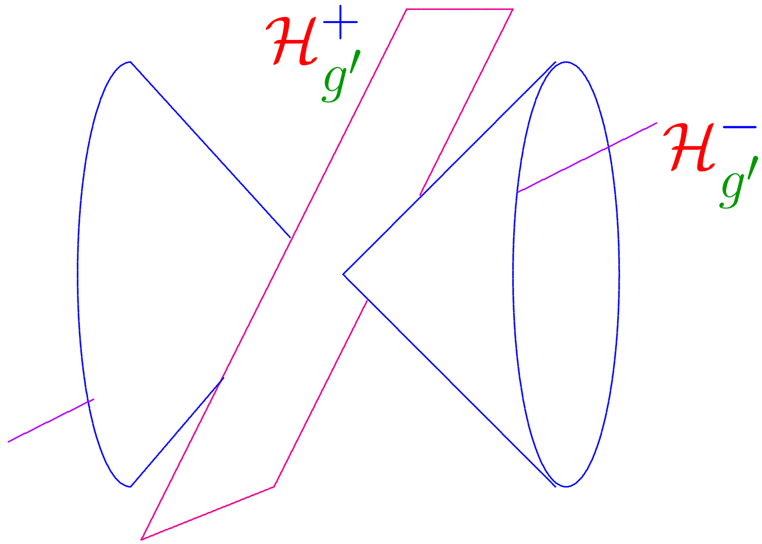

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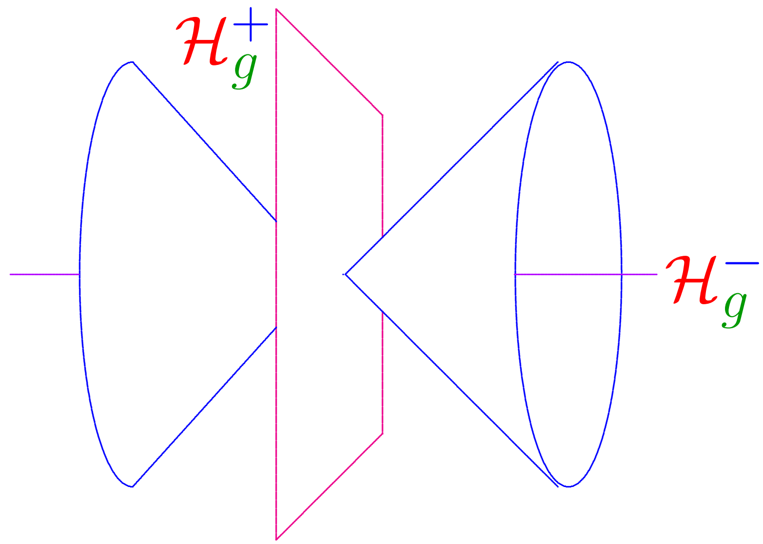

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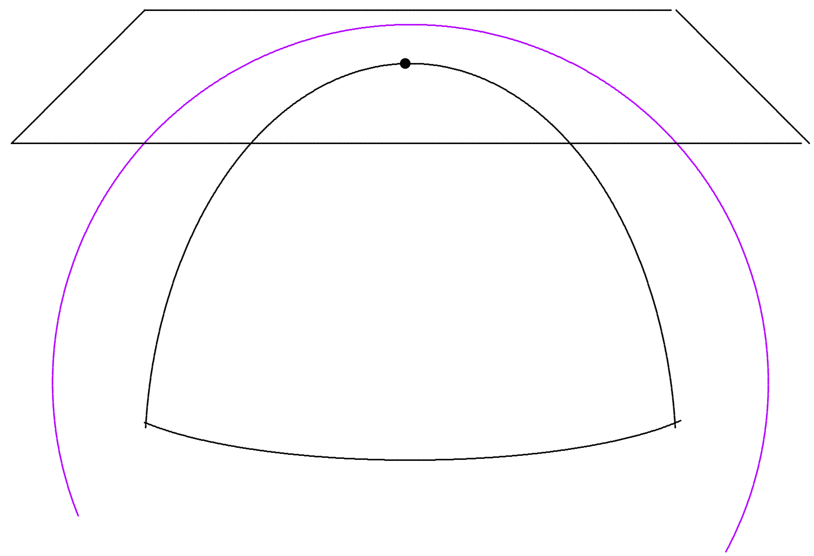

