Four-Manifolds,

Einstein Metrics, &

Differential Topology

Claude LeBrun
Stony Brook University

Rademacher Lectures
University of Pennsylvania
Four-Manifolds,

Einstein Metrics, &

Differential Topology, I

Colloquium:

Einstein Metrics
and Geometrization

October 19, 2016
University of Pennsylvania
Let \((M^n, g)\) be a Riemannian \(n\)-manifold, \(p \in M\).
Let \((M^n, g)\) be a Riemannian \(n\)-manifold, \(p \in M\). Metric defines locally shortest curves, called geodesics. Following geodesics from \(p\) defines a map

\[
\exp : T_pM \rightarrow M
\]
Let \((M^n, g)\) be a Riemannian \(n\)-manifold, \(p \in M\). Metric defines locally shortest curves, called geodesics. Following geodesics from \(p\) defines a map

\[
\exp : T_pM \to M
\]

which is a diffeomorphism on a neighborhood of 0:
Let \((M^n, g)\) be a Riemannian \(n\)-manifold, \(p \in M\). Metric defines locally shortest curves, called geodesics. Following geodesics from \(p\) defines a map

\[
\exp : T_p M \rightarrow M
\]

which is a diffeomorphism on a neighborhood of 0:

Now choosing \(T_p M \cong \mathbb{R}^n\) via some orthonormal basis gives us special coordinates on \(M\).
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = d\mu_{\text{Euclidean}}, \]
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \right] d\mu_{\text{Euclidean}}, \]
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + \ldots \right] d\mu_{\text{Euclidean}}, \]
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]

where \( r \) is the Ricci tensor.
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]

where \( r \) is the \textit{Ricci tensor} \( r_{jk} = \mathcal{R}^i_{jk} \).
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]

where \( r \) is the Ricci tensor \( r_{jk} = \mathcal{R}^i_{\ jik} \).

The Ricci curvature
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]

where \( r \) is the Ricci tensor \( r_{jk} = \mathcal{R}^i_{jk} \).

The Ricci curvature is by definition the function on the unit tangent bundle.
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]

where \( r \) is the Ricci tensor \( r_{jk} = R^i_{jik} \).

The Ricci curvature is by definition the function on the unit tangent bundle

\[ STM = \{ v \in TM \mid g(v, v) = 1 \} \]
In these “geodesic normal” coordinates the metric volume measure is given by

\[
d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}},
\]

where \( r \) is the Ricci tensor \( r_{jk} = R^{i}_{jk} \).

The Ricci curvature is by definition the function on the unit tangent bundle

\[
STM = \{ v \in TM \mid g(v, v) = 1 \}
\]

given by

\[
v \mapsto r(v, v).
\]
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]

where \( r \) is the Ricci tensor \( r_{jk} = \mathcal{R}^i_{jik} \).

Volume of narrow cone
In these “geodesic normal” coordinates the metric volume measure is given by

\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]

where \( r \) is the Ricci tensor \( r_{jk} = \mathcal{R}^i_{\ jik} \).

Volume of narrow cone vs. Euclidean expectation:

\[
\frac{\text{vol}_g(C_\varepsilon(p, v, \Omega))}{\text{Euclidean answer}} \approx 1 - r(v, v) \frac{n\varepsilon^2}{6(n+2)} + O(\varepsilon^3)
\]
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature.
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$. 
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.,

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$. 
**Definition.** A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.}

As punishment ...
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

$\lambda$ called Einstein constant.
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

$\lambda$ called Einstein constant.

Has same sign as the scalar curvature

$$s = r^j_j = \mathcal{R}^{ij} ij.$$
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

$\lambda$ called Einstein constant.

Has same sign as the scalar curvature

$$s = r^j_j = R^{ij}i_j.$$
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$. 

---
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

In dimension $n > 2$, equivalent to

$$r = fg$$
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

\[ r = \lambda g \]

for some constant $\lambda \in \mathbb{R}$.

In dimension $n > 2$, equivalent to

\[ r = \frac{s}{n} g \]
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

In dimension $n > 2$, equivalent to

$$r = \frac{s}{n} g$$

because of “contracted Bianchi identity”

$$\nabla \cdot r = \nabla \frac{s}{2}.$$
**Definition.** A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$. 

In dimension $n > 2$, equivalent to

$$r = \frac{s}{n} g$$
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

\[ \because \text{In dimension } n > 2, \text{ equivalent to} \]

$$\dot{r} = 0$$
**Definition.** A *Riemannian metric* $g$ is said to be *Einstein* if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

∴ In dimension $n > 2$, equivalent to

$$\dot{r} = 0$$

where

$$\dot{r} := r - \frac{s}{n} g$$

is the trace-free Ricci tensor.
**Definition.** A Riemannian metric $g$ is said to be **Einstein** if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$. 
**Definition.** A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow

“Mathematicians are like Frenchmen: tell them something, they translate it into their own language, and, before you know it, it’s something entirely different.”

— J.W. von Goethe
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.,

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$. 
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e. 

