Four-Manifolds,

Einstein Metrics, &

Differential Topology

Claude LeBrun
Stony Brook University

Rademacher Lectures
University of Pennsylvania
Four-Manifolds,

*Einstein Metrics, &*

*Differential Topology, II*

Kähler Paradigms in a Riemannian World

October 20, 2016
University of Pennsylvania
Hitchin-Thorpe Inequality:

\[
(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{\|\tilde{r}\|^2}{2} \right) d\mu_g
\]

Einstein \Rightarrow \quad = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g

**Theorem** (Hitchin-Thorpe Inequality). If smooth compact oriented \( M^4 \) admits Einstein \( g \), then

\[
(2\chi + 3\tau)(M) \geq 0,
\]

with equality only if \( \Lambda^+ \) is flat on \((M, g)\). The latter happens only if \((M, g)\) finitely covered by a flat \( T^4 \) or by a Calabi-Yau \( K3 \).
Corollary. Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. 
Corollary. Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. Then $M$ does not admit Einstein metrics.
Corollary. Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. Then $M$ does not admit Einstein metrics.

Kodaira: $\exists$ complex surfaces that are homotopy equivalent to $K3$, but which have $c_1 \neq 0$. 
Corollary. Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. Then $M$ does not admit Einstein metrics.

Kodaira: $\exists$ complex surfaces that are homotopy equivalent to $K3$, but which have $c_1 \neq 0$.

(Of course, still have $c_1^2 = 2\chi + 3\tau = 0$.)
Corollary. Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. Then $M$ does not admit Einstein metrics.

Kodaira: ∃ complex surfaces that are homotopy equivalent to $K3$, but which have $c_1 \neq 0$.

(Of course, still have $c_1^2 = 2\chi + 3\tau = 0$.)

For any integer $\ell$, ∃ examples where $2\ell | c_1$. 
Corollary. Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. Then $M$ does not admit Einstein metrics.

Kodaira: $\exists$ complex surfaces that are homotopy equivalent to $K3$, but which have $c_1 \neq 0$.

(Of course, still have $c_1^2 = 2\chi + 3\tau = 0$.)

For any integer $\ell$, $\exists$ examples where $2\ell | c_1$.

Later today: Pairwise non-diffeomorphic, even though all are homeomorphic to $K3$. 
Corollary. Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. Then $M$ does not admit Einstein metrics.

Kodaira: $\exists$ complex surfaces that are homotopy equivalent to $K3$, but which have $c_1 \neq 0$.

(Of course, still have $c_1^2 = 2\chi + 3\tau = 0$.)

For any integer $\ell$, $\exists$ examples where $2\ell | c_1$.

Later today: Pairwise non-diffeomorphic, even though all are homeomorphic to $K3$.

$\because$ Topological manifold $|K3|$ has infinitely many smooth structures, but only one of these admits Einstein metrics.
However, don’t get too discouraged...
Many complex surfaces do admit Einstein metrics.
Many complex surfaces do admit Einstein metrics.

For example: Fermat surface of degree $\ell$ in $\mathbb{CP}^3$

$$t^\ell + u^\ell + v^\ell + w^\ell = 0$$
Many complex surfaces do admit Einstein metrics.

For example: Fermat surface of degree $\ell$ in $\mathbb{CP}^3$

\[ t^\ell + u^\ell + v^\ell + w^\ell = 0 \]
Many complex surfaces do admit Einstein metrics.

For example: Fermat surface of degree $\ell$ in $\mathbb{CP}_3$

$$t^\ell + u^\ell + v^\ell + w^\ell = 0$$
Many complex surfaces do admit Einstein metrics.

For example: Fermat surface of degree $\ell$ in $\mathbb{CP}^3$

$$t^\ell + u^\ell + v^\ell + w^\ell = 0$$

All carry Einstein metrics which are Kähler.
Many complex surfaces do admit Einstein metrics.

For example: Fermat surface of degree $\ell$ in $\mathbb{CP}_3$

\[ t^\ell + u^\ell + v^\ell + w^\ell = 0 \]

All carry Kähler-Einstein metrics.
Many complex surfaces do admit Einstein metrics.

For example: Fermat surface of degree $\ell$ in $\mathbb{CP}^3$

$$t^{\ell} + u^{\ell} + v^{\ell} + w^{\ell} = 0$$
Many complex surfaces do admit Einstein metrics.

For example: Fermat surface of degree $\ell$ in $\mathbb{CP}^3$

\[ t^\ell + u^\ell + v^\ell + w^\ell = 0 \]

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$M$</th>
<th>Einstein $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbb{CP}_2$</td>
<td>+</td>
</tr>
</tbody>
</table>

Fubini-Study
Many complex surfaces do admit Einstein metrics.

For example: **Fermat surface of degree** $\ell$ **in** $\mathbb{CP}_3$

\[ t^\ell + u^\ell + v^\ell + w^\ell = 0 \]

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$M$</th>
<th>Einstein $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbb{CP}_2$</td>
<td>+</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{CP}_1 \times \mathbb{CP}_1$</td>
<td>+</td>
</tr>
</tbody>
</table>

round $\times$ round
Many complex surfaces do admit Einstein metrics.

For example: Fermat surface of degree $\ell$ in $\mathbb{CP}^3$

$$t^\ell + u^\ell + v^\ell + w^\ell = 0$$

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$M$</th>
<th>Einstein $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbb{CP}^2$</td>
<td>$+$</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{CP}^1 \times \mathbb{CP}^1$</td>
<td>$+$</td>
</tr>
<tr>
<td>3</td>
<td>$\mathbb{CP}^2 # 6\overline{\mathbb{CP}^2}$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Siu/Tian
Many complex surfaces do admit Einstein metrics.

For example: Fermat surface of degree $\ell$ in $\mathbb{CP}_3$

$$t^\ell + u^\ell + v^\ell + w^\ell = 0$$

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$M$</th>
<th>Einstein $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbb{CP}_2$</td>
<td>+</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{CP}_1 \times \mathbb{CP}_1$</td>
<td>+</td>
</tr>
<tr>
<td>3</td>
<td>$\mathbb{CP}_2 # 6\mathbb{CP}_2$</td>
<td>+</td>
</tr>
<tr>
<td>4</td>
<td>$K3$</td>
<td>0</td>
</tr>
</tbody>
</table>
Many complex surfaces do admit Einstein metrics.

For example: Fermat surface of degree $\ell$ in $\mathbb{CP}^3$

$$t^\ell + u^\ell + v^\ell + w^\ell = 0$$

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$M$</th>
<th>Einstein $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbb{CP}^2$</td>
<td>$+$</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{CP}^1 \times \mathbb{CP}^1$</td>
<td>$+$</td>
</tr>
<tr>
<td>3</td>
<td>$\mathbb{CP}^2 # 6 \mathbb{CP}^2$</td>
<td>$+$</td>
</tr>
<tr>
<td>4</td>
<td>$K3$</td>
<td>0</td>
</tr>
<tr>
<td>$\geq 5$</td>
<td>“general type”</td>
<td>$-$</td>
</tr>
</tbody>
</table>

Aubin/Yau
Many complex surfaces do admit Einstein metrics.

For example: Fermat surface of degree \( \ell \) in \( \mathbb{CP}^3 \)

\[ t^\ell + u^\ell + v^\ell + w^\ell = 0 \]

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>( M )</th>
<th>Einstein ( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \mathbb{CP}^2 )</td>
<td>+</td>
</tr>
<tr>
<td>2</td>
<td>( \mathbb{CP}_1 \times \mathbb{CP}_1 )</td>
<td>+</td>
</tr>
<tr>
<td>3</td>
<td>( \mathbb{CP}_2 # 6 \overline{\mathbb{CP}_2} )</td>
<td>+</td>
</tr>
<tr>
<td>4</td>
<td>( K3 )</td>
<td>0</td>
</tr>
<tr>
<td>( \geq 5 )</td>
<td>“general type”</td>
<td>-</td>
</tr>
</tbody>
</table>

\[ \pi_1 = 0, \quad \chi = \ell(\ell^2 - 4\ell + 6), \quad \tau = -\frac{1}{3}\ell(\ell^2 - 4), \quad \text{spin} \Leftrightarrow \ell \text{ even.} \]
Case of high degree $\ell \geq 5$: 
Case of high degree $\ell \geq 5$:

**Theorem (Aubin/Yau).** Compact complex manifold $(M^{2m}, J)$ admits compatible Kähler-Einstein metric with $\lambda < 0 \iff "c_1 < 0."$
Case of high degree $\ell \geq 5$:

**Theorem (Aubin/Yau).** Compact complex manifold $(M^{2m}, J)$ admits compatible Kähler-Einstein metric with $\lambda < 0 \iff -c_1(M)$ a Kähler class.
Case of high degree $\ell \geq 5$:

**Theorem (Aubin/Yau).** Compact complex manifold $(M^{2m}, J)$ admits compatible Kähler-Einstein metric with $\lambda < 0 \iff \exists$ holomorphic embedding

$$j : M \hookrightarrow \mathbb{CP}_k$$

such that $c_1(M)$ is negative multiple of $j^*c_1(\mathbb{CP}_k)$. 
Case of high degree $\ell \geq 5$:

**Theorem** (Aubin/Yau). Compact complex manifold $(M^{2m}, J)$ admits compatible Kähler-Einstein metric with $\lambda < 0 \iff \exists$ holomorphic embedding

$$j : M \hookrightarrow \mathbb{CP}_k$$

such that $c_1(M)$ is negative multiple of $j^*c_1(\mathbb{CP}_k)$.

*(Kodaira embedding theorem)*
Many complex surfaces do admit Einstein metrics.
Many complex surfaces do admit Einstein metrics.

**Question.** If \((M^4, J)\) is a compact complex surface, when does \(M\) admit an *Einstein metric* \(g\) (unrelated to \(J\))?
Many complex surfaces do admit Einstein metrics.

**Question.** If $(M^4, J)$ is a compact complex surface, when does $M$ admit an Einstein metric $g$ (unrelated to $J$)?

**Question.** When this happens, must $g$ be Kähler (but perhaps adapted to some other $J$)?
Many complex surfaces do admit Einstein metrics.

**Question.** If \((M^4, J)\) is a compact complex surface, when does \(M\) admit an Einstein metric \(g\) (unrelated to \(J\))?

**Question.** When this happens, must \(g\) be Kähler (but perhaps adapted to some other \(J\))?

These questions will be our main focus...
Question. Which smooth compact 4-manifolds $M^4$ admit Einstein metrics?
Question. Which smooth compact 4-manifolds $M^4$ admit Einstein metrics?

Complex geometry is rich source of examples.
Question. Which smooth compact 4-manifolds $M^4$ admit Einstein metrics?

Complex geometry is rich source of examples.

On suitable 4-manifolds, Seiberg-Witten theory allows one to mimic Kähler geometry when treating non-Kähler metrics.
**Question.** Which smooth compact 4-manifolds $M^4$ admit Einstein metrics?

Complex geometry is rich source of examples.

On suitable 4-manifolds, Seiberg-Witten theory allows one to mimic Kähler geometry when treating non-Kähler metrics.

**Our Focus.** Suppose $(M^4, J)$ is a compact complex surface. When does $M^4$ admit an Einstein metric $g$, perhaps completely unrelated to $J$?
Kodaira Classification
Kodaira Classification of Complex Surfaces


Kodaira Classification of Complex Surfaces

Most important invariant: Kodaira dimension.
Kodaira Classification of Complex Surfaces

Most important invariant: Kodaira dimension.