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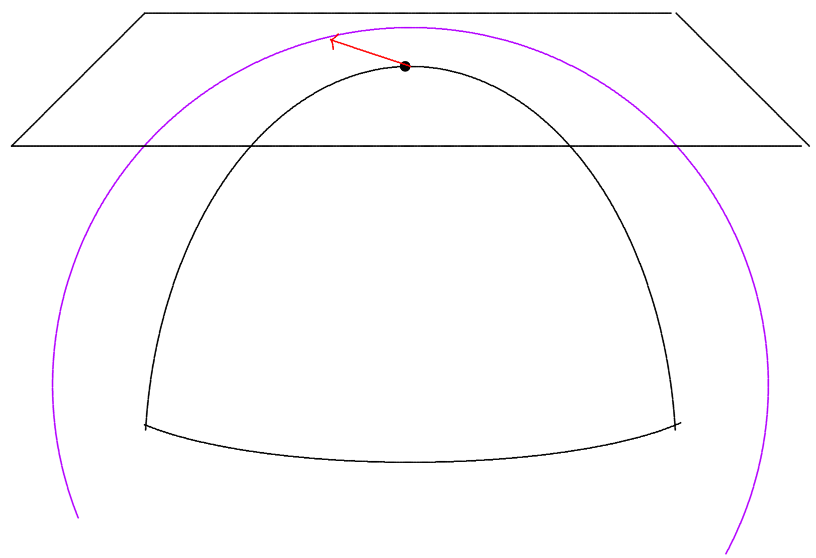

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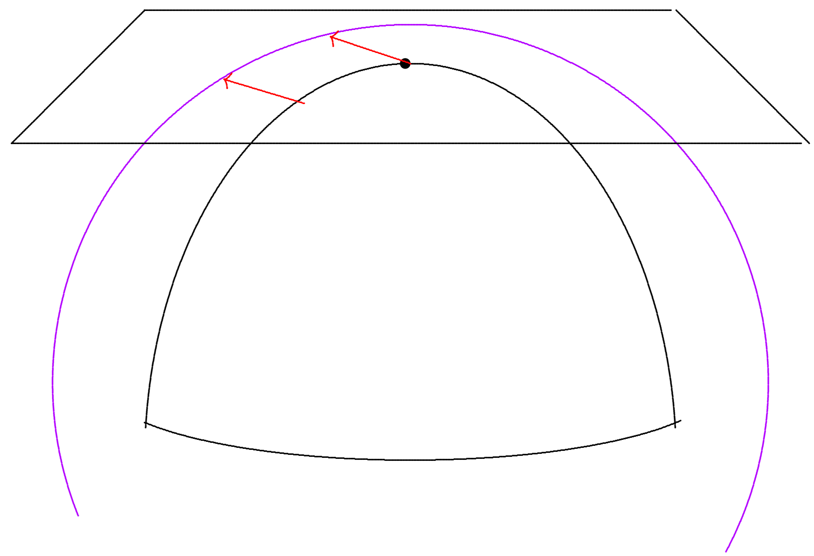

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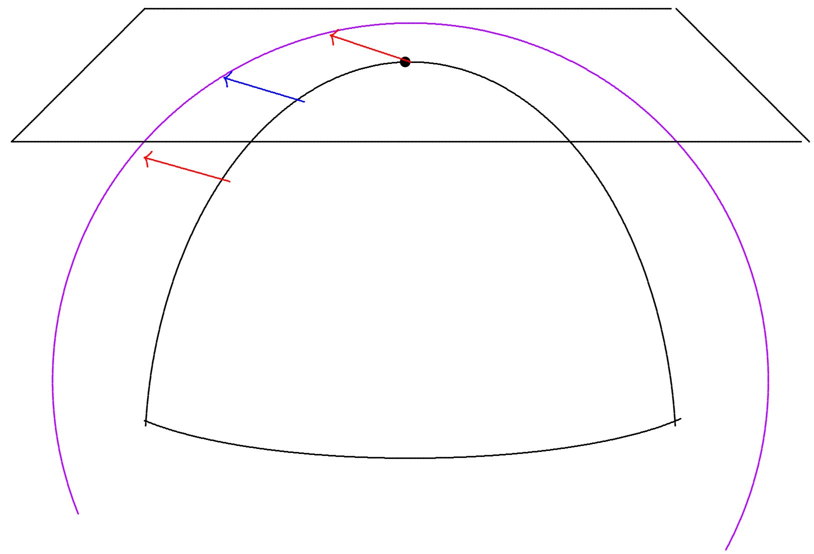

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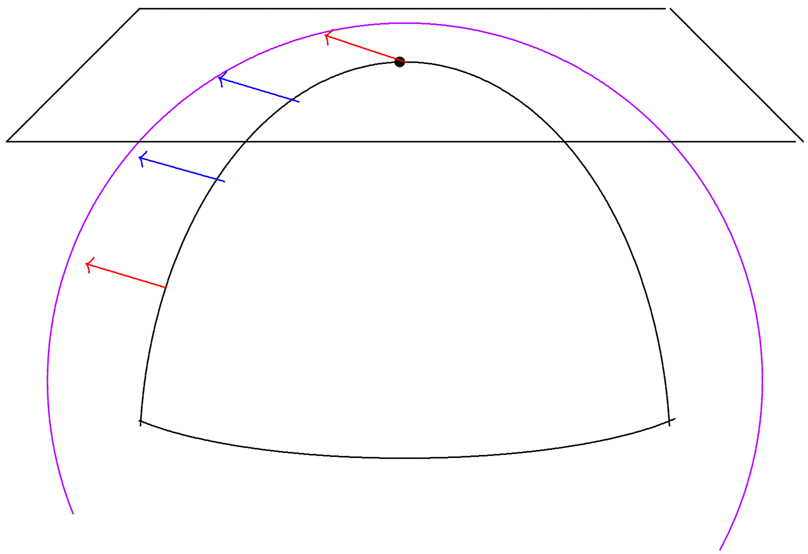

