Four-Manifolds,

Einstein Metrics, &

Differential Topology

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Rademacher Lectures
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Differential Topology, III

Moduli Spaces of Einstein Metrics

October 21, 2016
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have a solution (Φ, A) for every metric g on M.

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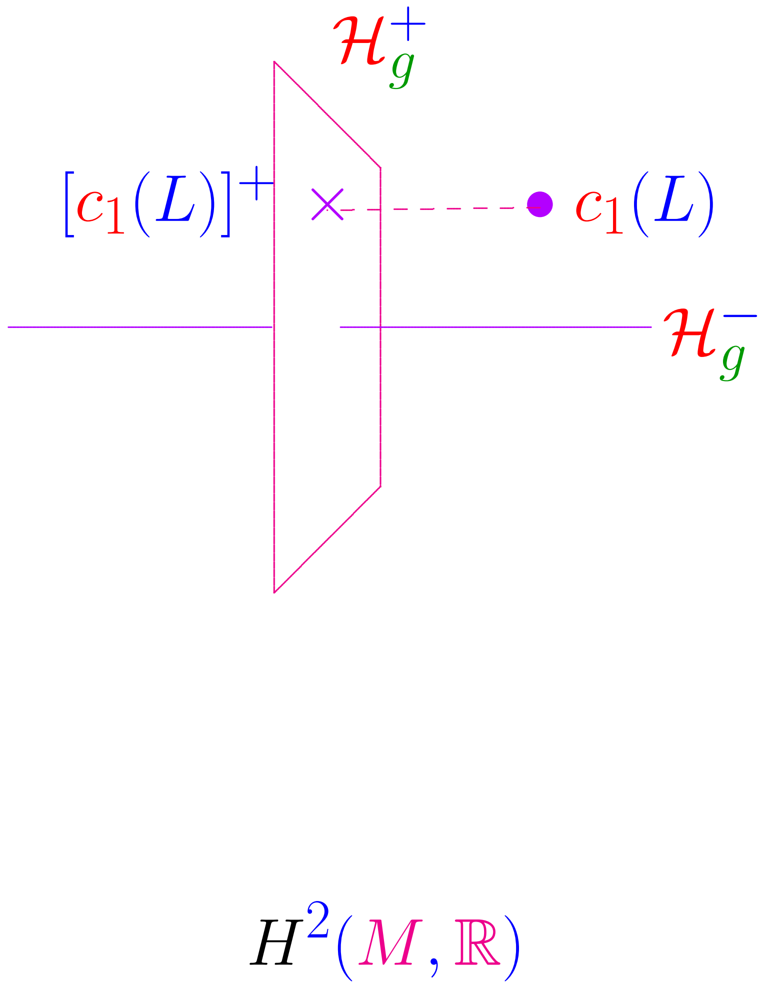
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The existence of SW solutions $\forall g$ may occur even when not detected by invariant we have discussed.

Example. Bauer-Furuta invariant...

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"Kähler Paradigms in a Riemannian World"

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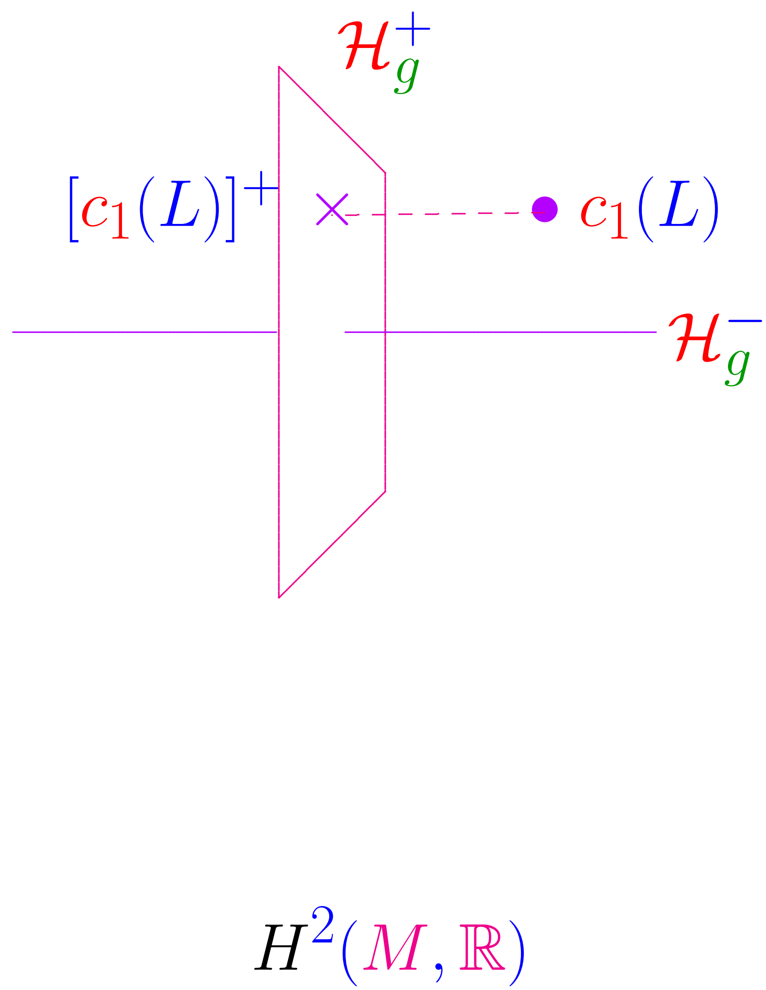

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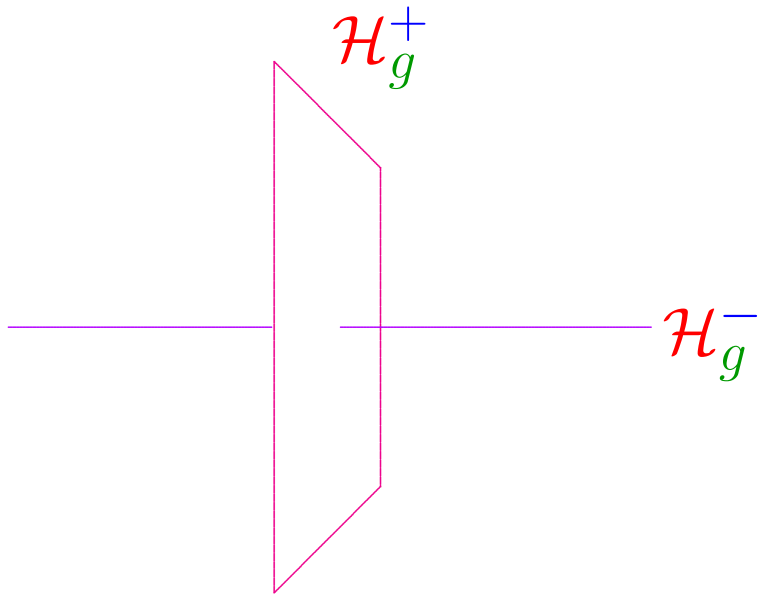

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But there is a metric-independent improvement!