Given $(M^4, J)$ compact complex surface,
Kodaira Classification of Complex Surfaces

Most important invariant: Kodaira dimension.

Given $(M^4, J)$ compact complex surface, set

\[
\text{Kod}(M) = \limsup_{\ell \to +\infty} \frac{\log \dim \Gamma(M, \mathcal{O}(K \otimes \ell))}{\log \ell}
\]
Kodaira Classification of Complex Surfaces

Most important invariant: Kodaira dimension.

Given $(M^4, J)$ compact complex surface, set

$$\text{Kod}(M) = \limsup_{\ell \to +\infty} \frac{\log \dim \Gamma(M, \mathcal{O}(K \otimes \ell))}{\log \ell}$$

where $K = \Lambda^{2,0}$ is canonical line bundle.
**Kodaira Classification** of Complex Surfaces

Most important invariant: Kodaira dimension.

Given \((M^4, J)\) compact complex surface, set

\[
\text{Kod}(M) = \limsup_{\ell \to +\infty} \frac{\log \dim \Gamma(M, \mathcal{O}(K \otimes \ell))}{\log \ell}
\]

where \(K = \Lambda^{2,0}\) is canonical line bundle.

Then \(\text{Kod}(M, J) \in \{-\infty, 0, 1, 2\}\)
Kodaira Classification of Complex Surfaces

Most important invariant: Kodaira dimension.

Given \((M^4, J)\) compact complex surface, set

\[
\text{Kod}(M) = \limsup_{\ell \to +\infty} \frac{\log \dim \Gamma(M, \mathcal{O}(K \otimes \ell))}{\log \ell}
\]

where \(K = \Lambda^{2,0}\) is canonical line bundle.

Then \(\text{Kod}(M, J) \in \{-\infty, 0, 1, 2\}\) is exactly

\[
\max \dim_{\mathbb{C}} \text{Image}(M \to \mathbb{CP}_N)
\]
Kodaira Classification of Complex Surfaces

Most important invariant: Kodaira dimension.

Given \((M^4, J)\) compact complex surface, set

\[
\text{Kod}(M) = \limsup_{\ell \to +\infty} \frac{\log \dim \Gamma(M, \mathcal{O}(K \otimes \ell))}{\log \ell}
\]

where \(K = \Lambda^{2,0}\) is canonical line bundle.

Then \(\text{Kod}(M, J) \in \{-\infty, 0, 1, 2\}\) is exactly

\[
\max \dim \mathbb{C} \text{ Image}(M \dashrightarrow \mathbb{CP}_N)
\]

over maps defined by holomorphic sections of \(K \otimes \ell\).
Blowing up:
Blowing up:

If $N$ is a complex surface,
Blowing up:

If $N$ is a complex surface, may replace $p \in N$
Blowing up:

If $N$ is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$
Blowing up:

If $N$ is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \mathbb{CP}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.
Blowing up:

If $N$ is a complex surface, may replace $p \in N$ with $\mathbb{C}P_1$ to obtain blow-up

$$M \approx N \# \mathbb{C}P_2$$

in which added $\mathbb{C}P_1$ has normal bundle $\mathcal{O}(-1)$. 
Blowing up:

If $N$ is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \mathbb{CP}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$. 

![Diagram showing blow-up process](image-url)
Blowing up:

If $N$ is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \mathbb{CP}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$. 
A complex surface $X$ is called minimal.
A complex surface $X$ is called minimal if it is not the blow-up of another complex surface.
A complex surface \( X \) is called \textit{minimal} if it is not the blow-up of another complex surface.
A complex surface $X$ is called \textit{minimal} if it is not the blow-up of another complex surface.
A complex surface $X$ is called **minimal** if it is not the blow-up of another complex surface.

Any complex surface $M$ can be obtained from a minimal surface $X$. 
A complex surface $X$ is called **minimal** if it is not the blow-up of another complex surface.

Any complex surface $M$ can be obtained from a minimal surface $X$ by blowing up a finite number of times:

$$M \approx X \# k\overline{\mathbb{C}P_2}$$
A complex surface $X$ is called minimal if it is not the blow-up of another complex surface.

Any complex surface $M$ can be obtained from a minimal surface $X$ by blowing up a finite number of times:

$$M \approx X \# k \mathbb{CP}_2$$

One says that $X$ is minimal model of $M$. 

A complex surface $X$ is called \textit{minimal} if it is not the blow-up of another complex surface.

Any complex surface $M$ can be obtained from a minimal surface $X$ by blowing up a finite number of times:

$$M \approx X \# k \mathbb{CP}^2$$

One says that $X$ is \textit{minimal model} of $M$.

The minimal model $X$ of $M$ is \textit{unique} if

$$\text{Kod}(M) \geq 0.$$
A complex surface $X$ is called minimal if it is not the blow-up of another complex surface.

Any complex surface $M$ can be obtained from a minimal surface $X$ by blowing up a finite number of times:

$$M \approx X \# k\overline{\mathbb{CP}^2}$$

One says that $X$ is minimal model of $M$.

The minimal model $X$ of $M$ is unique if

$$\text{Kod}(M) \geq 0.$$ 

Moreover, always have

$$\text{Kod}(X) = \text{Kod}(M),$$
A complex surface $X$ is called minimal if it is not the blow-up of another complex surface.

Any complex surface $M$ can be obtained from a minimal surface $X$ by blowing up a finite number of times:

$$M \approx X \# k\mathbb{CP}^2$$

One says that $X$ is minimal model of $M$.

The minimal model $X$ of $M$ is unique if

$$\text{Kod}(M) \geq 0.$$ 

Moreover, always have

$$\text{Kod}(X) = \text{Kod}(M),$$

and Kod invariant under deformations.
Kodaira Classification of Minimal Surfaces
Kodaira Classification of Minimal Surfaces

For $b_1$ even:
Kodaira Classification of Minimal Surfaces

For $b_1$ even:

<table>
<thead>
<tr>
<th>Kod($X$)</th>
<th>$X$</th>
<th>$c_1^2(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Kodaira Classification of Minimal Surfaces

For $b_1$ even:

<table>
<thead>
<tr>
<th>$\text{Kod}(X)$</th>
<th>$X$</th>
<th>$c_1^2(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>$\mathbb{CP}_2$, and $\mathbb{CP}_1$ bundles over curves</td>
<td>$+, 0, -$</td>
</tr>
</tbody>
</table>
Kodaira Classification of Minimal Surfaces

For $b_1$ even:

<table>
<thead>
<tr>
<th>$\text{Kod}(X)$</th>
<th>$X$</th>
<th>$c_1^2(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>$\mathbb{CP}_2$, and $\mathbb{CP}_1$ bundles over curves</td>
<td>$+, 0, -$</td>
</tr>
<tr>
<td>$0$</td>
<td>$K3$, $T^4$, and quotients</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Kodaira Classification of Minimal Surfaces

For $b_1$ even:

<table>
<thead>
<tr>
<th>$\text{Kod}(X)$</th>
<th>$X$</th>
<th>$c_1^2(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>$\mathbb{CP}_2$, and $\mathbb{CP}_1$ bundles over curves</td>
<td>$+, 0, -$</td>
</tr>
<tr>
<td>0</td>
<td>$K3$, $T^4$, and quotients</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>most elliptic fibrations over curves</td>
<td>0</td>
</tr>
</tbody>
</table>
**Kodaira Classification of Minimal Surfaces**

For $b_1$ even:

<table>
<thead>
<tr>
<th>$\text{Kod}(X)$</th>
<th>$X$</th>
<th>$c_1^2(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>$\mathbb{CP}_2$, and $\mathbb{CP}_1$ bundles over curves</td>
<td>$+, 0, -$</td>
</tr>
<tr>
<td>0</td>
<td>$K3$, $T^4$, and quotients</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>most elliptic fibrations over curves</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>“general type”</td>
<td>$+$</td>
</tr>
</tbody>
</table>
Kodaira Classification of Minimal Surfaces
Kodaira Classification of Minimal Surfaces

For $b_1$ odd:
Kodaira Classification of Minimal Surfaces

For $b_1$ odd:

<table>
<thead>
<tr>
<th>$\text{Kod}(X)$</th>
<th>$X$</th>
<th>$c_1^2(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Kodaira Classification of Minimal Surfaces

For $b_1$ odd:

<table>
<thead>
<tr>
<th>$\text{Kod}(X)$</th>
<th>$X$</th>
<th>$c_1^2(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>“Type VII”</td>
<td>$0, -$</td>
</tr>
</tbody>
</table>
Kodaira Classification of Minimal Surfaces

For $b_1$ odd:

<table>
<thead>
<tr>
<th>$\text{Kod}(X)$</th>
<th>$X$</th>
<th>$c_1^2(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>“Type VII”</td>
<td>0, −</td>
</tr>
<tr>
<td>0</td>
<td>certain $T^2$ bundles over $T^2$</td>
<td>0</td>
</tr>
</tbody>
</table>
Kodaira Classification of Minimal Surfaces

For $b_1$ odd:

<table>
<thead>
<tr>
<th>$\text{Kod}(X)$</th>
<th>$X$</th>
<th>$c_1^2(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>“Type VII”</td>
<td>0, $-$</td>
</tr>
<tr>
<td>0</td>
<td>certain $T^2$ bundles over $T^2$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>certain elliptic fibrations over curves</td>
<td>0</td>
</tr>
</tbody>
</table>
Theorem.
Theorem. Let $(M^4, J)$ be a compact complex surface,
Theorem. Let \((M^4, J)\) be a compact complex surface, and suppose that \(M\) admits an Einstein metric \(g\).
Theorem. Let \((M^4, J)\) be a compact complex surface, and suppose that \(M\) admits an Einstein metric \(g\) (not assumed to be related to \(J\) in any way).
**Theorem.** Let \((M^4, J)\) be a compact complex surface, and suppose that \(M\) admits an Einstein metric \(g\) (not assumed to be related to \(J\) in any way). Then either
Theorem. Let \((M^4, J)\) be a compact complex surface, and suppose that \(M\) admits an Einstein metric \(g\) (not assumed to be related to \(J\) in any way). Then either

- \(M\) is diffeomorphic to \(S^2 \times S^2\);
Theorem. Let \((M^4, J)\) be a compact complex surface, and suppose that \(M\) admits an Einstein metric \(g\) (not assumed to be related to \(J\) in any way). Then either

- \(M \approx S^2 \times S^2\); or
Theorem. Let $(M^4, J)$ be a compact complex surface, and suppose that $M$ admits an Einstein metric $g$ (not assumed to be related to $J$ in any way). Then either

- $M \cong S^2 \times S^2$; or
- $M \cong \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, where $0 \leq k \leq 8$; or
Theorem. Let \((M^4, J)\) be a compact complex surface, and suppose that \(M\) admits an Einstein metric \(g\) (not assumed to be related to \(J\) in any way). Then either

- \(M \cong S^2 \times S^2\); or
- \(M \cong \mathbb{CP}_2 \# k\mathbb{CP}_2\), where \(0 \leq k \leq 8\); or
- \(M\) is is finitely covered by \(T^4\); or
Theorem. Let \((M^4, J)\) be a compact complex surface, and suppose that \(M\) admits an Einstein metric \(g\) (not assumed to be related to \(J\) in any way). Then either

- \(M \cong S^2 \times S^2\); or
- \(M \cong \mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}\), where \(0 \leq k \leq 8\); or
- \(M\) is is finitely covered by \(T^4\); or
- \(M\) is is finitely covered by \(K3\); or
Theorem. Let $(M^4, J)$ be a compact complex surface, and suppose that $M$ admits an Einstein metric $g$ (not assumed to be related to $J$ in any way). Then either

- $M \approx S^2 \times S^2$; or
- $M \approx \mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}$, where $0 \leq k \leq 8$; or
- $M$ is is finitely covered by $T^4$; or
- $M$ is is finitely covered by $K3$; or
- $(M, J)$ is of general type.
Theorem. Let \((M^4, J)\) be a compact complex surface, and suppose that \(M\) admits an Einstein metric \(g\) (not assumed to be related to \(J\) in any way). Then either

- \(M \cong S^2 \times S^2\); or
- \(M \cong \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}\), where \(0 \leq k \leq 8\); or
- \(M\) is is finitely covered by \(T^4\); or
- \(M\) is is finitely covered by \(K3\); or
- \((M, J)\) is of general type.