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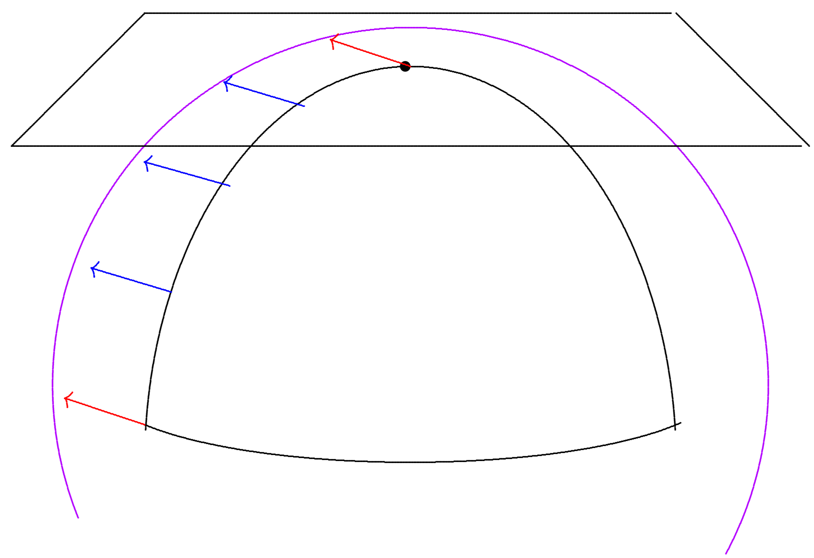

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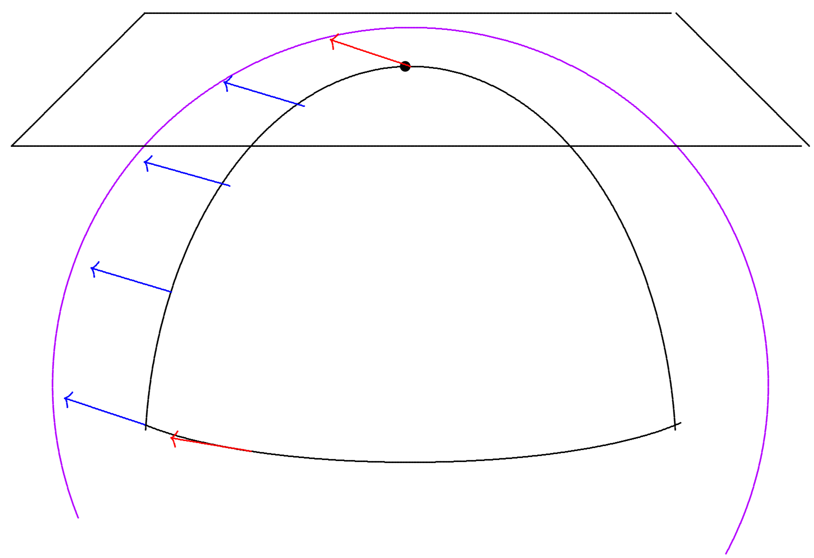

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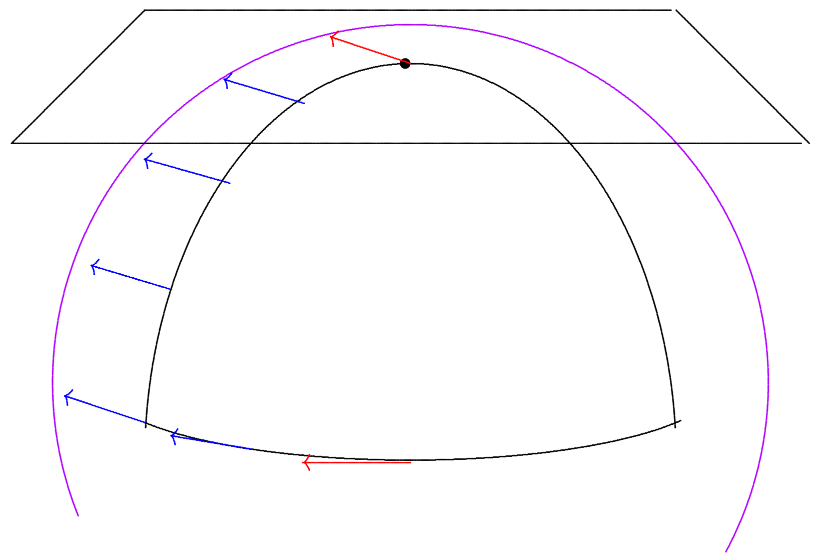