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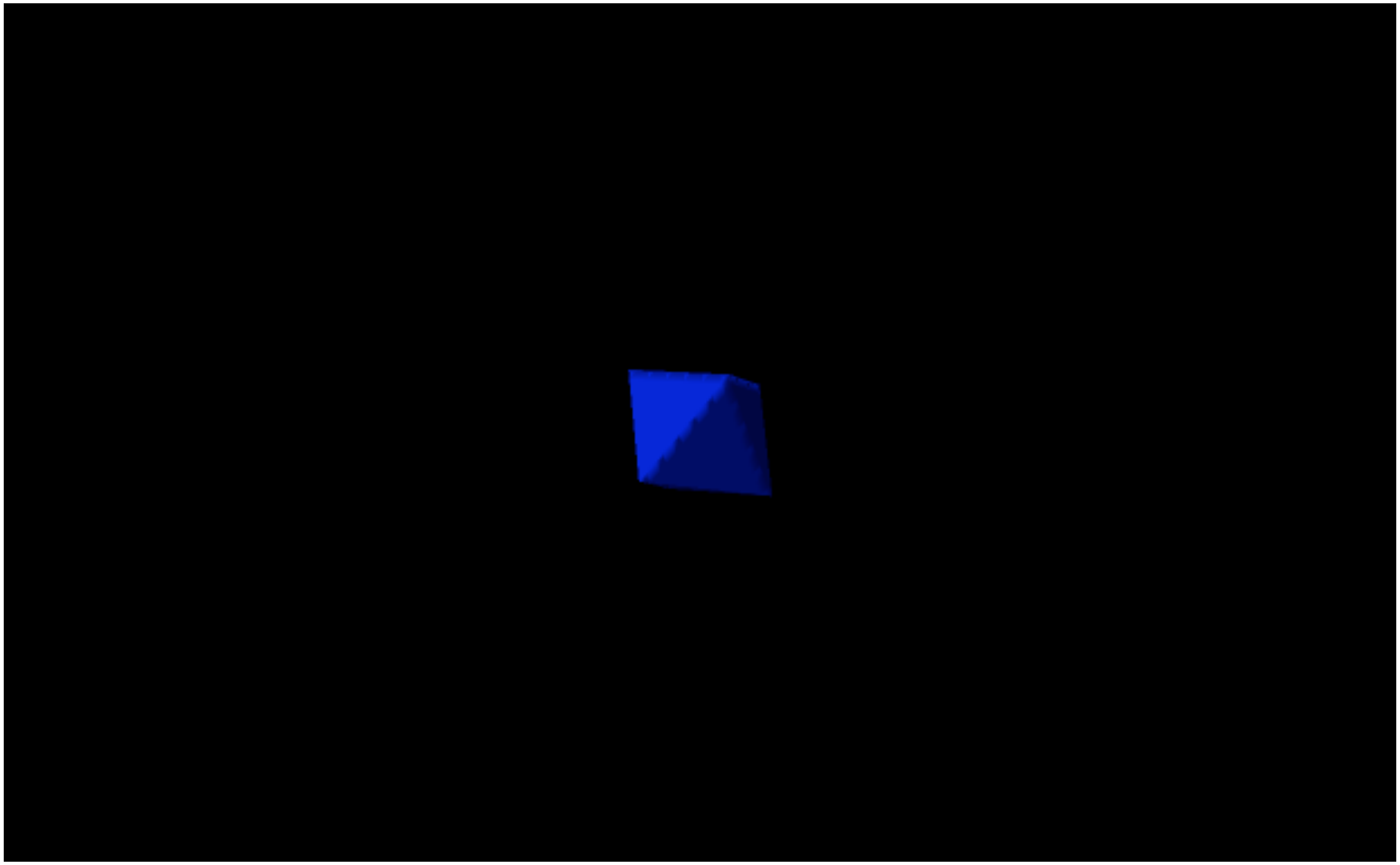

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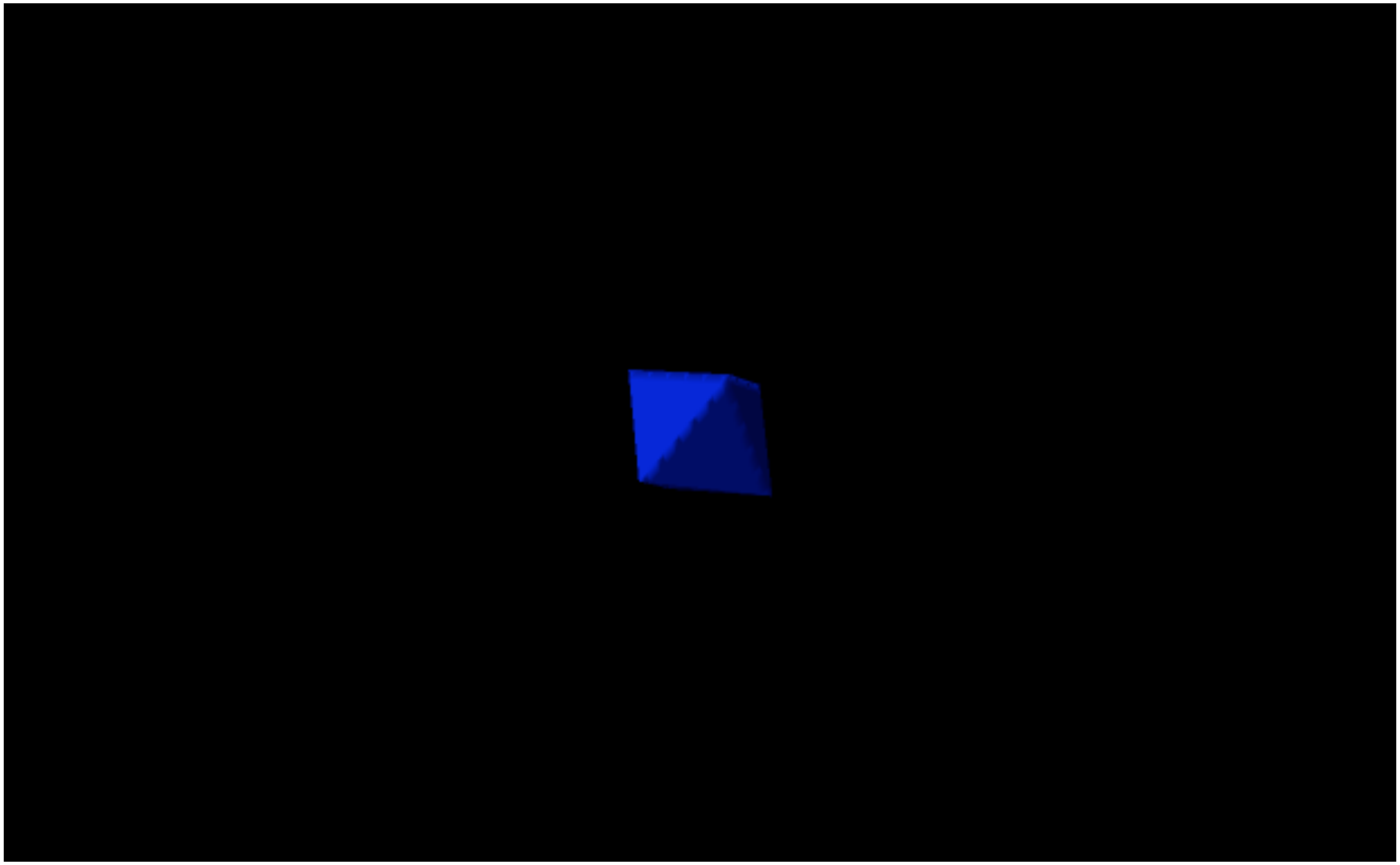