Moreover, \(M\) admits Kähler metrics, and so in particular admits symplectic structures.
Theorem. Let \((M^4, J)\) be a compact complex surface, and suppose that \(M\) admits an Einstein metric \(g\) (not assumed to be related to \(J\) in any way). Then either
- \(M \approx S^2 \times S^2\); or
- \(M \approx \mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}\), where \(0 \leq k \leq 8\); or
- \(M\) is is finitely covered by \(T^4\); or
- \(M\) is is finitely covered by \(K3\); or
- \((M, J)\) is of general type.

Moreover, \(M\) admits Kähler metrics, and so in particular admits symplectic structures.

Symplectic structure:

2-form \(\omega\) with \(d\omega = 0\) and \(\omega \wedge \omega > 0\).
Theorem. Let \((M^4, J)\) be a compact complex surface, and suppose that \(M\) admits an Einstein metric \(g\) (not assumed to be related to \(J\) in any way). Then either

- \(M \cong S^2 \times S^2\); or
- \(M \cong \mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2\), where \(0 \leq k \leq 8\); or
- \(M\) is finitely covered by \(T^4\); or
- \(M\) is finitely covered by \(K3\); or
- \((M, J)\) is of general type.

Moreover, \(M\) admits Kähler metrics, and so in particular admits symplectic structures.
**Theorem.** Let $(M^4, J)$ be a compact complex surface, and suppose that $M$ admits an Einstein metric $g$ (not assumed to be related to $J$ in any way). Then either

- $M \approx S^2 \times S^2$; or
- $M \approx \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, where $0 \leq k \leq 8$; or
- $M$ is is finitely covered by $T^4$; or
- $M$ is is finitely covered by $K3$; or
- $(M, J)$ is of general type.

Moreover, $M$ admits Kähler metrics, and so in particular admits symplectic structures.

**Proof:**
**Theorem.** Let $(M^4, J)$ be a compact complex surface, and suppose that $M$ admits an Einstein metric $g$ (not assumed to be related to $J$ in any way). Then either

- $M \approx S^2 \times S^2$; or
- $M \approx \mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}$, where $0 \leq k \leq 8$; or
- $M$ is is finitely covered by $T^4$; or
- $M$ is is finitely covered by $K3$; or
- $(M, J)$ is of general type.

Moreover, $M$ admits Kähler metrics, and so in particular admits symplectic structures.

Proof: Hitchin-Thorpe!
Theorem. Let \((M^4, J)\) be a compact complex surface, and suppose that \(M\) admits an Einstein metric \(g\) (not assumed to be related to \(J\) in any way). Then either

- \(M \cong S^2 \times S^2\); or
- \(M \cong \mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}\), where \(0 \leq k \leq 8\); or
- \(M\) is finitely covered by \(T^4\); or
- \(M\) is finitely covered by \(K3\); or
- \((M, J)\) is of general type.

Moreover, \(M\) admits Kähler metrics, and so in particular admits symplectic structures.

Proof: Hitchin-Thorpe!

\[ c_1^2 = 2\chi + 3\tau \] decreases under blowing up.
**Theorem.** Let \((M^4, J)\) be a compact complex surface, and suppose that \(M\) admits an Einstein metric \(g\) (not assumed to be related to \(J\) in any way). Then either

- \(M \approx S^2 \times S^2\); or
- \(M \approx \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}\), where \(0 \leq k \leq 8\); or
- \(M\) is is finitely covered by \(T^4\); or
- \(M\) is is finitely covered by \(K3\); or
- \((M, J)\) is of general type.

Moreover, \(M\) admits Kähler metrics, and so in particular admits symplectic structures.

**Proof:** Hitchin-Thorpe!

\(c_1^2 = 2\chi + 3\tau\) decreases under blowing up.

\(\therefore\) Minimal model must have \(c_1^2 \geq 0\ldots\)
This statement is not sharp, and will be improved.
This statement is not sharp, and will be improved.

Will also discuss results in the converse direction.
This statement is not sharp, and will be improved.

Will also discuss results in the converse direction.

But first we need to develop some new tools!
This statement is not sharp, and will be improved.

Will also discuss results in the converse direction.

But first we need to develop some new tools!

Let’s think more about Riemannian 4-manifolds. . .
Special nature of dimension 4:
Special nature of dimension 4:

The Lie group $\text{SO}(4)$ is not simple:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$
Special nature of dimension 4:

The Lie group $\textbf{SO}(4)$ is not simple:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

$$\mathbb{R}^4 = \mathbb{H} = \{\text{quaternions}\}$$
Special nature of dimension 4:

The Lie group $\textbf{SO}(4)$ is not simple:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

\[ \mathbb{R}^4 = \mathbb{H} = \{\text{quaternions}\} \]

\[ \text{Sp}(1) = S^3 \subset \mathbb{H}^\times \text{ multiplicative group}. \]
Special nature of dimension 4:

The Lie group $\text{SO}(4)$ is not simple:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

$\mathbb{R}^4 = \mathbb{H} = \{\text{quaternions}\}$

$\text{Sp}(1) = S^3 \subset \mathbb{H} \times \text{multiplicative group.}$

Left & right multiplication $\rightsquigarrow$
Special nature of dimension 4:

The Lie group $\mathbf{SO}(4)$ is not simple:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

$\mathbb{R}^4 = \mathbb{H} = \{\text{quaternions}\}$

$\mathbf{Sp}(1) = S^3 \subset \mathbb{H} \times$ multiplicative group.

Left & right multiplication $\leadsto$

$$\mathbb{Z}_2 \hookrightarrow \mathbf{Sp}(1) \times \mathbf{Sp}(1) \downarrow \mathbf{SO}(4)$$
Special nature of dimension 4:

The Lie group $\textbf{SO}(4)$ is \textit{not simple}:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

$$\tilde{\textbf{SO}}(4) = \textbf{Sp}(1) \times \textbf{Sp}(1)$$
Special nature of dimension 4:

The Lie group $\text{SO}(4)$ is not simple:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

$$\tilde{\text{SO}}(4) = \text{Sp}(1) \times \text{Sp}(1)$$

$$\tilde{\text{SO}}(3) = \text{Sp}(1)$$
Special nature of dimension 4:

The Lie group $\textbf{SO}(4)$ is not simple:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

$$\text{Spin}(4) = \text{Sp}(1) \times \text{Sp}(1)$$

$$\text{Spin}(3) = \text{Sp}(1)$$
Special nature of dimension 4:

The Lie group $\text{SO}(4)$ is not simple:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$
Special nature of dimension 4:

The Lie group \( \text{SO}(4) \) is \textit{not simple}:

\[
\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).
\]

On oriented \((M^4, g)\), \(\implies\)

\[
\Lambda^2 = \Lambda^+ \oplus \Lambda^-
\]

where \(\Lambda^\pm\) are \((\pm 1)\)-eigenspaces of

\[
\star : \Lambda^2 \to \Lambda^2,
\]

\[
\star^2 = 1.
\]
Special nature of dimension 4:

The Lie group $\mathbf{SO}(4)$ is not simple:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented $(M^4, g)$, $\implies$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^{\pm}$ are $(\pm 1)$-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

$\Lambda^+$ self-dual 2-forms.

$\Lambda^-$ anti-self-dual 2-forms.
Recall that $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing
Recall that $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$( [\varphi], [\psi] ) \quad \mapsto \quad \int_M \varphi \wedge \psi$$
Recall that $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$(\left[\varphi\right], \left[\psi\right]) \mapsto \int_M \varphi \wedge \psi$$

Diagonalize:
Recall that $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \rightarrow \mathbb{R} \quad \mapsto \int_M \varphi \wedge \psi$$

Diagonalize:

$$\begin{pmatrix}
+1 \\
\ldots \\
+1 \\
-1 \\
\ldots \\
-1
\end{pmatrix}.$$
Recall that \( \tau(M) = b_+(M) - b_-(M) \) defined in terms of intersection pairing

\[
H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \rightarrow \mathbb{R}
\]

\[
( [\varphi] , [\psi] ) \mapsto \int_M \varphi \wedge \psi
\]

Diagonalize:

\[
\begin{bmatrix}
+1 \\
\vdots \\
+1 \\
b_+(M) \\
b_-(M)
\end{bmatrix}
\]
Hodge theory:

\[ H^2(M, \mathbb{R}) = \{ \varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, \ d\star \varphi = 0 \}. \]
Hodge theory:

\[ H^2(M, \mathbb{R}) = \{ \varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, \; d \star \varphi = 0 \}. \]

Since \( \star \) is involution of RHS, \( \implies \)

\[ H^2(M, \mathbb{R}) = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-, \]
Hodge theory:

\[ H^2(M, \mathbb{R}) = \{ \varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, \ d \ast \varphi = 0 \}. \]

Since \( \ast \) is involution of RHS, \( \implies \)

\[ H^2(M, \mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g, \]

where

\[ \mathcal{H}^\pm_g = \{ \varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0 \} \]

self-dual & anti-self-dual harmonic forms.
Hodge theory:

\[ H^2(M, \mathbb{R}) = \{ \varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, \ d\star\varphi = 0 \}. \]

Since \( \star \) is involution of RHS, \( \implies \)

\[ H^2(M, \mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g, \]

where

\[ \mathcal{H}^\pm_g = \{ \varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0 \} \]

self-dual & anti-self-dual harmonic forms. Then

\[ b^\pm(M) = \dim \mathcal{H}^\pm_g. \]
\[ \mathcal{H}^+_g \] \[ \mathcal{H}^-_g \]

\[ H^2(M, \mathbb{R}) \]
\{ a \mid a \cdot a = 0 \} \subset H^2(M, \mathbb{R})
\( \{ a \mid a \cdot a = 0 \} \subset H^2(M, \mathbb{R}) \)
\{ a \mid a \cdot a = 0 \} \subset H^2(M, \mathbb{R})
Special nature of dimension 4:

The Lie group $\mathbf{SO}(4)$ is \textit{not simple}:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented $(M^4, g)$, $\implies$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^{\pm}$ are $(\pm 1)$-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

$\Lambda^+$ self-dual 2-forms.