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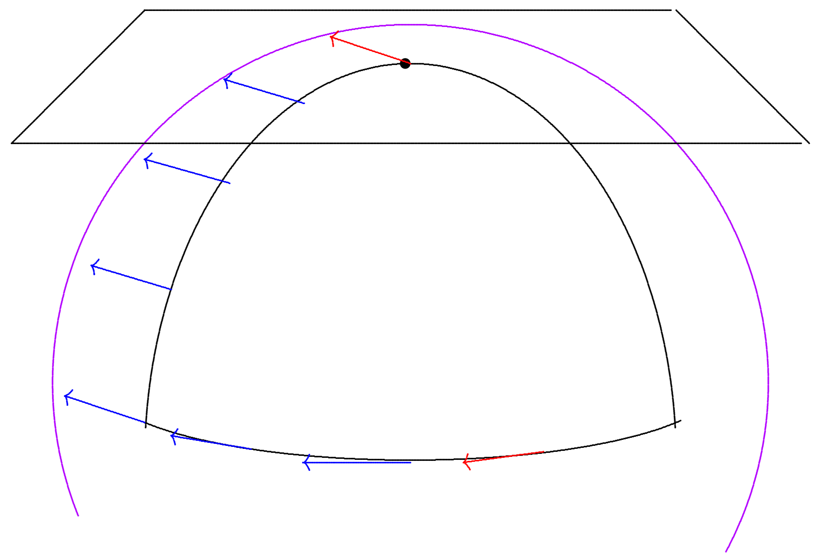

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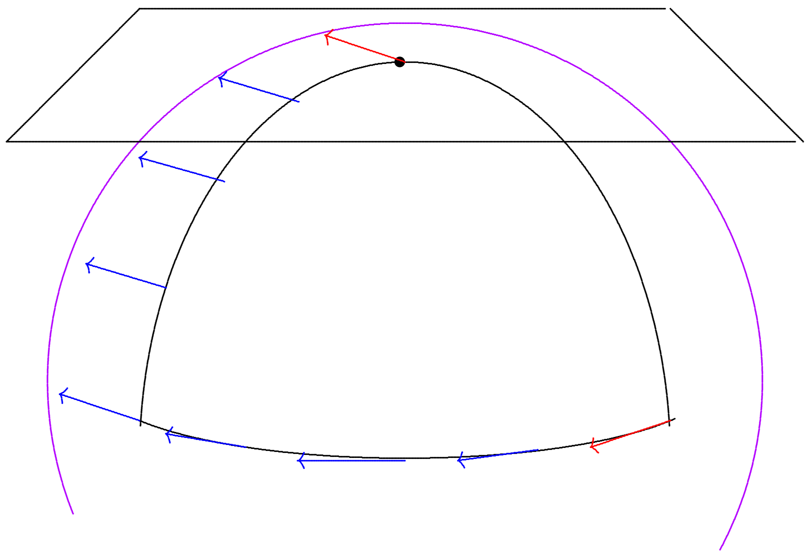

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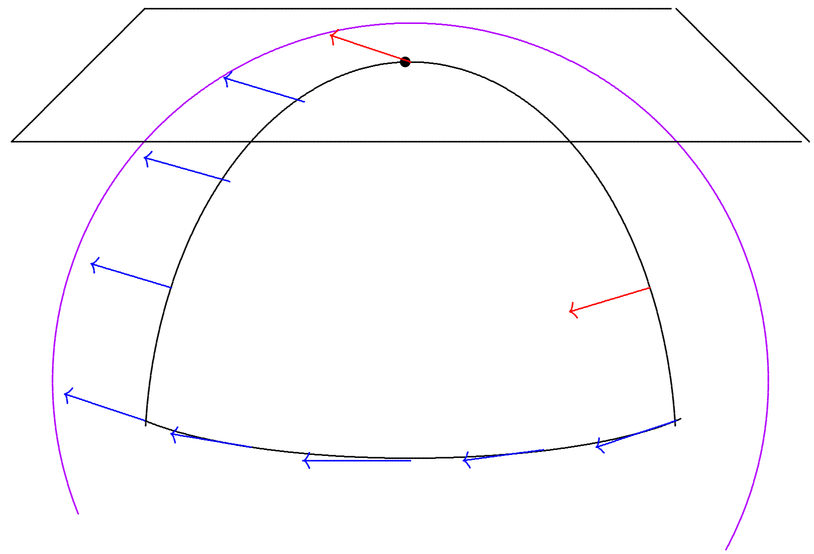