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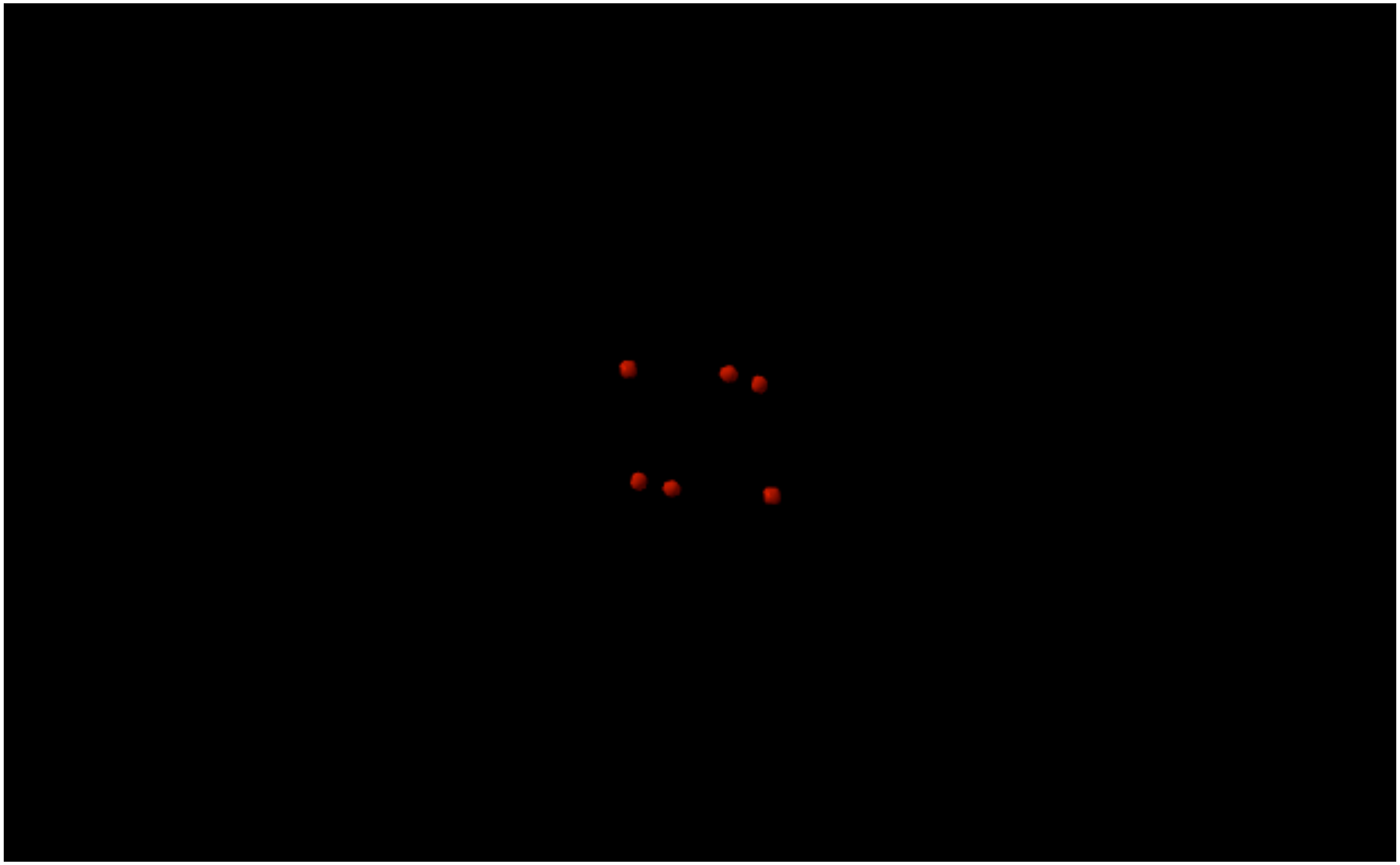

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and equality can only occur if g is CSCK, with $s < 0$.

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For both, equality holds iff $M = X$*, and* g *is Kähler-Einstein with* $\lambda < 0$*.*

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Einstein $\Longrightarrow$ $\quad (2\chi + 3\tau)(M) > \frac{2}{3}{c_1}^2(X)$

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Einstein $\Longrightarrow$ ${c_1}^2(M) > \frac{2}{3}{c_1}^2(X)$

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Einstein $\Longrightarrow$ $\quad {c_1}^2(X) - k > \frac{2}{3}{c_1}^2(X)$

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Einstein $\Longrightarrow$ $\frac{1}{3}{c_1}^2(X) > k$

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So being "very" non-minimal is an obstruction.

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When $n = 2m = 4$, such M are the minimal complex surfaces of general type such that

$$\nexists \mathbb{CP}_1 \overset{\mathcal{O}}{\hookrightarrow} M$$

of homological self-intersection -2.

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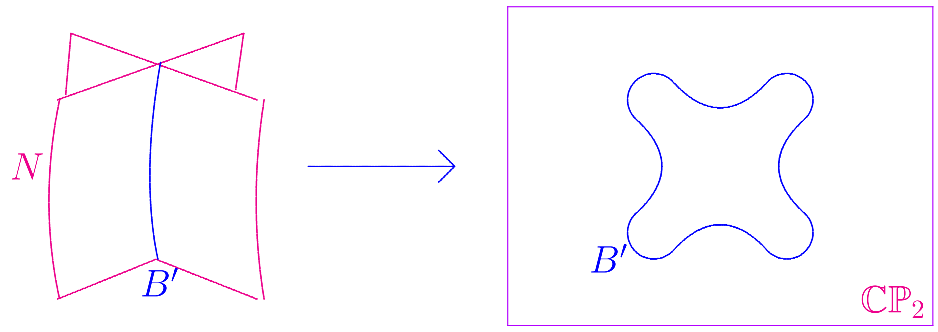