$\Lambda^-$ anti-self-dual 2-forms.
Kähler metrics:

\[(M^4, g) \text{ Kähler } \iff \text{holonomy } \subset U(2)\]
Kähler metrics:

$$(M^4, g) \text{ Kähler} \iff \text{holonomy } \subset \mathbf{U}(2)$$
Kähler metrics:

\((M^4, g) \text{ Kähler } \iff \text{ holonomy } \subset U(2)\)
Kähler metrics:

$$(M^4, g) \text{ Kähler } \iff \text{ holonomy } \subset U(2)$$
Kähler metrics:

$$(M^4, g) \text{ Kähler } \iff \text{holonomy } \subset \mathbf{U}(2)$$
Kähler metrics:

\((M^4, g) \text{ Kähler } \iff \text{holonomy } \subset \mathbf{U}(2)\)
Kähler metrics:

\[(M^4, g) \text{ Kähler } \iff \text{ holonomy } \subset U(2)\]
Kähler metrics:

\[(M^4, g) \text{ Kähler } \iff \text{holonomy } \subset U(2)\]
Kähler metrics:

\[(M^4, g) \text{ Kähler } \iff \text{holonomy } \subset \text{U}(2)\]
Kähler metrics:

\((M^4, g)\) Kähler \iff\ holonomy \subset U(2)\)
Kähler metrics:

\[ (M^4, g) \text{ Kähler } \iff \text{ holonomy } \subset U(2) \]
Kähler metrics:

$$(M^4, g) \text{ Kähler} \iff \text{holonomy } \subset \mathbb{U}(2)$$
Kähler metrics:

\((M^4, g)\) Kähler \iff \text{holonomy} \subset U(2)\)
Kähler metrics:

\((M^4, g)\) Kähler \iff \text{holonomy} \subset U(2)\)
Kähler metrics:

\((M^4, g)\) Kähler \iff\ holonomy \subset \text{U}(2)\)
Special nature of dimension 4:

\[
\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \\
\mathfrak{u}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2)
\]
Special nature of dimension 4:

\[
\begin{align*}
\mathfrak{so}(4) &= \mathfrak{so}(3) \oplus \mathfrak{so}(3) \\
\mathfrak{u}(2) &= \mathfrak{u}(1) \oplus \mathfrak{su}(2)
\end{align*}
\]

\[
\begin{align*}
\text{SO}(4) &\leftarrow \text{Sp}(1) \times \text{Sp}(1) \\
\text{U}(2) &\leftarrow \text{U}(1) \times \text{SU}(2)
\end{align*}
\]
Kähler metrics:

\((M^4, g)\) Kähler \iff\ holonomy \subset \mathbf{U}(2)\)
Kähler metrics:

$$(M^4, g) \text{ Kähler } \iff \text{holonomy } \subset \text{U}(2)$$

\iff \exists \text{ almost-complex structure } J \text{ with } \nabla J = 0 \text{ and } g(J \cdot, J \cdot) = g.

\iff J \text{ is integrable and } \exists J\text{-invariant closed 2-form } \omega \text{ given by } \omega = g(J \cdot, \cdot).
Kähler metrics:

\((M^4, g) \text{ Kähler} \iff \text{holonomy} \subset \mathbb{U}(2)\)

\iff \exists \text{ almost-complex structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g.

\iff J \text{ is integrable and } \exists J\text{-invariant closed 2-form } \omega \text{ given by } \omega = g(J\cdot, \cdot). \quad \text{“Kähler form”}
Special nature of dimension 4:

$$\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

$$\mathfrak{u}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2)$$
Special nature of dimension 4:

\[ \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \]
\[ \mathfrak{u}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2) \]

Corresponds to:

\[ \Lambda^2 = \Lambda^+ \oplus \Lambda^- \]
\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^{1,1}_0 \]
Special nature of dimension 4:

\[ \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \]
\[ \mathfrak{u}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2) \]

Corresponds to:

\[ \Lambda^2 = \Lambda^+ \oplus \Lambda^- \]
\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda_{0,1}^{1,1} \]

\[ \Lambda^2_C = \Lambda^{2,0}_C \oplus \Lambda_{C}^{1,1} \oplus \Lambda^{0,2} \]
Special nature of dimension 4:

\[ \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \]
\[ \mathfrak{u}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2) \]

Corresponds to:

\[ \Lambda^2 = \Lambda^+ \oplus \Lambda^- \]
\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda_{0,1}^{1,1} \]

\[ \Lambda^{2,0}_\mathbb{C} = \Lambda^{2,0} \oplus \Lambda_{\mathbb{C}}^{1,1} \oplus \Lambda^{0,2} \]
\[ dz^1 \wedge dz^2 \quad dz^j \wedge d\bar{z}^k \quad d\bar{z}^1 \wedge d\bar{z}^2 \]
Special nature of dimension 4:

\[ \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \]
\[ \cup \quad \cup \quad \| \]
\[ \mathfrak{u}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2) \]

Corresponds to:

\[ \Lambda^2 = \Lambda^+ \oplus \Lambda^- \]
\[ \cup \quad \cup \quad \| \]
\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda_0^{1,1} \]

\[ \Lambda^+_{\mathbb{C}} = \Lambda^{2,0} \oplus \mathbb{C}\omega \oplus \Lambda^{0,2} \]
Riemann curvature of $g$

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

\[
\begin{array}{ccc}
\Lambda^+ & W_+ & \frac{s}{12} \\
\Lambda^- & \hat{\mathcal{R}} & W_- + \frac{s}{12}
\end{array}
\]

where

- $s$ = scalar curvature
- $\hat{\mathcal{R}}$ = trace-free Ricci curvature
- $W_+$ = self-dual Weyl curvature \((conformally invariant)\)
- $W_-$ = anti-self-dual Weyl curvature

""
Kähler case:

$$\Lambda^{1,1} = \mathbb{R} \omega \oplus \Lambda^-$$
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R} \omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R} \omega \oplus \mathbb{R}e(\Lambda^{2,0}) \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \mathbb{R}e(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \iff \mathcal{R} \in \text{End}(\Lambda^{1,1}) \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ W_+ + \frac{s}{12} = \begin{pmatrix} 0 \\ 0 \\ \ast \end{pmatrix} \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R} \omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R} \omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ W_+ + \frac{s}{12} = \begin{pmatrix} 0 & \frac{s}{4} \\ 0 & \frac{s}{4} \end{pmatrix} \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \text{Re}(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ W_+ = \begin{pmatrix} -\frac{s}{12} & \frac{s}{12} \\ -\frac{s}{12} & \frac{s}{6} \end{pmatrix} \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R} \omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R} \omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ |W_+|^2 = \frac{s^2}{24} \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R} \omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R} \omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ \mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & \hat{r} \\
\hat{r} & W_- + \frac{s}{12}
\end{pmatrix} \]

Curvature \( \Lambda^+ \)  
Curvature \( \Lambda^- \)
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ \mathcal{R}(\omega) =: \rho \]

Curvature \( \Lambda^+ \) \( \leftrightarrow \rho \).
Kähler metrics:

\((M^4, g)\) Kähler \iff \text{holonomy} \subset \text{U}(2)

\iff \exists \text{almost-complex structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g.

\iff J \text{ is integrable and } \exists J\text{-invariant closed 2-form } \omega \text{ given by } \omega = g(J\cdot, \cdot); \text{ called the “Kähler form.”}
Kähler metrics:

\((M^4, g)\) Kähler \iff\ holonomy \subset U(2)

\iff\ \exists\ almost-complex structure \(J\) with \(\nabla J = 0\)
and \(g(J \cdot, J \cdot) = g\).

\iff\ \(J\) is integrable and \(\exists\ \) \(J\)-invariant closed 2-form \(\omega\) given by \(\omega = g(J \cdot, \cdot)\); called the “Kähler form.”

Kähler magic:

There is a closed 2-form \(\rho\)
Kähler metrics:

\[(M^4, g) \text{ Kähler } \iff \text{holonomy } \subset U(2)\]

\[\iff \exists \text{ almost-complex structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g.\]

\[\iff J \text{ is integrable and } \exists J\text{-invariant closed 2-form } \omega \text{ given by } \omega = g(J\cdot, \cdot); \text{ called the "Kähler form."} \]

Kähler magic:

There is a closed 2-form \(\rho\) given by

\[\rho = r(J\cdot, \cdot)\]
Kähler metrics:

$$(M^4, g) \text{ Kähler } \iff \text{holonomy } \subset \text{U}(2)$$

$$\iff \exists \text{ almost-complex structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g.$$  

$$\iff \exists \text{ J-integrable closed 2-form } \omega \text{ given by } \omega = g(J\cdot, \cdot); \text{ called the “Kähler form.”}$$  

Kähler magic:

There is a closed 2-form $\rho$ given by

$$\rho = r(J\cdot, \cdot)$$

and called the “Ricci form.”
Kähler metrics:

\((M^4, g)\) Kähler \iff \text{holonomy} \subset U(2)\n
\iff \exists \text{almost-complex structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g.

\iff J \text{ is integrable and } \exists J\text{-invariant closed 2-form } \omega \text{ given by } \omega = g(J\cdot, \cdot); \text{ called the "Kähler form."}

Kähler magic:

There is a closed 2-form \(\rho\) given by

\[\rho = r(J\cdot, \cdot)\]

and called the "Ricci form." Moreover, \(i\rho\) is exactly the curvature of canonical line bundle \(K = \Lambda^{2;0}\).
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R} \omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R} \omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ \rho = \mathcal{R}(\omega) = r(J \cdot, \cdot). \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ \rho = \mathcal{R}(\omega) = r(J\cdot, \cdot). \]

\[ \rho \wedge \omega = \frac{s}{4} \omega \wedge \omega \]
\textbf{Kähler case:}

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ \rho = \mathcal{R}(\omega) = r(J\cdot, \cdot). \]

\[ \rho \wedge \omega = \frac{s}{4}\omega \wedge \omega = \frac{s}{2}d\mu \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R} \omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R} \omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ \rho = \mathcal{R}(\omega) = r(J \cdot, \cdot). \]

\[ \rho \wedge \omega = \frac{s}{4} \omega \wedge \omega = \frac{s}{2} d\mu \]

\[ [\rho] = 2\pi c_1 \]
Kähler case:

\[
\Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \\
\Lambda^+ = \mathbb{R}\omega \oplus \text{Re}(\Lambda^{2,0})
\]

\[\nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[
\rho = \mathcal{R}(\omega) = r(J\cdot, \cdot).
\]

\[
\rho \wedge \omega = \frac{s}{4}\omega \wedge \omega = \frac{s}{2}d\mu
\]

\[
[\rho] = 2\pi c_1(TM, J)
\]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ \rho = \mathcal{R}(\omega) = r(J\cdot,\cdot). \]

\[ \rho \wedge \omega = \frac{s}{4} \omega \wedge \omega = \frac{s}{2} d\mu \]

\[ [\rho] = 2\pi c_1(TM,J) \in H^2(M,\mathbb{R}) \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ \int_M s \, d\mu = 4\pi \ c_1 \cdot [\omega]. \]

\[ \rho \wedge \omega = \frac{s}{4} \omega \wedge \omega = \frac{s}{2} d\mu \]

\[ [\rho] = 2\pi c_1(TM, J) \in H^2(M, \mathbb{R}) \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ \int_M s \, d\mu = 4\pi \, c_1 \cdot [\omega]. \]
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \]

\[ \int_M s \, d\mu = 4\pi \, c_1 \cdot [\omega]. \]