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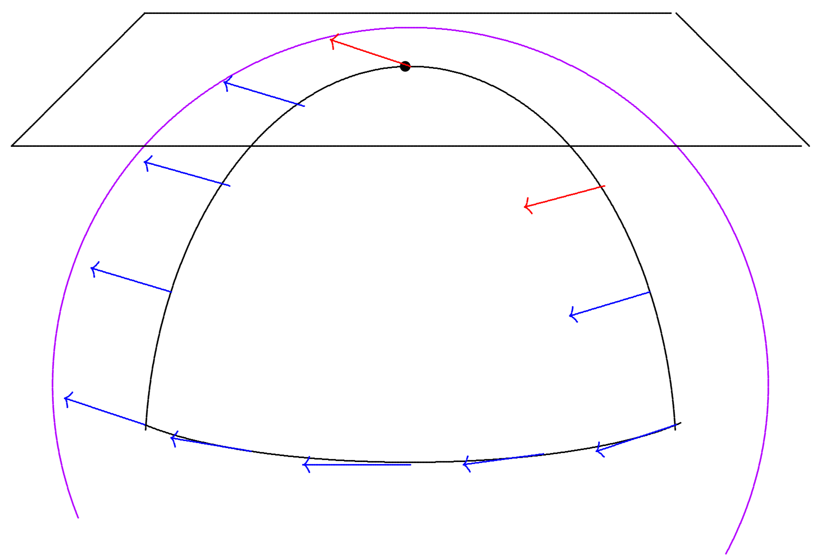

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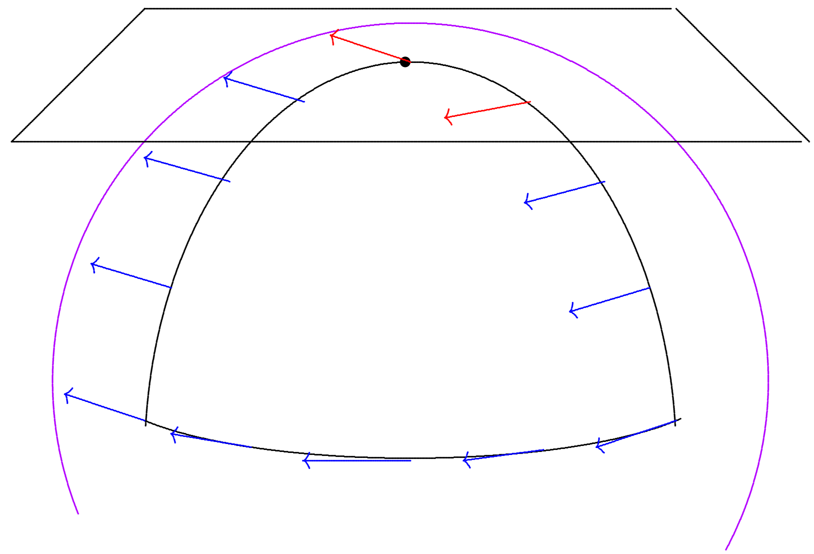

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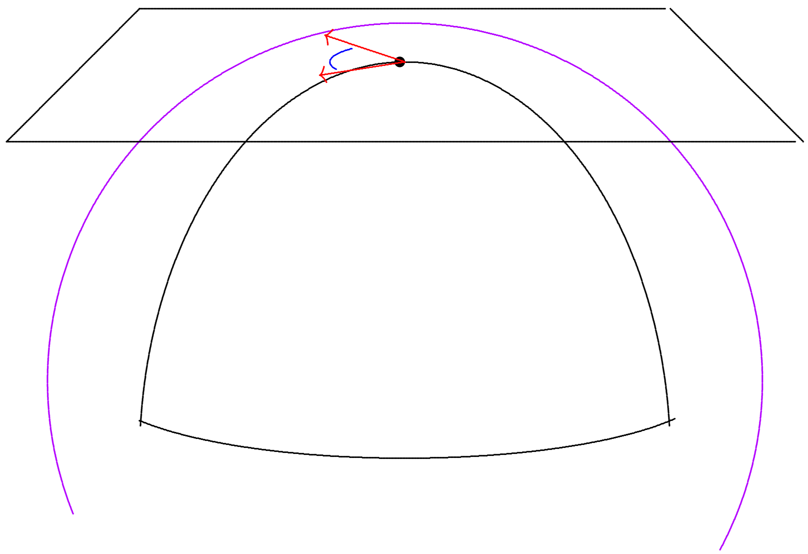

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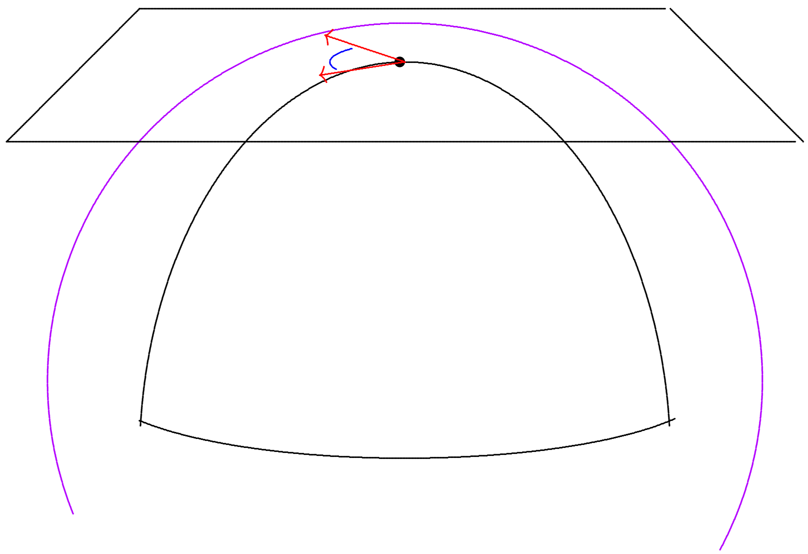