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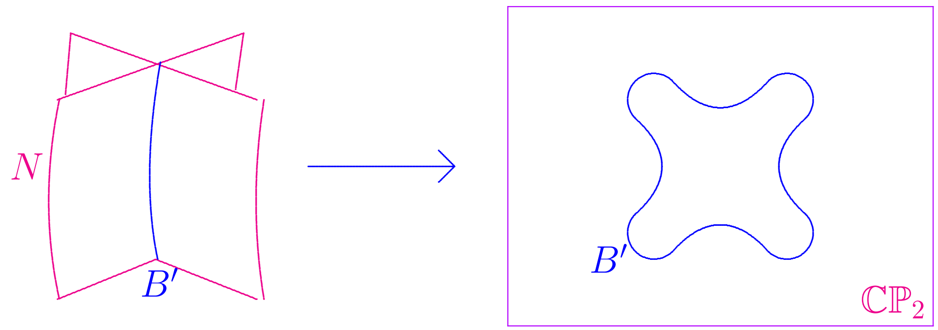
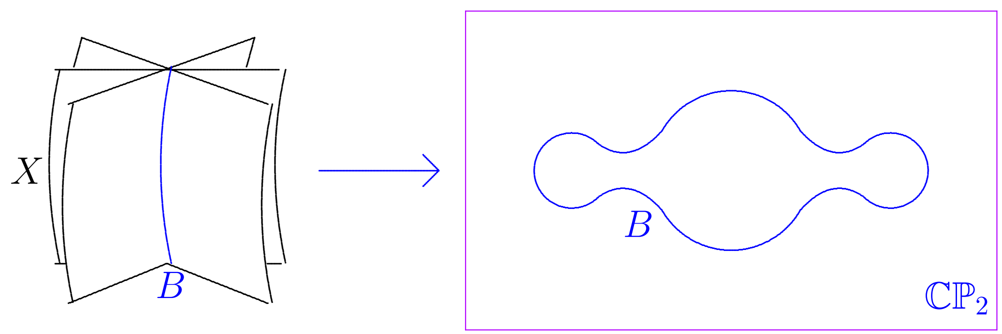


Aubin/Yau $\Longrightarrow$ N carries Einstein metric.

Now let X be a triple cyclic cover $\mathbb{CP}_2$, ramified at a smooth sextic



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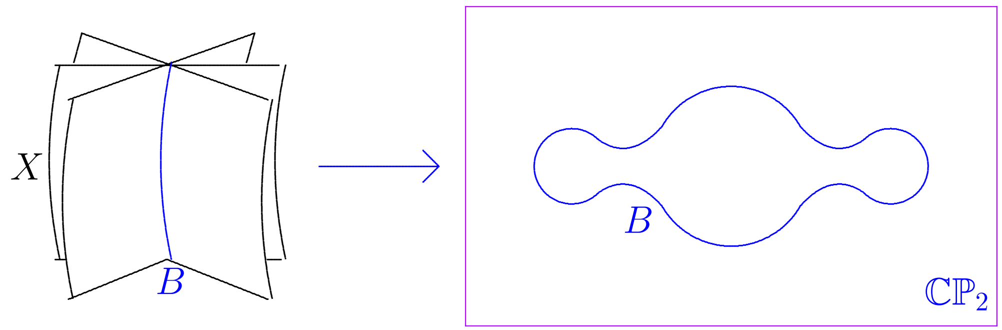

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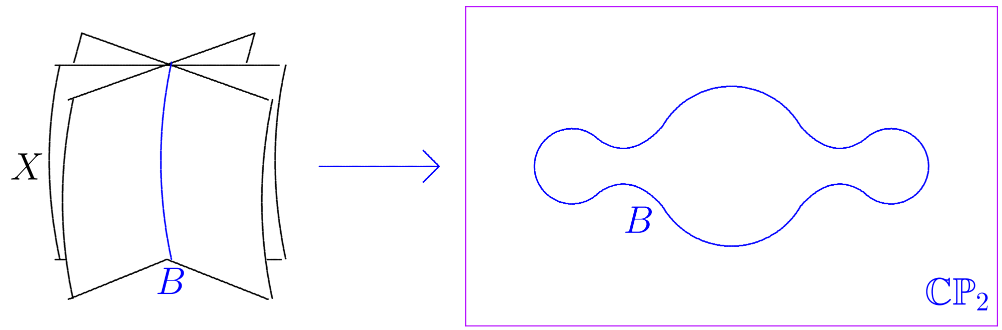

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Here

$$c_1{}^2(X) = 3$$

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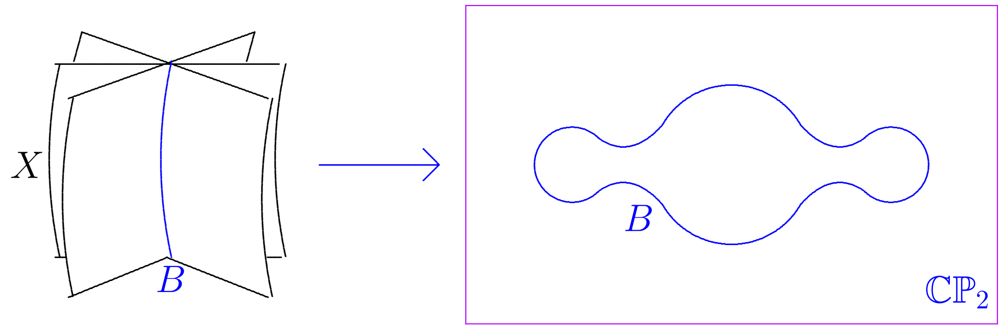
In example:

$$\begin{aligned} {c_1}^2(X) &= 3 \\ k = 1 &= {c_1}^2(X)/3 \end{aligned}$$

X is triple cover $\mathbb{CP}_2$ ramified at sextic



$$M = X \# \overline{\mathbb{CP}}_2 .$$

Theorem $\Longrightarrow$ *no* Einstein metric on M.

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Moral: Existence depends on diffeotype!

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$$M \quad = \quad T^4, \qquad K3, \qquad \mathcal{H}^4/\Gamma, \qquad \mathbb{C}\mathcal{H}_2/\Gamma.$$

On T^4 and $K3$, Einstein $\Longleftrightarrow$ hyper-Kähler.

If only mod out by $\mathit{Diff}_H(M)$, use period map.