So Cauchy-Schwarz \implies

\[ \int_M s^2 \, d\mu \geq 32\pi^2 \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} \]

because \[ \int_M d\mu = [\omega]^2/2. \]
$H^2(M, \mathbb{R})$
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \mathbb{R}e(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ \int_M s^2 \, d\mu \geq 32\pi^2 |c_1^+|^2 \]

with equality iff \( s \) is constant.
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R} \omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R} \omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ \int_M s^2 \, d\mu \geq 32\pi^2 |c_1^+|^2 \]

with equality iff \( s \) is constant. Similarly,

\[ \int_M |r|^2 \, d\mu \geq 8\pi^2 \left( |c_1^+|^2 + |c_1^-|^2 \right) \]

with equality iff \( s \) is constant.
Kähler case:

\[ \Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^- \]

\[ \Lambda^+ = \mathbb{R}\omega \oplus \Re(\Lambda^{2,0}) \]

\[ \nabla J = 0 \implies \mathcal{R} \in \text{End}(\Lambda^{1,1}) \implies \]

\[ \int_M s^2 \, d\mu \geq 32\pi^2 |c_1^+|^2 \]

with equality iff \( s \) is constant. Similarly,

\[ \int_M |r|^2 \, d\mu \geq 8\pi^2 \left( |c_1^+|^2 + |c_1^-|^2 \right) \]

with equality iff \( s \) is constant. \( \text{ (Calabi 1982)} \)
Theorem (L).
Theorem (L). Let $(M^4, J)$ be a compact complex surface
Theorem (L). Let $(M^4, J)$ be a compact complex surface with $Kod \neq -\infty$.
Theorem (L). Let \((M^4, J)\) be a compact complex surface with \(\text{Kod} \neq -\infty\) and \(b_1\) even.
Theorem (L). Let $(M^4, J)$ be a compact complex surface with $Kod \neq -\infty$ and $b_1$ even. Let $g$ be any Riemannian metric on $M$. 
Theorem (L). Let $(M^4, J)$ be a compact complex surface with $\text{Kod} \neq -\infty$ and $b_1$ even. Let $g$ be any Riemannian metric on $M$. Then,

the curvature of $g$ satisfies
Theorem (L). Let $(M^4, J)$ be a compact complex surface with $\text{Kod} \neq -\infty$ and $b_1$ even. Let $g$ be any Riemannian metric on $M$. Then,

the curvature of $g$ satisfies

$$\int_M s^2 \, d\mu \geq 32\pi^2 |c_1^+|^2$$
Theorem (L). Let \((M^4, J)\) be a compact complex surface with \(\text{Kod} \neq -\infty\) and \(b_1\) even. Let \(g\) be any Riemannian metric on \(M\). Then, the curvature of \(g\) satisfies

\[
\int_M s^2 \, d\mu \geq 32\pi^2 |c_1^+|^2 \\
\int_M |r|^2 \, d\mu \geq 8\pi^2 \left( |c_1^+|^2 + |c_1^-|^2 \right)
\]
Theorem (L). Let $(M^4, J)$ be a compact complex surface with $Kod \neq -\infty$ and $b_1$ even. Let $g$ be any Riemannian metric on $M$. Then, possibly after moving $J$ by a self-diffeomorphism of $M$, the curvature of $g$ satisfies

$$
\int_M s^2 \, d\mu \geq 32\pi^2 |c_1^+|^2 \\
\int_M |r|^2 \, d\mu \geq 8\pi^2 \left( |c_1^+|^2 + |c_1^-|^2 \right)
$$
Theorem (L). Let $(M^4, J)$ be a compact complex surface with $Kod \neq -\infty$ and $b_1$ even. Let $g$ be any Riemannian metric on $M$. Then, possibly after moving $J$ by a self-diffeomorphism of $M$, the curvature of $g$ satisfies

\[
\int_M s^2 \, d\mu \geq 32\pi^2 |c_1^+|^2
\]
\[
\int_M |r|^2 \, d\mu \geq 8\pi^2 \left(|c_1^+|^2 + |c_1^-|^2\right)
\]

with equality iff $g$ is constant-scalar-curvature Kähler.
Theorem (L). Let \((M^4, J)\) be a compact complex surface with \(\text{Kod} \neq -\infty\) and \(b_1\) even. Let \(g\) be any Riemannian metric on \(M\). Then, possibly after moving \(J\) by a self-diffeomorphism of \(M\), the curvature of \(g\) satisfies

\[
\int_M s^2 \, d\mu \geq 32\pi^2 |c_1^+|^2
\]

\[
\int_M |r|^2 \, d\mu \geq 8\pi^2 \left( |c_1^+|^2 + |c_1^-|^2 \right)
\]

with equality iff \(g\) is constant-scalar-curvature Kähler (for some \(J'\) with same \(c_1\) as \(J\)).
Theorem (L). Let $(M^4, J)$ be a compact complex surface with $\text{Kod} \neq -\infty$ and $b_1$ even. Let $g$ be any Riemannian metric on $M$. Then, possibly after moving $J$ by a self-diffeomorphism of $M$, the curvature of $g$ satisfies

$$\int_M s^2 \, d\mu \geq 32\pi^2 |c_1^+|^2$$

$$\int_M |r|^2 \, d\mu \geq 8\pi^2 \left( |c_1^+|^2 + |c_1^-|^2 \right)$$

with equality iff $g$ is constant-scalar-curvature Kähler (for some $J'$ with same $c_1$ as $J$).

“Kähler Paradigms in a Riemannian World”
Theorem (L). Let \((M^4, J)\) be a compact complex surface with \(K\text{od} \neq -\infty\) and \(b_1\) even. Let \(g\) be any Riemannian metric on \(M\). Then, possibly after moving \(J\) by a self-diffeomorphism of \(M\), the curvature of \(g\) satisfies

\[
\int_M s^2 \, d\mu \geq 32\pi^2 |c_1^+|^2
\]

\[
\int_M |r|^2 \, d\mu \geq 8\pi^2 \left( |c_1^+|^2 + |c_1^-|^2 \right)
\]

with equality iff \(g\) is constant-scalar-curvature Kähler (for some \(J'\) with same \(c_1\) as \(J\)).

Self-diffeomorphism unneeded if \(b_+ > 1\) or \(c_1^2 \geq 0\).
Proof involves a non-linear Dirac equation...
Spin structures:
Spin structures:

\( w_2(TM) \) is obstruction to spin structure on \( M \):
Spin structures:

$w_2(TM)$ is obstruction to spin structure on $M$:

Double cover of $SO(4)$ bundle of oriented orthonormal frames by principal bundle for group

$$\text{Spin}(4) = \text{Sp}(1) \times \text{Sp}(1).$$
Spin structures:

$w_2(TM)$ is obstruction to spin structure on $M$:

Double cover of $SO(4)$ bundle of oriented orthonormal frames by principal bundle for group

$$\text{Spin}(4) = \text{Sp}(1) \times \text{Sp}(1).$$

Standard representation of $\text{Sp}(1) = \text{SU}(2)$
Spin structures:

\( w_2(TM) \) is obstruction to spin structure on \( M \):

Double cover of \( SO(4) \) bundle of oriented orthonormal frames by principal bundle for group

\[ \text{Spin}(4) = \text{Sp}(1) \times \text{Sp}(1). \]

Standard representation of \( \text{Sp}(1) = \text{SU}(2) \Rightarrow \)

Spinor bundles \( S_+ \) and \( S_- \):
Spin structures:

$w_2(TM)$ is obstruction to spin structure on $M$:

Double cover of $SO(4)$ bundle of oriented orthonormal frames by principal bundle for group

$$\text{Spin}(4) = \text{Sp}(1) \times \text{Sp}(1).$$

Standard representation of $\text{Sp}(1) = \text{SU}(2)$ \[\implies\]

Spinor bundles $S_+$ and $S_-$:

$$\mathbb{H} = \mathbb{C}^2 \rightarrow S_\pm \quad \downarrow \quad M$$
More geometrically:
More geometrically:

The bundle $S(\Lambda^+) \text{ over any oriented } (M^4, g)$
More geometrically:

The bundle $S(\Lambda^+) \text{ over any oriented } (M^4, g)$
More geometrically:

The bundle $S(\Lambda^+)$ over any oriented $(M^4, g)$ can be viewed as a $\mathbb{CP}_1$-bundle.
More geometrically:

The bundle $S(\Lambda^+)$ over any oriented $(M^4, g)$ can be viewed as a $\mathbb{CP}_1$-bundle.

If $w_2 = 0$ ($M$ spin), then

$$S(\Lambda^+) = \mathbb{P}(\mathbb{S}_+)$$

$$\wedge^2 \mathbb{S}_+ = \mathbb{C}$$
More geometrically:

The bundle $S(\Lambda^+)$ over any oriented $(M^4, g)$ can be viewed as a $\mathbb{CP}_1$-bundle.

If $w_2 = 0$ ($M$ spin), then

$$S(\Lambda^+) = \mathbb{P}(S_+)$$
$$\wedge^2 S_+ = \mathbb{C}$$
$$S(\Lambda^-) = \mathbb{P}(S_-)$$
$$\wedge^2 S_- = \mathbb{C}$$
\[ \Lambda^1_C = \text{Hom}(S_+, S_-) \]
\[ \Lambda^1_C = \text{Hom}(S_+, S_-) \]

so get natural Clifford multiplication map
\[ \Lambda^1_{\mathbb{C}} = \text{Hom}(S_+, S_-) \]

so get natural Clifford multiplication map

\[ \bullet : \Lambda^1 \otimes S_+ \rightarrow S_- \]
\[
\Lambda^1_C = \text{Hom}(S_+, S_-)
\]
so get natural Clifford multiplication map

\[
\bullet : \Lambda^1 \otimes S_+ \rightarrow S_-.
\]

Also have covariant derivative
\[ \Lambda^1_{\mathbb{C}} = \text{Hom}(S_+, S_-) \]

so get natural Clifford multiplication map

\[ \bullet : \Lambda^1 \otimes S_+ \rightarrow S_. \]

Also have covariant derivative

\[ \nabla : \Gamma(S_+) \rightarrow \Gamma(\Lambda^1 \otimes S_+) \]
\[ \Lambda^1_C = \text{Hom}(\mathcal{S}_+, \mathcal{S}_-) \]

so get natural Clifford multiplication map

\[ \circ : \Lambda^1 \otimes \mathcal{S}_+ \rightarrow \mathcal{S}_- \]

Also have covariant derivative

\[ \nabla : \Gamma(\mathcal{S}_+) \rightarrow \Gamma(\Lambda^1 \otimes \mathcal{S}_+) \]

Compose to get Dirac operator \( D \):
\[ \Lambda^1_C = \text{Hom}(S_+, S_-) \]

so get natural Clifford multiplication map

\[ \bullet : \Lambda^1 \otimes S_+ \to S_- . \]

Also have covariant derivative

\[ \nabla : \Gamma(S_+) \to \Gamma(\Lambda^1 \otimes S_+) \]

Compose to get Dirac operator \( D \):

\[
\begin{array}{ccc}
\Gamma(S_+) & \xrightarrow{D} & \Gamma(S_-) \\
\nabla & \bullet & \nabla \\
\Gamma(\Lambda^1 \otimes S_+) & \xrightarrow{\bullet} & \Gamma(S_-)
\end{array}
\]
Atiyah-Singer: Dirac operator
Atiyah-Singer: Dirac operator

\[ D : \Gamma(S_+) \rightarrow \Gamma(S_-) \]
Atiyah-Singer: Dirac operator

\[ D : \Gamma(S_+) \to \Gamma(S_-) \]

is elliptic, with \( \text{ind}(D) = -\tau(M)/8. \)
Atiyah-Singer: Dirac operator

\[ D : \Gamma(S_+) \to \Gamma(S_-) \]

is elliptic, with \( \text{ind}(D) = -\tau(M)/8 \).