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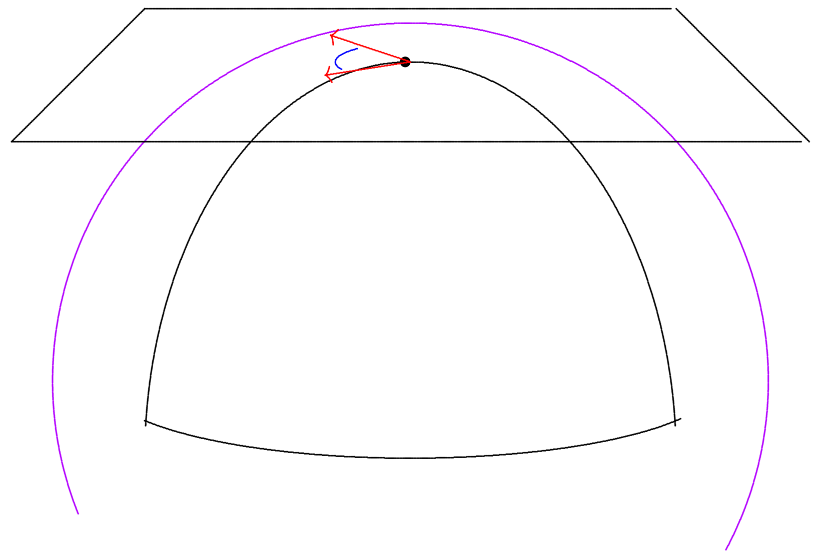

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Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

$$\begin{array}{c|c|c|}
 & \Lambda^{+*} & \Lambda^{-*} \\ \hline
\Lambda^{+} & W_+ + \frac{s}{12} & \mathring{r} \\ \hline
\Lambda^{-} & \mathring{r} & W_- + \frac{s}{12} \\ \hline
\end{array}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature *(conformally invariant)*

W_- = anti-self-dual Weyl curvature ″

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(M^4, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(2)$

$\Longleftrightarrow$ $\exists$ almost-complex structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

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and called the "Ricci form." Moreover, $i\rho$ is exactly the curvature of canonical line bundle $K = \Lambda^{2,0}$.

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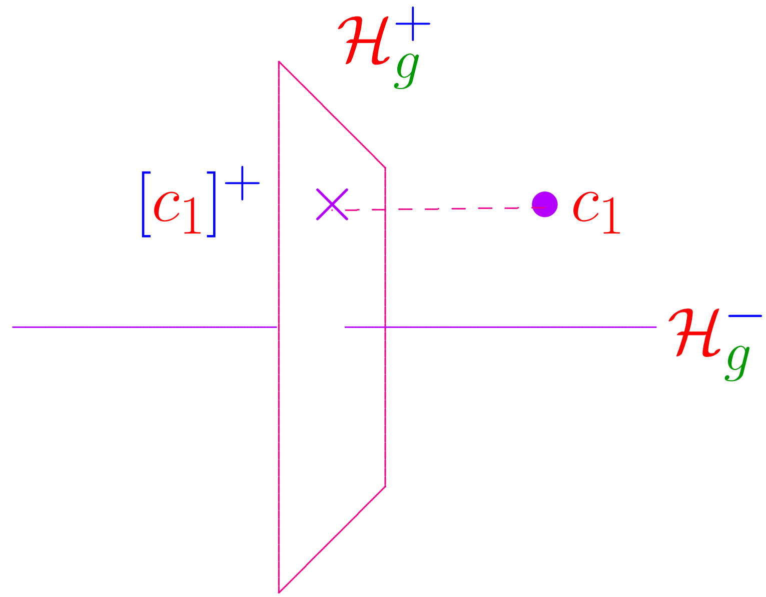
So Cauchy-Schwarz $\Longrightarrow$

$$\int_M s^2\, d\mu \geq 32\pi^2 \frac{(c_1 \cdot [\omega])^2}{[\omega]^2}$$

because $\int_M d\mu = [\omega]^2/2$.



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"Kähler Paradigms in a Riemannian World"

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Self-diffeomorphism unneeded if $b_+ > 1$ or $c_1{}^2 \geq 0$.

Proof involves a non-linear Dirac equation. . .