Period map:

$$\{\text{Metrics on } M\} \longrightarrow Gr^+_{b_+}[H^2(M, \mathbb{R})]$$

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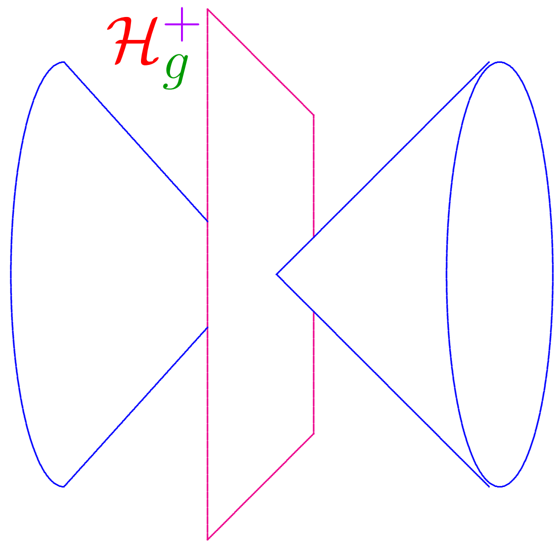
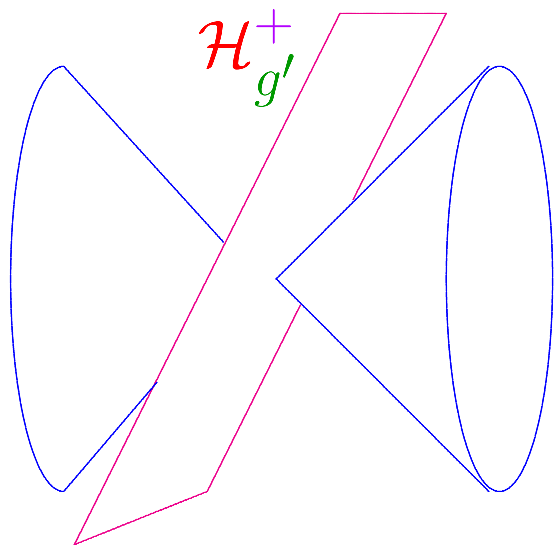


$H^2(M, \mathbb{R})$



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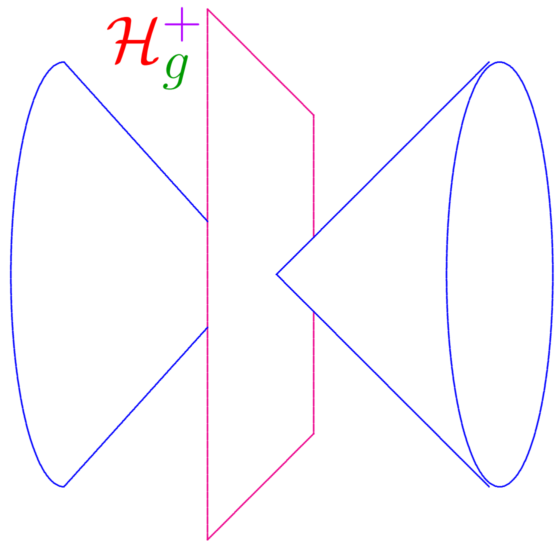

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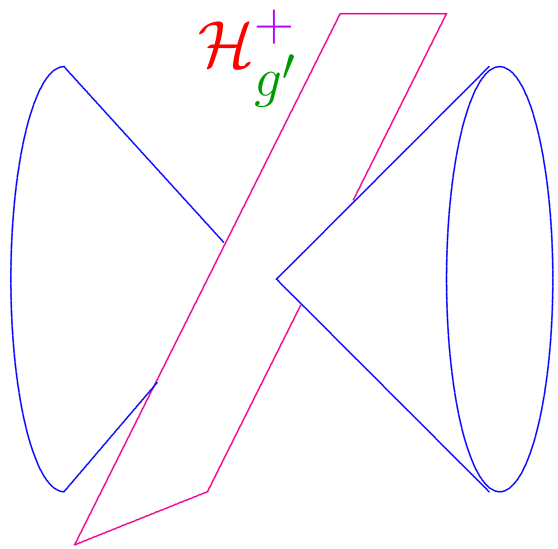

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Smooth: complement of codimension-3 subset.

When a complex surface (M, J) admits an Einstein metric g, also admits a symplectic structure ω.

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When $\lambda \geq 0$, we will now see that symplectic alone suffices to imply strong results.

Theorem (L).

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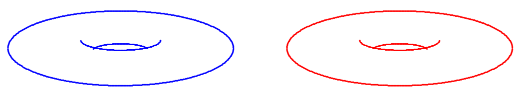

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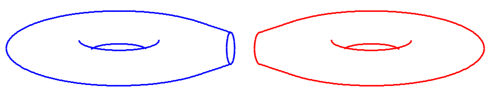

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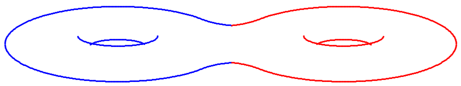

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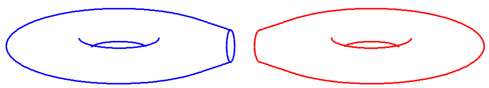

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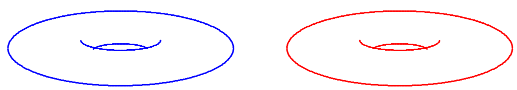

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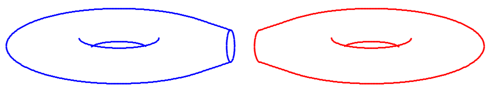

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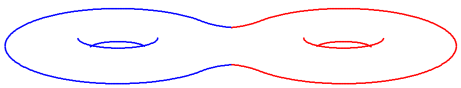

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Del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.

$$
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Definitive list . . .

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But we understand some cases better than others!

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Know an Einstein metric on each manifold.

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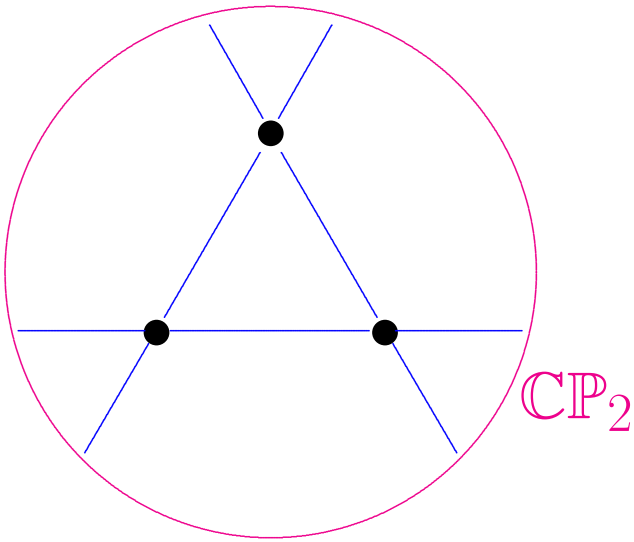

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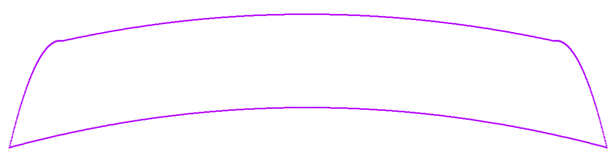

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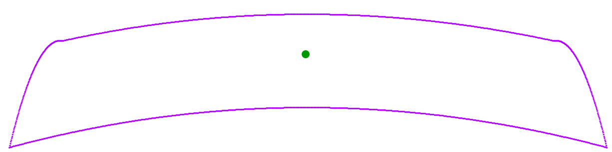