**Theorem (Rochlin).** For any smooth compact spin \( M^4 \), \( \tau(M) \equiv 0 \mod 16 \).
Atiyah-Singer: Dirac operator

\[ D : \Gamma(S_+) \to \Gamma(S_-) \]

is elliptic, with \( \text{ind}(D) = -\tau(M)/8 \).

**Theorem** (Rochlin). *For any smooth compact spin \( M^4 \), \( \tau(M) \equiv 0 \) mod 16.*

**Example.** \( \tau(K3) = -16 \).
Atiyah-Singer: Dirac operator

\[ D : \Gamma(S_+) \rightarrow \Gamma(S_-) \]

is elliptic, with \( \text{ind}(D) = -\tau(M)/8. \)
Atiyah-Singer: Dirac operator

\[ D : \Gamma(S_+) \to \Gamma(S_-) \]

is elliptic, with \( \text{ind}(D) = -\tau(M)/8 \).

Weitzenböck formula: \( \forall \Phi \in \Gamma(S_+) \),
Atiyah-Singer: Dirac operator

\[ D : \Gamma(S_+) \to \Gamma(S_-) \]

is elliptic, with \( \text{ind}(D) = -\tau(M)/8 \).

Weitzenböck formula: \( \forall \Phi \in \Gamma(S_+) \),

\[ \langle \Phi, D^* D \Phi \rangle = \frac{1}{2} \Delta |\Phi|^2 + |\nabla \Phi|^2 + \frac{s}{4} |\Phi|^2 \]
Atiyah-Singer: Dirac operator

\[ D : \Gamma(S_+) \rightarrow \Gamma(S_-) \]

is elliptic, with \( \text{ind}(D) = -\tau(M)/8 \).

Weitzenböck formula: \( \forall \Phi \in \Gamma(S_+) \),

\[
\langle \Phi, D^* D \Phi \rangle = \frac{1}{2} \Delta |\Phi|^2 + |\nabla \Phi|^2 + \frac{s}{4} |\Phi|^2
\]

**Proposition** (Lichnerowicz). If \( M^4 \) compact spin, with \( \tau \neq 0 \), then \( \exists \) metric \( g \) on \( M \) with \( s > 0 \).
Atiyah-Singer: Dirac operator

\[ D : \Gamma(S_+) \to \Gamma(S_-) \]

is elliptic, with \( \text{ind}(D) = -\tau(M)/8 \).

Weitzenböck formula: \( \forall \Phi \in \Gamma(S_+) \),

\[ \langle \Phi, D^* D \Phi \rangle = \frac{1}{2} \Delta |\Phi|^2 + |\nabla \Phi|^2 + \frac{s}{4} |\Phi|^2 \]

**Proposition** (Lichnerowicz). *If \( M^4 \) compact spin, with \( \tau \neq 0 \), then \( \nexists \) metric \( g \) on \( M \) with \( s > 0 \).*

**Example.** \( \nexists \) metric of \( s > 0 \) on \( K3 \).
Spin$^c$ structures:
Spin$^c$ structures:

\[ w_2(TM^4) \in H^2(M, \mathbb{Z}_2) \]

in image of

\[ H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}_2) \]
Spin$^c$ structures:

$$w_2(TM^4) \in H^2(M, \mathbb{Z}_2)$$

in image of

$$H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}_2)$$

$$\implies \exists \text{ Hermitian line bundles }$$

$$L \to M$$

with

$$c_1(L) \equiv w_2(TM) \mod 2.$$
Spin\(^c\) structures:

\[ w_2(TM^4) \in H^2(M, \mathbb{Z}_2) \]
in image of
\[ H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}_2) \]
\[ \implies \exists \text{ Hermitian line bundles} \]
\[ L \rightarrow M \]

with
\[ c_1(L) \equiv w_2(TM) \mod 2. \]

Given \( g \) on \( M \), \[ \implies \exists \text{ rank-2 Hermitian vector bundles} \]
\[ V_\pm \rightarrow M \]
Spin\textsuperscript{c} structures:

\[ w_2(TM^4) \in H^2(M, \mathbb{Z}_2) \]

in image of

\[ H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}_2) \]

\[ \implies \exists \text{ Hermitian line bundles} \]

\[ L \to M \]

with

\[ c_1(L) \equiv w_2(TM) \mod 2. \]

Given \( g \) on \( M \),

\[ \implies \exists \text{ rank-2 Hermitian vector bundles} \]

\[ V_\pm \to M \text{ which formally satisfy} \]

\[ V_\pm = S_\pm \otimes L^{1/2}, \]
Spin\textsuperscript{c} structures:

\[ w_2(TM^4) \in H^2(M, \mathbb{Z}_2) \]

in image of

\[ H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}_2) \]

\[ \Rightarrow \exists \text{ Hermitian line bundles }\]

\[ L \rightarrow M \]

with

\[ c_1(L) \equiv w_2(TM) \mod 2. \]

Given \( g \) on \( M \), \[ \Rightarrow \exists \text{ rank-2 Hermitian vector bundles } V_\pm \rightarrow M \text{ which formally satisfy } \]

\[ V_\pm = S_\pm \otimes L^{1/2}, \]

where \( S_\pm \) are the (locally defined) left- and right-handed spinor bundles of \((M, g)\).
Key Example

Let $J$ be any almost-complex structure on $M$. 
Key Example

Let $J$ be any almost-complex structure on $M$.

Let $L = \Lambda^{0,2}$ be its anti-canonical line bundle.
Key Example

Let $J$ be any almost-complex structure on $M$.

Let $L = \Lambda^{0,2}$ be its anti-canonical line bundle.

∀$g$ on $M$, the bundles

\[ V_+ = \Lambda^{0,0} \oplus \Lambda^{0,2} \]
\[ V_- = \Lambda^{0,1} \]
**Key Example**

Let $J$ be any almost-complex structure on $M$.

Let $L = \Lambda^{0,2}$ be its anti-canonical line bundle.

∀$g$ on $M$, the bundles

$$\mathbb{V}_+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$$

$$\mathbb{V}_- = \Lambda^{0,1}$$

can formally be written as

$$\mathbb{V}_\pm = S_\pm \otimes L^{1/2},$$
Key Example

Let $J$ be any almost-complex structure on $M$.

Let $L = \Lambda^{0,2}$ be its anti-canonical line bundle.

∀$g$ on $M$, the bundles

\[
V_+ = \Lambda^{0,0} \oplus \Lambda^{0,2}
\]
\[
V_- = \Lambda^{0,1}
\]

can formally be written as

\[
V_\pm = S_\pm \otimes L^{1/2},
\]

where $S_\pm$ are left & right-handed spinor bundles.
Key Example

Let $J$ be any almost-complex structure on $M$.

Let $L = \Lambda^{0,2}$ be its anti-canonical line bundle.

For all $g$ on $M$, the bundles
\[
\mathcal{V}_+ = \Lambda^{0,0} \oplus \Lambda^{0,2} \\
\mathcal{V}_- = \Lambda^{0,1}
\]
can formally be written as
\[
\mathcal{V}_\pm = \mathcal{S}_\pm \otimes L^{1/2},
\]
where $\mathcal{S}_\pm$ are left & right-handed spinor bundles.

A spin$^c$ structure arises from some $J$ if and only if
\[
c_1^2(L) = (2\chi + 3\tau)(M).
\]
Every unitary connection $A$ on $L$
Every unitary connection $A$ on $L$ induces spin$^c$ Dirac operator
Every unitary connection $A$ on $L$ induces spin$^c$ Dirac operator

$$D_A : \Gamma(\mathcal{V}_+) \to \Gamma(\mathcal{V}_-)$$
Every unitary connection $A$ on $L$ induces spin$^c$ Dirac operator

$$D_A : \Gamma(V_+) \to \Gamma(V_-)$$

generalizing $\bar{\partial} + \bar{\partial}^*$. 

Every unitary connection $A$ on $L$ induces spin$^c$ Dirac operator

$$D_A : \Gamma(\mathcal{V}_+) \to \Gamma(\mathcal{V}_-)$$

generalizing $\bar{\partial} + \bar{\partial}^*$. 

Weitzenböck formula: $\forall \Phi \in \Gamma(\mathcal{V}_+)$,

$$\langle \Phi, D_A^* D_A \Phi \rangle = \frac{1}{2} \Delta |\Phi|^2 + |\nabla_A \Phi|^2 + \frac{s}{4} |\Phi|^2$$
Every unitary connection $A$ on $L$ induces spin$^c$ Dirac operator

$$D_A : \Gamma(\mathbb{V}_+) \to \Gamma(\mathbb{V}_-)$$

generalizing $\bar{\partial} + \bar{\partial}^*$. 

Weitzenböck formula: $\forall \Phi \in \Gamma(\mathbb{V}_+)$,

$$\langle \Phi, D_A^* D_A \Phi \rangle = \frac{1}{2} \Delta |\Phi|^2 + |\nabla_A \Phi|^2 + \frac{s}{4} |\Phi|^2$$

$$+ 2\langle -iF_A^+, \sigma(\Phi) \rangle$$
Every unitary connection $A$ on $L$ induces spin$^c$ Dirac operator

$$D_A : \Gamma(V_+) \to \Gamma(V_-)$$

generalizing $\bar{\partial} + \bar{\partial}^*$. 

Weitzenböck formula: $\forall \Phi \in \Gamma(V_+)$, 

$$\langle \Phi, D_A^* D_A \Phi \rangle = \frac{1}{2} \Delta |\Phi|^2 + |\nabla A \Phi|^2 + \frac{s}{4} |\Phi|^2$$
$$+ 2 \langle -i F_A^+, \sigma(\Phi) \rangle$$

where $F_A^+$ = self-dual part curvature of $A$, 

Every unitary connection $A$ on $L$ induces spin$^c$ Dirac operator

$$D_A : \Gamma(\mathbb{V}_+) \to \Gamma(\mathbb{V}_-)$$

generalizing $\bar{\partial} + \bar{\partial}^*$. 

Weitzenböck formula: $\forall \Phi \in \Gamma(\mathbb{V}_+),$

$$\langle \Phi, D_A^* D_A \Phi \rangle = \frac{1}{2} \Delta |\Phi|^2 + |\nabla_A \Phi|^2 + \frac{s}{4} |\Phi|^2$$

$$+ 2 \langle -i F_A^+, \sigma(\Phi) \rangle$$

where $F_A^+$ = self-dual part curvature of $A$, and

$\sigma : \mathbb{V}_+ \to \Lambda^+$ is a natural real-quadratic map,

$$|\sigma(\Phi)| = \frac{1}{2 \sqrt{2}} |\Phi|^2.$$
Witten:

consider both $\Phi$ and $A$ as unknowns,
Witten:

consider both $\Phi$ and $A$ as unknowns, subject to Seiberg-Witten equations

$$D_A \Phi = 0$$

$$F^+_A = i\sigma(\Phi).$$
Witten:

consider both $\Phi$ and $A$ as unknowns,
subject to *Seiberg-Witten equations*

$$D_A \Phi = 0$$

$$F^+_A = i\sigma(\Phi).$$

Non-linear, but elliptic
Witten:

consider both $\Phi$ and $A$ as unknowns,

subject to Seiberg-Witten equations

\[
D_A \Phi = 0 \\
F_A^+ = i\sigma(\Phi).
\]

Non-linear, but elliptic once ‘gauge-fixing’

\[
d^*(A - A_0) = 0
\]

imposed to eliminate automorphisms of $L \to M$. 
Weitzenböck formula becomes

\[ 0 = 2\Delta|\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]
Weitzenböck formula becomes

\[ 0 = 2\Delta|\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional} \ldots \]
Weitzenböck formula becomes

$$0 = 2\Delta |\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4$$

$$\implies$$ moduli space compact, finite-dimensional...