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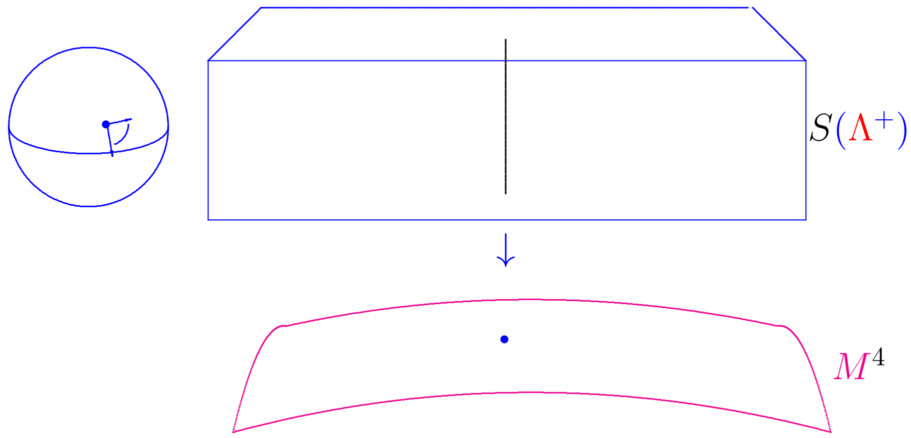

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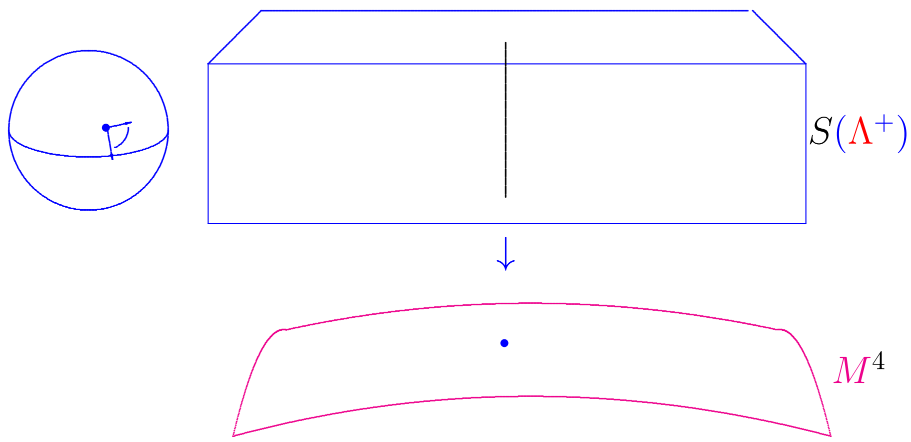

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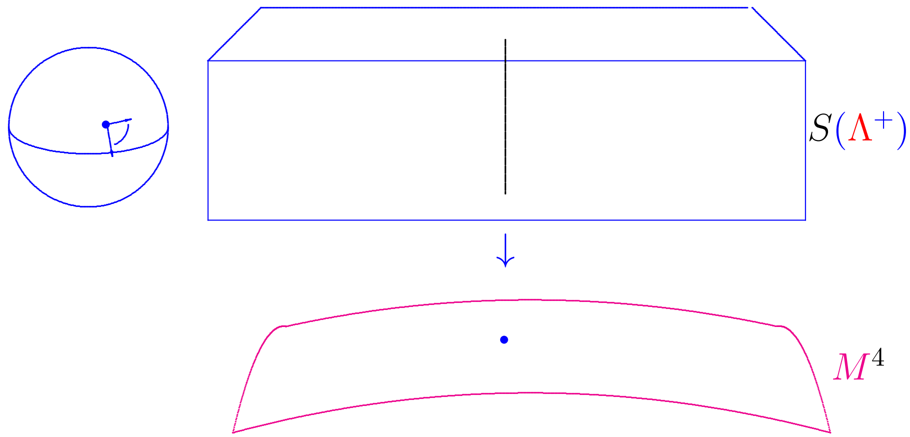

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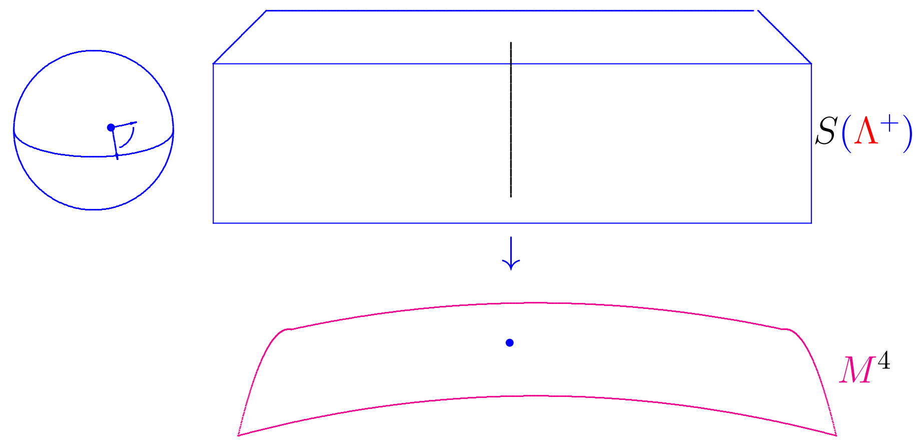

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Example. $\tau(K3) = -16$.

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Example. $\nexists$ metric of $s > 0$ on $K3$.

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where $\mathbb{S}_\pm$ are the (locally defined) left- and right-handed spinor bundles of (M, g).

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$$d^*(A - A_0) = 0$$

imposed to eliminate automorphisms of $L \to M$.