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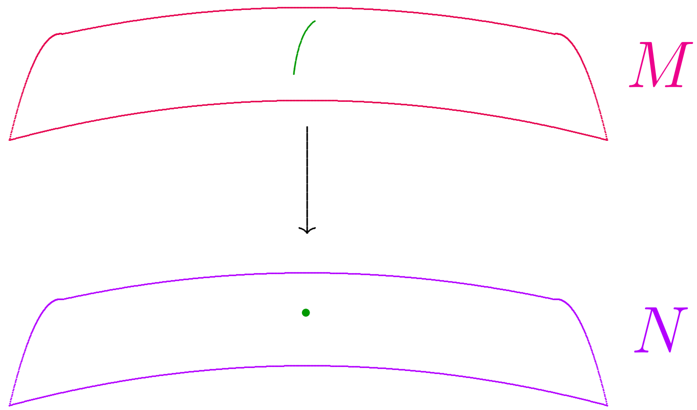

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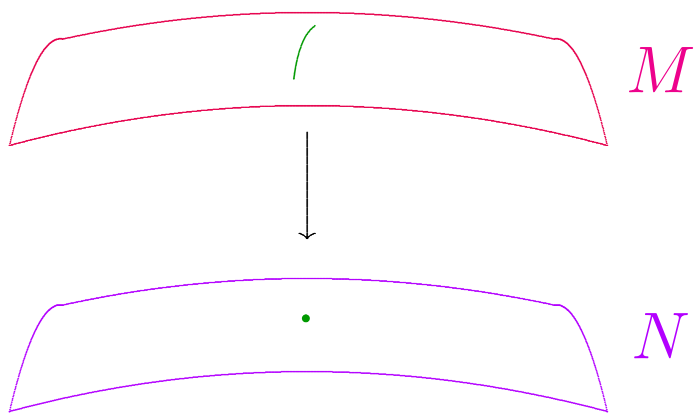

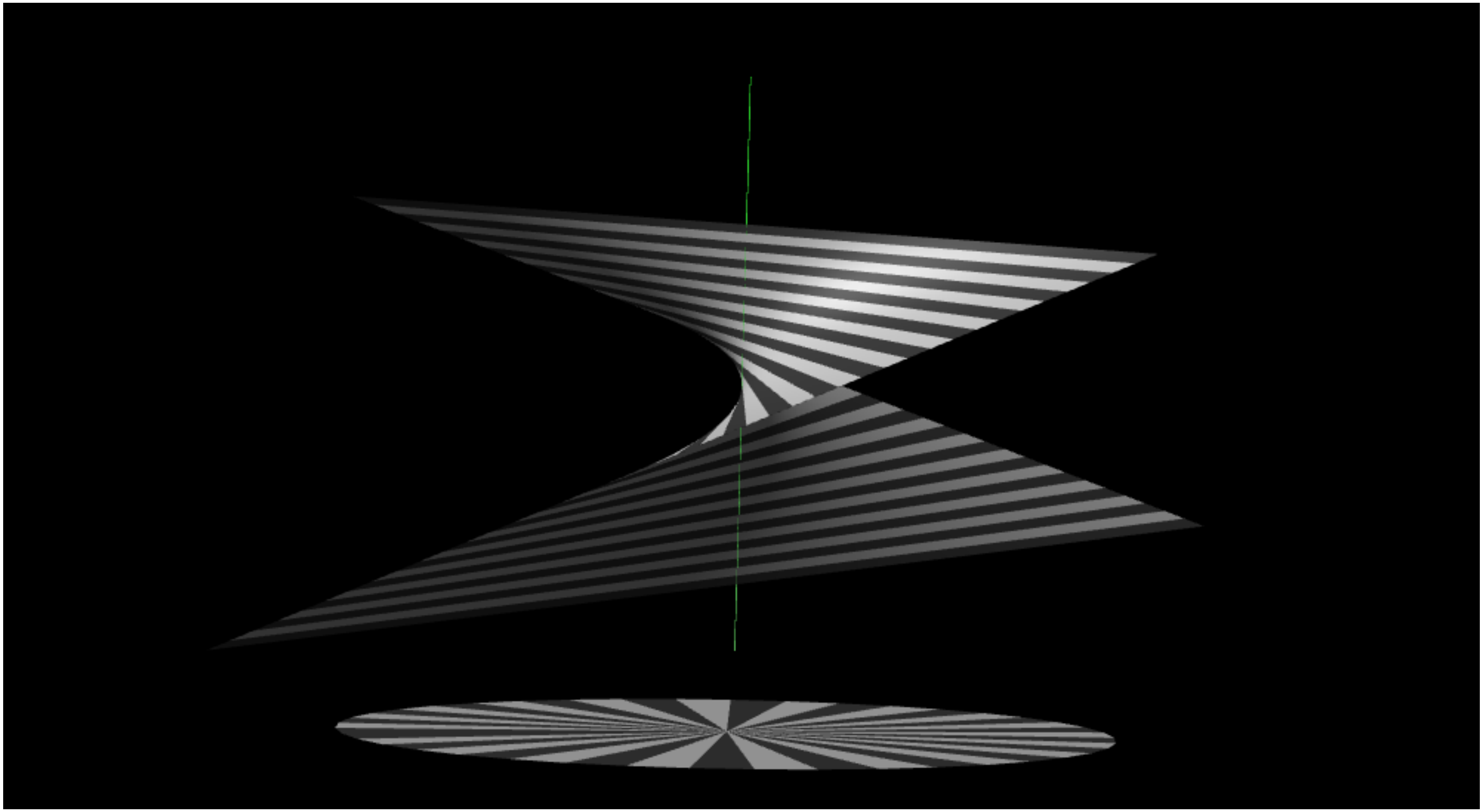

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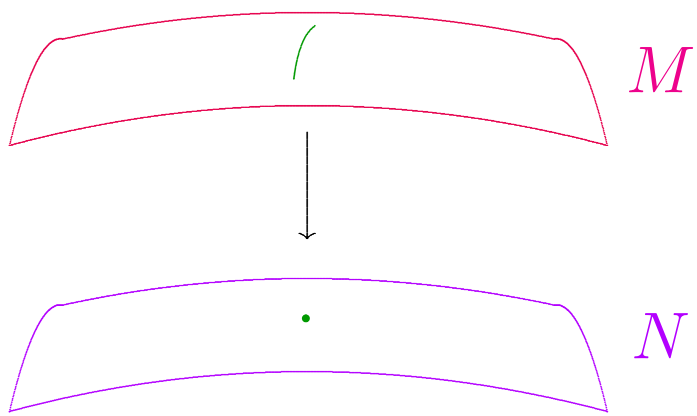