**Compactness:**
Weitzenböck formula becomes

\[ 0 = 2\Delta |\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional...} \]

**Compactness:** Implies $C^0$ bound on $\Phi$: 
Weitzenböck formula becomes

\[ 0 = 2\Delta |\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional...} \]

**Compactness:** Implies \( C^0 \) bound on \( \Phi \):

At maximum of \( \Phi \), \( \Delta |\Phi|^2 \geq 0 \), so
Weitzenböck formula becomes

\[ 0 = 2\Delta|\Phi|^2 + 4|\nabla_A\Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\implies \text{moduli space compact, finite-dimensional...}

**Compactness:** Implies \( C^0 \) bound on \( \Phi \):

At maximum of \( \Phi \), \( \Delta|\Phi|^2 \geq 0 \), so

\[ 0 \geq s|\Phi|^2 + |\Phi|^4 \]
Weitzenböck formula becomes

\[ 0 = 2\Delta |\Phi|^2 + 4|\nabla A\Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional} \ldots \]

**Compactness:** Implies \( C^0 \) bound on \( \Phi \):

At maximum of \( \Phi \), \( \Delta |\Phi|^2 \geq 0 \), so

\[ 0 \geq s|\Phi|^2 + |\Phi|^4 \]

and hence \( |\Phi|^2 \leq -s \), unless \( \Phi \equiv 0 \). Hence
Weitzenböck formula becomes

\[ 0 = 2\Delta|\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

implies moduli space compact, finite-dimensional...

**Compactness:** Implies $C^0$ bound on $\Phi$:

At maximum of $\Phi$, $\Delta|\Phi|^2 \geq 0$, so

\[ 0 \geq s|\Phi|^2 + |\Phi|^4 \]

and hence $|\Phi|^2 \leq -s$, unless $\Phi \equiv 0$. Hence

\[ |\Phi| \leq \sqrt{\max |s_-|} \]

everywhere!
Weitzenböck formula becomes

\[ 0 = 2\Delta |\Phi|^2 + 4\nabla A|\Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional...} \]

**Compactness:** Implies $C^0$ bound on $\Phi$:

At maximum of $\Phi$, $\Delta |\Phi|^2 \geq 0$, so

\[ 0 \geq s|\Phi|^2 + |\Phi|^4 \]

and hence $|\Phi|^2 \leq -s$, unless $\Phi \equiv 0$. Hence

\[ |\Phi| \leq \sqrt{\max|s_-|} \]

everywhere!

Bootstrapping with gauge-fixed equations, one gets $L^p_k$ bounds for $(\Phi, A)$ for all $k, p$. 
Weitzenböck formula becomes

$$0 = 2\Delta |\Phi|^2 + 4|\nabla A \Phi|^2 + s|\Phi|^2 + |\Phi|^4$$
Weitzenböck formula becomes

\[ 0 = 2\Delta|\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional} \ldots \]
Weitzenböck formula becomes

$$0 = 2\Delta |\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4$$

$$\implies$$ moduli space compact, finite-dimensional...

**Dimension:** Index of gauge-fixed system is
Weitzenböck formula becomes

\[ 0 = 2\Delta |\Phi|^2 + 4|\nabla A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional}. . . \]

**Dimension:** Index of gauge-fixed system is

\[ \frac{c_1^2(L) - (2\chi + 3\tau)(M)}{4} \]
Weitzenböck formula becomes

\[ 0 = 2\Delta |\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional...} \]

**Dimension:** Index of gauge-fixed system is

\[ \frac{c_1^2(L) - (2\chi + 3\tau)(M)}{4} \]

For a given spin\(^c\) structure and fixed metric \(g\), this is the dimension of pre-image of any regular value of map defined by gauge-fixed SW equations.
Weitzenböck formula becomes

\[ 0 = 2\Delta|\Phi|^2 + 4|\nabla_A\Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional...} \]

**Dimension:** Index of gauge-fixed system is

\[ \frac{c_1^2(L) - (2\chi + 3\tau)(M)}{4} \]

For a given spin\(^c\) structure and fixed metric \(g\), this is the dimension of pre-image of any regular value of map defined by gauge-fixed SW equations.

Spin\(^c\) structure arises from some \(J \iff c_1^2(L) = 2\chi + 3\tau \iff \) Fredholm index is zero.
Weitzenböck formula becomes

\[ 0 = 2\Delta|\Phi|^2 + 4|\nabla A\Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\(
\implies \text{moduli space compact, finite-dimensional} \ldots
\)

**Dimension:** Index of gauge-fixed system is

\[
\frac{c_1^2(L) - (2\chi + 3\tau)(M)}{4}
\]

For a given spin\(^c\) structure and fixed metric \(g\), this is the dimension of pre-image of any regular value of map defined by gauge-fixed SW equations.

Spin\(^c\) structure arises from some \(J \iff c_1^2(L) = 2\chi + 3\tau \iff \text{Fredholm index is zero.}\)

SW invariant \(\in \mathbb{Z}_2\) means mod-2 mapping degree.
Weitzenböck formula becomes

$$0 = 2\Delta |\Phi|^2 + 4|\nabla A\Phi|^2 + s|\Phi|^2 + |\Phi|^4$$

$$\implies \text{moduli space compact, finite-dimensional} \ldots$$
Weitzenböck formula becomes

\[ 0 = 2\Delta |\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional...} \]

If \( b_+(M) \geq 2 \), then, as metric varies, moduli spaces are cobordant, so can construct invariants that sometimes predict existence of solutions.
Weitzenböck formula becomes

\[ 0 = 2\Delta |\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional} \ldots \]

If \( b_+(M) \geq 2 \), then, as metric varies, moduli spaces are cobordant, so can construct invariants that sometimes predict existence of solutions.

Specifically, if spin\(^c\) structure comes from some \( J \), Fredholm index is 0, and moduli spaces generically discrete. Counting solutions mod 2 gives \( \mathbb{Z}_2 \)-valued invariant.
Weitzenböck formula becomes

\[ 0 = 2\Delta|\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional} \ldots \]

If \( b_+(M) \geq 2 \), then, as metric varies, moduli spaces are cobordant, so can construct invariants that sometimes predict existence of solutions.

Specifically, if spin\(^c\) structure comes from some \( J \), Fredholm index is 0, and moduli spaces generically discrete. Counting solutions mod 2 gives \( \mathbb{Z}_2 \)-valued invariant.

This invariant is non-zero for complex surfaces of Kähler type (i.e. with \( b_1 \) even).
Weitzenböck formula becomes

\[ 0 = 2\Delta |\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 \]

\[ \implies \text{moduli space compact, finite-dimensional...} \]

If \( b_+(M) \geq 2 \), then, as metric varies, moduli spaces are cobordant, so can construct invariants that sometimes predict existence of solutions.

Specifically, if spin\(^c\) structure comes from some \( J \), Fredholm index is 0, and moduli spaces generically discrete. Counting solutions mod 2 gives \( \mathbb{Z}_2 \)-valued invariant.

This invariant is non-zero for complex surfaces of Kähler type (i.e. with \( b_1 \) even).

Implies non-existence of metrics \( g \) for which \( s > 0 \).
When $b_+(M) = 1$, theory is more complicated.
When $b_+(M) = 1$, theory is more complicated.

However, theory works the same way when
When $b_+(M) = 1$, theory is more complicated.

However, theory works the same way when

- $c_1^2(L) > 0$;
When $b_+(M) = 1$, theory is more complicated.

However, theory works the same way when
- $c_1^2(L) > 0$; or
- $c_1^2(L) = 0$, but $c_1(L) \neq 0 \in H_2(M, \mathbb{R})$. 

275
When $b_+(M) = 1$, theory is more complicated.

However, theory works the same way when

- $c_1^2(L) > 0$; or
- $c_1^2(L) = 0$, but $c_1(L) \neq 0 \in H^2(M, \mathbb{R})$. 
When $b_+(M) = 1$, theory is more complicated.

However, theory works the same way when

- $c_1^2(L) > 0$; or
- $c_1^2(L) = 0$, but $c_1(L) \neq 0 \in H^2(M, \mathbb{R})$.

Enough for us, by Hitchin-Thorpe Inequality.
When $b_+(M) = 1$, theory is more complicated.

However, theory works the same way when

- $c_1^2(L) > 0$; or
- $c_1^2(L) = 0$, but $c_1(L) \neq 0 \in H^2(M, \mathbb{R})$.

Enough for us, by Hitchin-Thorpe Inequality.

In this context,
When $b_+(M) = 1$, theory is more complicated.

However, theory works the same way when

- $c_1^2(L) > 0$; or
- $c_1^2(L) = 0$, but $c_1(L) \neq 0 \in H^2(M, \mathbb{R})$.

Enough for us, by Hitchin-Thorpe Inequality.

In this context,

- $SW = 0$ if $\text{Kod}(M) = -\infty$; and
When $b_+(M) = 1$, theory is more complicated.

However, theory works the same way when

- $c_1^2(L) > 0$; or
- $c_1^2(L) = 0$, but $c_1(L) \neq 0 \in H^2(M, \mathbb{R})$.

Enough for us, by Hitchin-Thorpe Inequality.

In this context,

- $SW = 0$ if $\text{Kod}(M) = -\infty$; and
- $SW \neq 0$ if $\text{Kod}(M) \geq 0$
When $b_+(M) = 1$, theory is more complicated.

However, theory works the same way when

- $c_1^2(L) > 0$; or
- $c_1^2(L) = 0$, but $c_1(L) \neq 0 \in H^2(M, \mathbb{R})$.

Enough for us, by Hitchin-Thorpe Inequality.