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Bootstrapping with gauge-fixed equations, one gets L^p_k bounds for (Φ, A) for all k, p.

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SW invariant $\in \mathbb{Z}_2$ means mod-2 mapping degree.

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Implies non-existence of metrics g for which $s > 0$.

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Same conclusion if M admits ω instead of J.

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Tomorrow: We will see that this is sharp!

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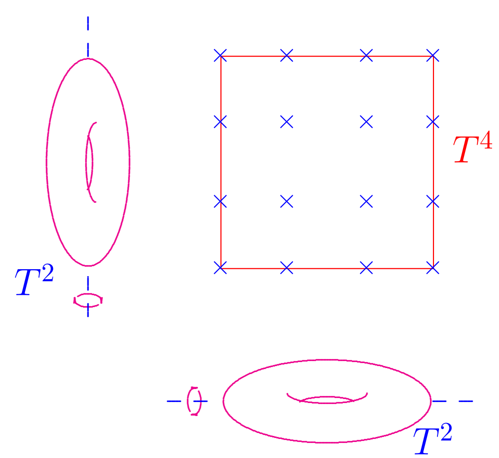
$K3$ = resolution of $T^4/\mathbb{Z}_2$:



When SW $\neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

J integrable, b_1 even $\Longrightarrow c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth M^4.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira “homotopy K3.”

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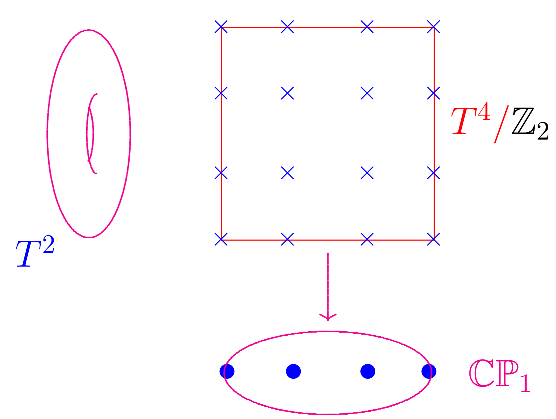

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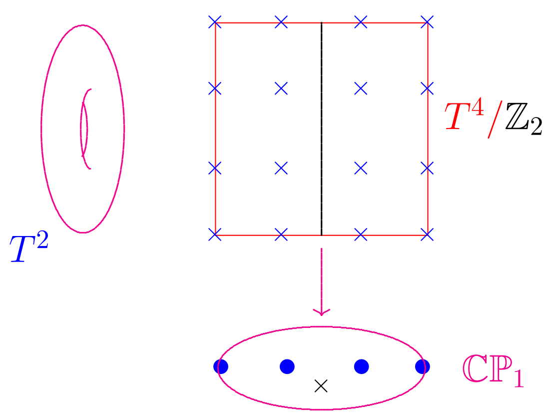

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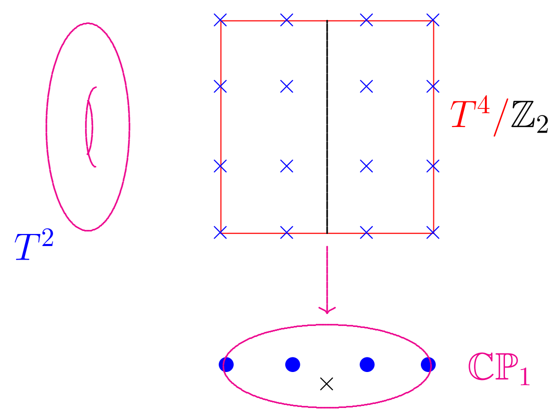
Replace chosen fiber T^2 with $T^2/\mathbb{Z}_{2\ell+1}$



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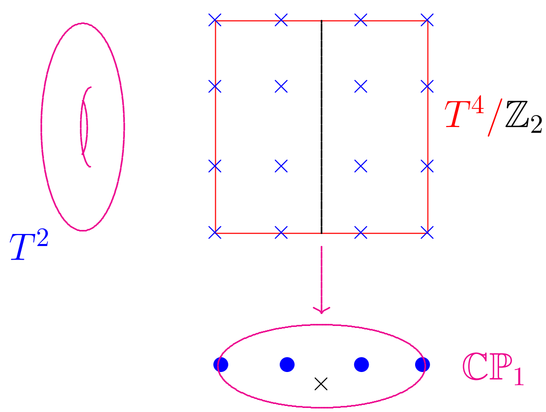
Key: $[T^2 \times (D^2 - \{0\})]/\mathbb{Z}_{2\ell+1} \cong T^2 \times (D^2 - \{0\})$.



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where $\mathfrak{f} \neq 0$ is homology class of new $T^2/\mathbb{Z}_{2\ell+1}$.

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As $\ell \to \infty$, get infinitely many different diffeomorphism types: if finite, divisibility would be bounded!

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Proposition. *The topological manifold $|K3|$ admits infinitely many smooth structures. Exactly one of these admits an Einstein metric.*

End, Part II