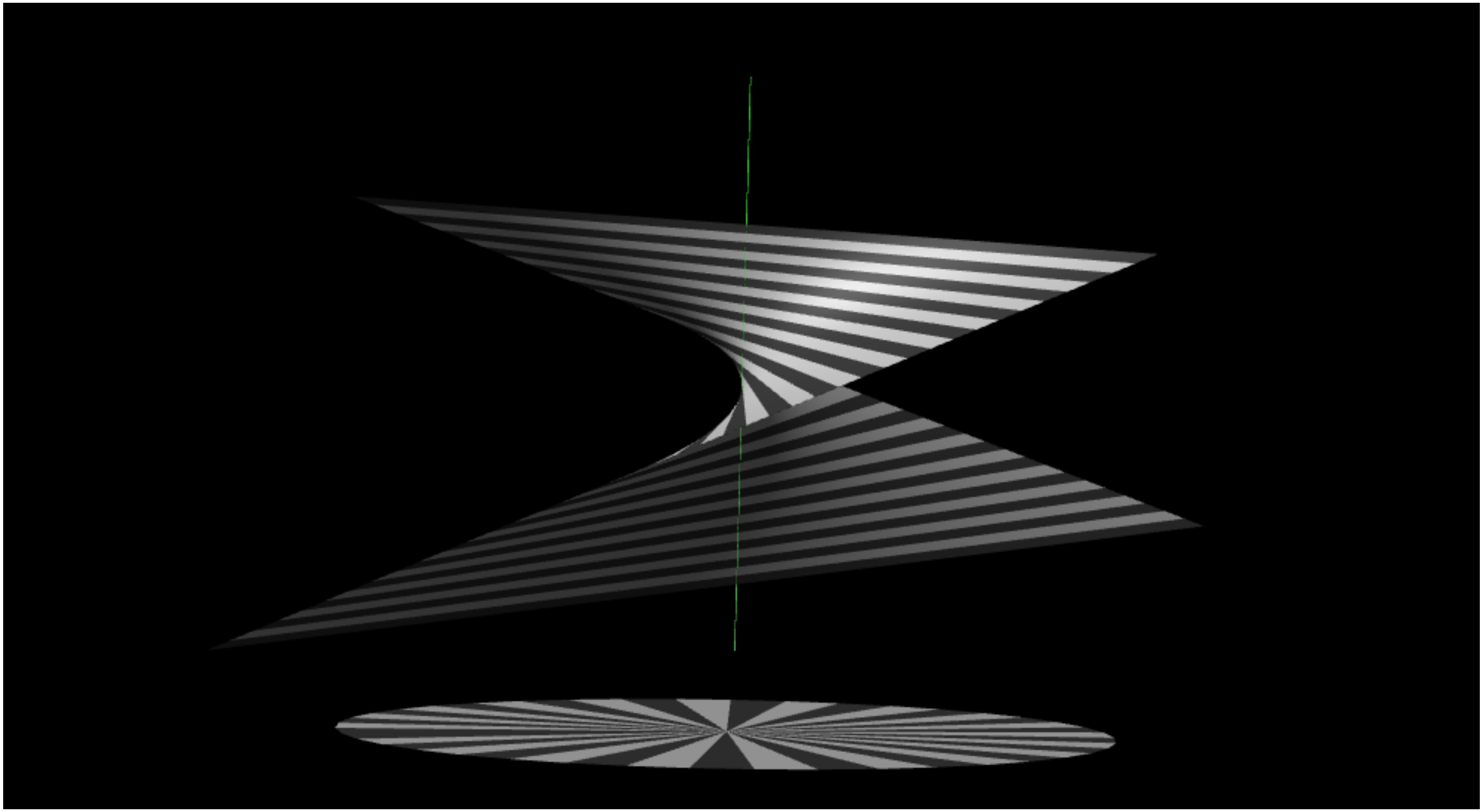

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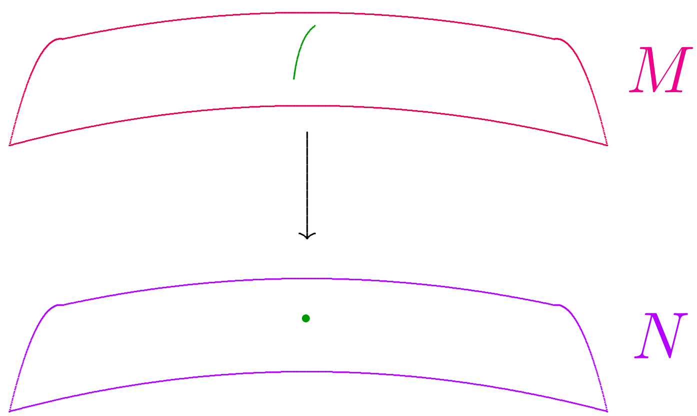