In this context,

- $SW = 0$ if $\text{Kod}(M) = -\infty$; and
- $SW \neq 0$ if $\text{Kod}(M) \geq 0$

for spin$^c$ structure given by complex structure.
Application:
Application:

**Theorem.** Suppose that $(M, J)$ is a compact complex surface.
Application:

**Theorem.** Suppose that $(M, J)$ is a compact complex surface. If the smooth compact 4-manifold $M$ admits an Einstein metric $g$,
Application:

**Theorem.** Suppose that \((M, J)\) is a compact complex surface. If the smooth compact 4-manifold \(M\) admits an Einstein metric \(g\) with \(\lambda > 0\),
Application:

**Theorem.** Suppose that $(M, J)$ is a compact complex surface. If the smooth compact 4-manifold $M$ admits an Einstein metric $g$ with $\lambda > 0$, then $Kod(M, J) = -\infty$. 
Application:

**Theorem.** Suppose that $(M, J)$ is a compact complex surface. If the smooth compact 4-manifold $M$ admits an Einstein metric $g$ with $\lambda > 0$, then $Kod(M, J) = -\infty$, and

$$M \approx_{\text{diff}} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \\ \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \end{cases}$$

$$0 \leq k \leq 8.$$
Application:

**Theorem.** Suppose that \((M, J)\) is a compact complex surface. If the smooth compact 4-manifold \(M\) admits an Einstein metric \(g\) with \(\lambda > 0\), then \(\text{Kod}(M, J) = -\infty\), and

\[
M \approx_{\text{diff}} \begin{cases} 
\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\
\end{cases}
\]
Application:

**Theorem.** Suppose that \((M, J)\) is a compact complex surface. If the smooth compact 4-manifold \(M\) admits an Einstein metric \(g\) with \(\lambda > 0\), then \(\text{Kod}(M, J) = -\infty\), and

\[
M \cong_{\text{diff}} \begin{cases} 
\mathbb{CP}_2 \# k\mathbb{CP}_2, & 0 \leq k \leq 8, \\
\text{or} \\
S^2 \times S^2
\end{cases}
\]
Application:

**Theorem.** Suppose that \((M, J)\) is a compact complex surface. If the smooth compact 4-manifold \(M\) admits an Einstein metric \(g\) with \(\lambda > 0\), then 
\[Kod(M, J) = -\infty,\] and
\[M \cong_{\text{diff}} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, & 0 \leq k \leq 8, \\ \text{or} \\ S^2 \times S^2 \end{cases}\]

Key point: SW \(\Rightarrow s > 0\) impossible when Kod = 2.
Application:

**Theorem.** Suppose that $(M, J)$ is a compact complex surface. If the smooth compact 4-manifold $M$ admits an Einstein metric $g$ with $\lambda > 0$, then $\text{Kod}(M, J) = -\infty$, and

$$M \cong_{\text{diff}} \begin{cases} \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}, & 0 \leq k \leq 8, \\ \text{or} \\ S^2 \times S^2 \end{cases}$$

Key point: $\text{SW} \Rightarrow s > 0$ impossible when $\text{Kod} = 2$.

Same conclusion if $M$ admits $\omega$ instead of $J$. 

Application:

**Theorem.** Suppose that \((M, J)\) is a compact complex surface. If the smooth compact 4-manifold \(M\) admits an Einstein metric \(g\) with \(\lambda > 0\), then \(\text{Kod}(M, J) = -\infty\), and

\[
M \cong_{\text{diff}} \begin{cases} 
\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\
\text{or} \\
S^2 \times S^2
\end{cases}
\]

**Tomorrow:** We will see that this is sharp!
For simplicity,
For simplicity, we now assume that
For simplicity, we now assume that

(*) Either $b_+(M) \geq 2$, 
For simplicity, we now assume that

(*): Either $b_+(M) \geq 2$, or $(2\chi + 3\tau)(M) \geq 0$. 
For simplicity, we now assume that

\((\ast)\) Either \(b_+(M) \geq 2\), or \((2\chi + 3\tau)(M) \geq 0\).

**Definition.** Let \(M\) be a smooth compact oriented 4-manifold satisfying (\(\ast\)),

For simplicity, we now assume that

(*) Either \( b_+(M) \geq 2 \), or \( (2\chi + 3\tau)(M) \geq 0 \).

**Definition.** Let \( M \) be a smooth compact oriented 4-manifold satisfying (*) , and suppose that \( M \) carries almost-complex structure \( J \).
For simplicity, we now assume that

(*) Either $b_+(M) \geq 2$, or $(2\chi + 3\tau)(M) \geq 0$.

**Definition.** Let $M$ be a smooth compact oriented 4-manifold satisfying (*), and suppose that $M$ carries almost-complex structure $J$ such that

$$SW \neq 0$$
For simplicity, we now assume that

(*) Either $b_+(M) \geq 2$, or $(2\chi + 3\tau)(M) \geq 0$.

**Definition.** Let $M$ be a smooth compact oriented 4-manifold satisfying (*), and suppose that $M$ carries almost-complex structure $J$ such that

$$SW \neq 0$$

for spin$^c$ structure induced by $J$. 
For simplicity, we now assume that

\((\ast)\) Either \(b_+(M) \geq 2\), or \((2\chi + 3\tau)(M) \geq 0\).

**Definition.** Let \(M\) be a smooth compact oriented 4-manifold satisfying \((\ast)\), and suppose that \(M\) carries almost-complex structure \(J\) such that

\[SW \neq 0\]

for spin\(^c\) structure induced by \(J\). Then

\[c_1(M, J) \in H^2(M, \mathbb{R})\]
For simplicity, we now assume that

\((*)\) Either \(b_+(M) \geq 2\), or \((2\chi + 3\tau)(M) \geq 0\).

**Definition.** Let \(M\) be a smooth compact oriented \(4\)-manifold satisfying \((*)\), and suppose that \(M\) carries almost-complex structure \(J\) such that \(SW \neq 0\)

for spin\(^c\) structure induced by \(J\). Then

\[c_1(M, J) \in H^2(M, \mathbb{R})\]

is called a basic class of \(M\).
Key property:
Key property:

Every basic class
Key property:

Every basic class

\[ b \in H^2(M, \mathbb{R}) \]
Key property:

Every basic class

\[ b \in H^2(M, \mathbb{R}) \]

arises from a spin\(^c\) structure
Key property:

Every basic class

\[ b \in H^2(M, \mathbb{R}) \]

arises from a spin\(^c\) structure such that the Seiberg-Witten equations
Key property:

Every basic class

\[ b \in H^2(M, \mathbb{R}) \]

arises from a spin\(^c\) structure such that the Seiberg-Witten equations

\[
D_A \Phi = 0 \\
F_A^+ = i\sigma(\Phi).
\]
Key property:

Every basic class

\[ b \in H^2(M, \mathbb{R}) \]

arises from a spin\(^c\) structure such that the Seiberg-Witten equations

\[
D_A \Phi = 0 \\
F_A^+ = i\sigma(\Phi).
\]

have a solution \((\Phi, A)\) for every metric \(g\) on \(M\).
Key property:

Every basic class

\[ b \in H^2(M, \mathbb{R}) \]

arises from a spin\(^c\) structure such that the Seiberg-Witten equations

\[
D_A \Phi = 0 \\
F_A^+ = i\sigma(\Phi).
\]

have a solution \((\Phi, A)\) for every metric \(g\) on \(M\).
Key property:

Every basic class
\[ b \in H^2(M, \mathbb{R}) \]
arises from a spin\(^c\) structure such that the Seiberg-Witten equations
\[ D_A \Phi = 0 \]
\[ F_A^+ = i\sigma(\Phi). \]
have a solution \((\Phi, A)\) for every metric \(g\) on \(M\).

If \((M, J)\) complex surface with \(b_1\) even, and either
Key property:

Every basic class

\[ b \in H^2(M, \mathbb{R}) \]

arises from a spin\(^c\) structure such that the Seiberg-Witten equations

\[
D_A \Phi = 0 \\
F_A^+ = i\sigma(\Phi).
\]

have a solution \((\Phi, A)\) for every metric \(g\) on \(M\).

If \((M, J)\) complex surface with \(b_1\) even, and either

\[ b_+(M) > 1 \]

or
Key property:

Every basic class

\[ b \in H^2(M, \mathbb{R}) \]

arises from a spin\(^c\) structure such that the Seiberg-Witten equations

\[
\begin{align*}
D_A \Phi &= 0 \\
F_A^+ &= i \sigma(\Phi).
\end{align*}
\]

have a solution \((\Phi, A)\) for every metric \(g\) on \(M\).

If \((M, J)\) complex surface with \(b_1\) even, and either

- \(b_+(M) > 1\) or
- \(\text{Kod}(M) \neq -\infty\) and \(c_1^2 \geq 0\),
Key property:

Every basic class

\[ b \in H^2(M, \mathbb{R}) \]

arises from a spin\(^c\) structure such that the Seiberg-Witten equations

\[
D_A \Phi = 0 \\
F_A^+ = i\sigma(\Phi).
\]

have a solution \((\Phi, A)\) for every metric \(g\) on \(M\).

If \((M, J)\) complex surface with \(b_1\) even, and either

- \(b_+(M) > 1\) or
- \(\text{Kod}(M) \neq -\infty\) and \(c_1^2 \geq 0\),

then \(c_1(M, J)\) is a basic class of \(M\).
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$. 
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$. 
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira “homotopy K3.”
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira “homotopy K3.”

$K3 = \text{resolution of } T^4/\mathbb{Z}_2$: 

![Diagram of $K3$ resolution](image)
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira “homotopy K3.”

$K3 = \text{resolution of } T^4/\mathbb{Z}_2$: 

\[ T^2 \quad T^4/\mathbb{Z}_2 \quad \mathbb{CP}_1 \]
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira “homotopy K3.”

$K3 = \text{resolution of } T^4/\mathbb{Z}_2$: 

![Diagram of $K3$ resolution](image)
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ is called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira “homotopy K3.”

Replace chosen fiber $T^2$ with $T^2/\mathbb{Z}_{2\ell+1}$
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira “homotopy K3.”

Key: $[T^2 \times (D^2 - \{0\})]/\mathbb{Z}_{2^{\ell+1}} \cong T^2 \times (D^2 - \{0\})$.
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira “homotopy K3.”

Homotopy equivalent to $K3$, but $c_1 = -2\ell \mathfrak{f}$, where $\mathfrak{f} \neq 0$ is homology class of new $T^2/\mathbb{Z}_{2\ell+1}$. 
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira “homotopy K3.”

Homotopy equivalent to $K3$, but $c_1 = -2\ell \mathfrak{f}$, where $\mathfrak{f} \neq 0$ is homology class of new $T^2/\mathbb{Z}_{2\ell+1}$.

Produces non-zero basic class divisible by $2\ell$. 

When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira “homotopy K3.”

Homotopy equivalent to $K3$, but $c_1 = -2\ell \hat{f}$, where $\hat{f} \neq 0$ is homology class of new $T^2/\mathbb{Z}2\ell+1$.

Produces non-zero basic class divisible by $2\ell$.

As $\ell \to \infty$, get infinitely many different diffeomorphism types:
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira “homotopy K3.”

Homotopy equivalent to $K3$, but $c_1 = -2\ell \mathfrak{f}$, where $\mathfrak{f} \neq 0$ is homology class of new $T^2/\mathbb{Z}_{2\ell+1}$.

Produces non-zero basic class divisible by $2\ell$.

As $\ell \to \infty$, get infinitely many different diffeomorphism types: if finite, divisibility would be bounded!
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira "homotopy $K3$.”

**Proposition.** The topological manifold $|K3|$ admits infinitely many smooth structures.
When $SW \neq 0$, corresponding $c_1(L) \in H^2(M, \mathbb{Z})$ called basic class.

$J$ integrable, $b_1$ even $\implies c_1(M, J)$ is a basic class.

Only finitely many basic classes on any smooth $M^4$.

On $K3$, only basic class is $0 \in H^2(M, \mathbb{Z})$.

Different for any Kodaira “homotopy K3.”

**Proposition.** The topological manifold $|K3|$ admits infinitely many smooth structures. Exactly one of these admits an Einstein metric.
End, Part II