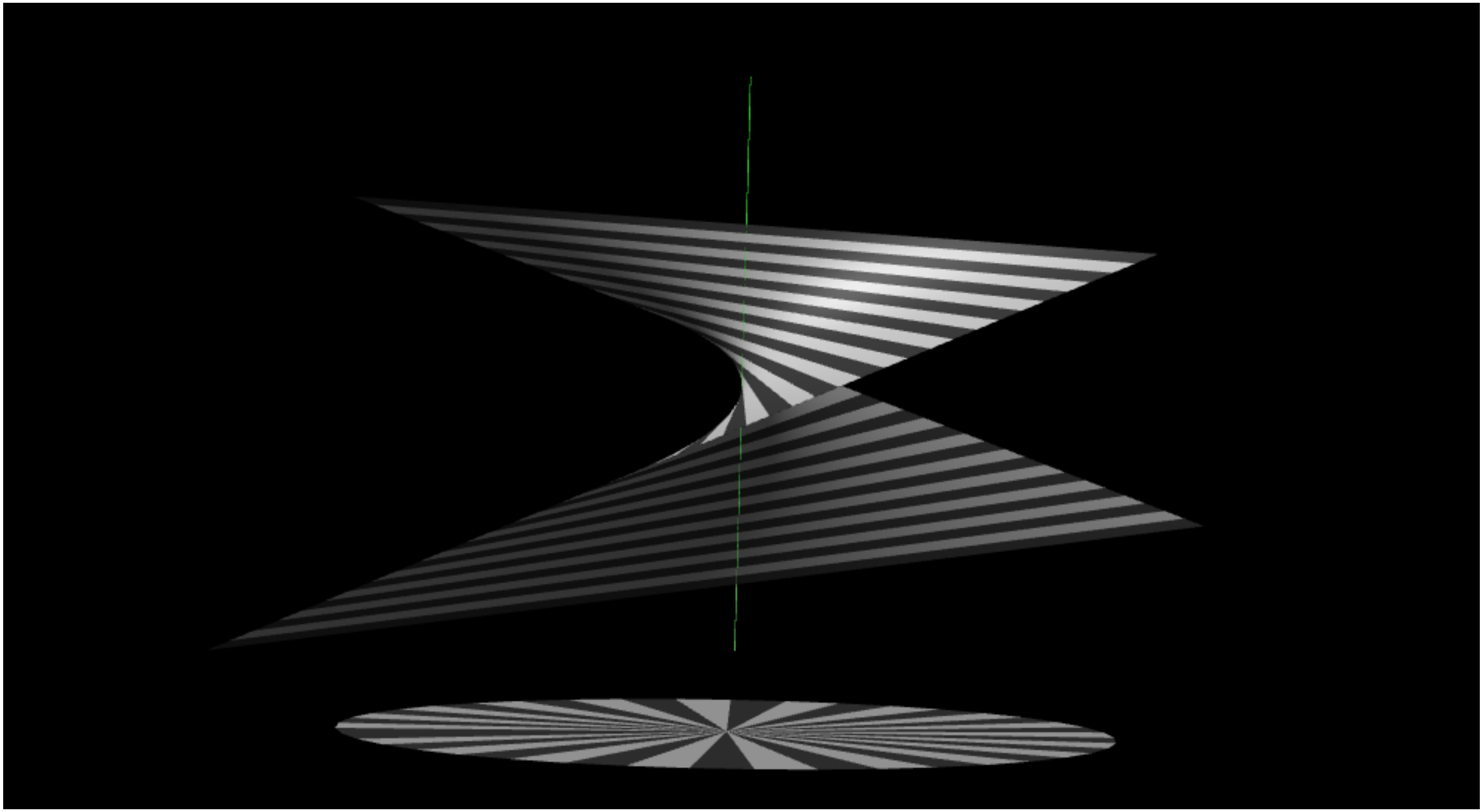

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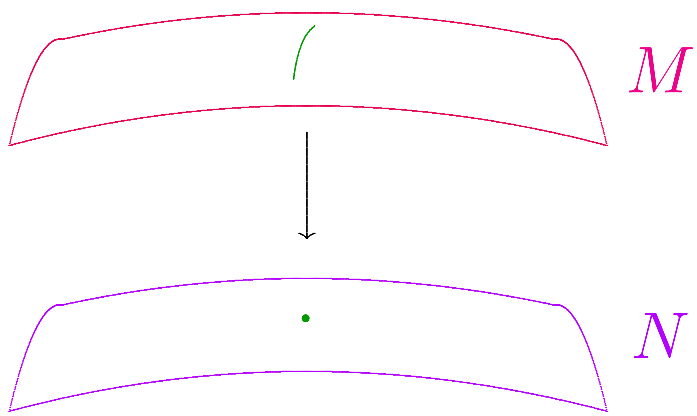

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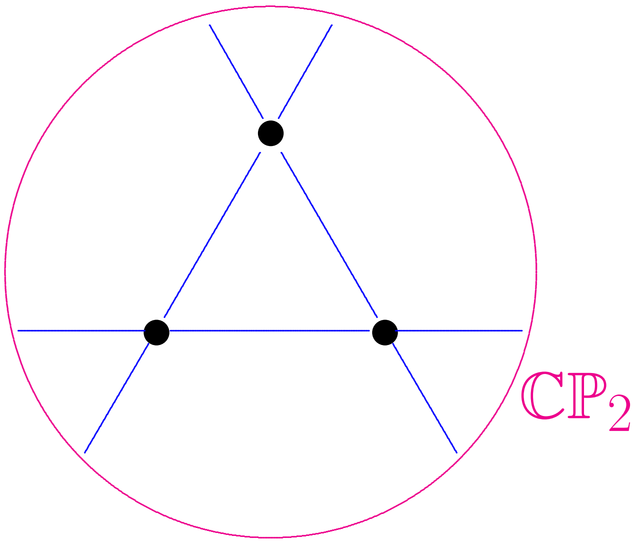

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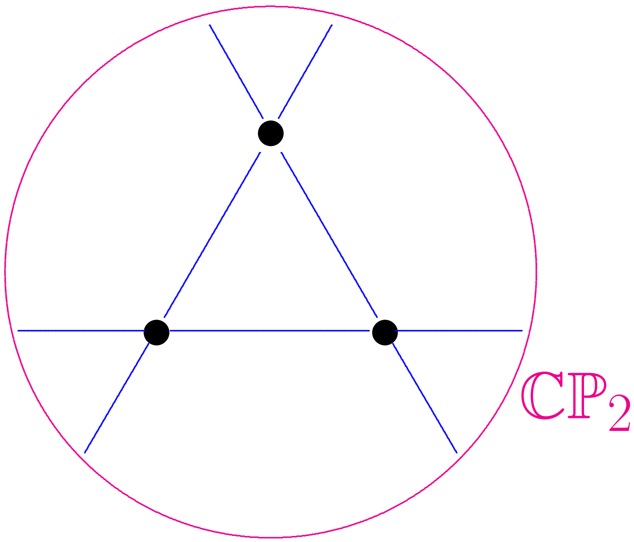

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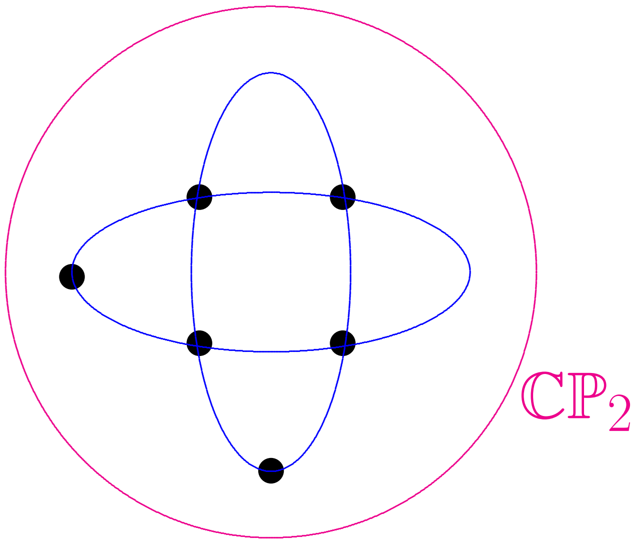

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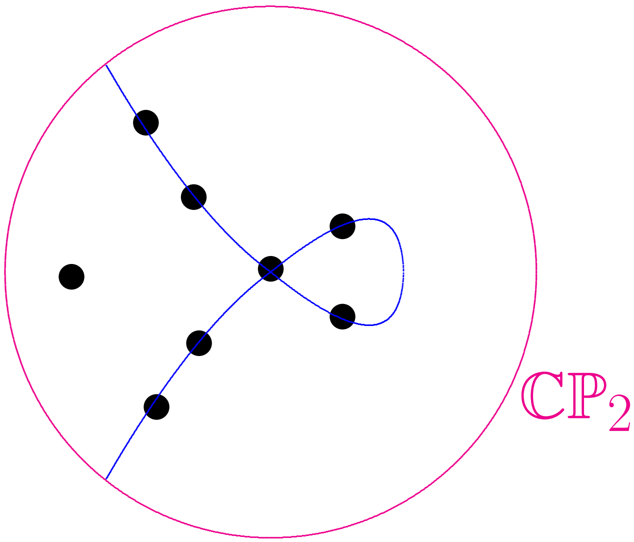

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Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber…

Uniqueness: Bando-Mabuchi, L 2012…

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Corollary. $\mathscr{E}^+_\omega(M)$ *is exactly one connected component of* $\mathscr{E}(M)$*.*

Method of Proof.

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Remark. If such metrics exist, $b_+(M) = 1$.

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Restrict to Einstein case; use results of [L '12]...