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

So why are we interested?
**Definition.** A Riemannian metric $g$ is said to be **Einstein** if it has constant Ricci curvature — i.e.,

$$ r = \lambda g $$

for some constant $\lambda \in \mathbb{R}$. 
**Definition.** A Riemannian metric $g$ is said to be **Einstein** if it has **constant Ricci curvature** — i.e. 

$$ r = \lambda g $$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

$n = 2, 3$: Einstein $\iff$ constant sectional
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

$n = 2, 3$: Einstein $\iff$ constant sectional

$n \geq 4$: Einstein $\iff, \not\iff$ constant sectional
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Determined system:

same number of equations as unknowns.
**Definition.** A Riemannian metric $g$ is said to be **Einstein** if it has constant Ricci curvature — i.e.

$$ r = \lambda g $$

for some constant $\lambda \in \mathbb{R}$. 

Generalizes constant sectional curvature condition, but weaker.

**Determined system:**
same number of equations as unknowns.

$g_{jk}$: $\frac{n(n+1)}{2}$ components.

$r_{jk}$: $\frac{n(n+1)}{2}$ components.
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Determined system:
same number of equations as unknowns.

$g_{jk}$: $\frac{n(n+1)}{2}$ components.

$r_{jk}$: $\frac{n(n+1)}{2}$ components.

(DeTurck-Kazdan)
**Definition.** A Riemannian metric $g$ is said to be **Einstein** if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

**Determined system:**
same number of equations as unknowns.

$g_{jk}$: $\frac{n(n+1)}{2}$ components.

$r_{jk}$: $\frac{n(n+1)}{2}$ components.
**Definition.** A Riemannian metric $g$ is said to be **Einstein** if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

**Determined system:**
same number of equations as unknowns.

- $g_{jk}$: $\frac{n(n+1)}{2}$ components.
- $r_{jk}$: $\frac{n(n+1)}{2}$ components.
- $R_{j klm}$: $\frac{n^2(n^2-1)}{12}$ components.
Definition. A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Determined system:
same number of equations as unknowns.

Elliptic non-linear PDE after gauge fixing.
**Definition.** A Riemannian metric $g$ is said to be **Einstein** if it has **constant Ricci curvature** — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

**Determined system:**

same number of equations as unknowns.

**Elliptic non-linear PDE after gauge fixing.**

$$\Delta x^j = 0 \implies r_{jk} = \frac{1}{2} \Delta g_{jk} + \text{lots}.$$
**Definition.** A Riemannian metric $g$ is said to be **Einstein** if it has constant Ricci curvature — i.e.

$$ r = \lambda g $$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Determined system:
same number of equations as unknowns.

Elliptic non-linear PDE after gauge fixing.
Solves a natural variational problem.
**Definition.** A Riemannian metric $g$ is said to be Einstein if it has constant Ricci curvature — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

**Determined system:**
same number of equations as unknowns.

**Elliptic non-linear PDE after gauge fixing.**

**Solves a natural variational problem.**

$$g \mapsto \int_M s_g d\mu_g$$
**Definition.** A Riemannian metric $g$ is said to be **Einstein** if it has **constant Ricci curvature** — i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

**Determined system:**
same number of equations as unknowns.

**Elliptic non-linear PDE after gauge fixing.**

Solves a natural **variational problem**.

$$g \mapsto \int_M s_g d\mu_g , \quad \text{Vol}(M, g) = \text{const.}$$
**Definition.** A Riemannian metric \( g \) is said to be Einstein if it has constant Ricci curvature — i.e.

\[ r = \lambda g \]

for some constant \( \lambda \in \mathbb{R} \).

Generalizes constant sectional curvature condition, but weaker.

**Determined system:**

same number of equations as unknowns.

**Elliptic non-linear PDE after gauge fixing.**

**Solves a natural variational problem.** (Hilbert)

\[ g \mapsto \int_M s_g d\mu_g, \quad \text{Vol}(M, g) = \text{const}. \]
**Question (Yamabe).** Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?
Question (Yamabe). Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?

What we know:
Question (Yamabe). Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?

What we know:

- When $n = 2$: Yes! (Riemann)
**Question**  (Yamabe). *Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?*

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\iff$ Poincaré conjecture.
**Question** (Yamabe). *Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?*

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\iff$ Poincaré conjecture. Hamilton, Perelman, . . . Yes!
Question (Yamabe). Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\iff$ Poincaré conjecture. Hamilton, Perelman, ... Yes!
- When $n = 5$: Probably! (Boyer-Galicki-Kollár)
**Question** (Yamabe). *Does every smooth compact simply-connected \( n \)-manifold admit an Einstein metric?*

What we know:

- **When** \( n = 2 \): Yes! (Riemann)
- **When** \( n = 3 \): \( \iff \) Poincaré conjecture. Hamilton, Perelman, . . . Yes!
- **When** \( n = 5 \): Probably! (Boyer-Galicki-Kollár)
- **When** \( n \geq 6 \): Maybe???
**Question** (Yamabe). *Does every smooth compact simply-connected $n$-manifold admit an Einstein metric?*

What we know:

- **When $n = 2$: Yes!** (Riemann)
- **When $n = 3$: $\iff$ Poincaré conjecture.**
  - Hamilton, Perelman, ... Yes!
- **When $n = 4$: No!** (Hitchin)
- **When $n = 5$: Probably!** (Boyer-Galicki-Kollár)
- **When $n \geq 6$: Maybe???
Dimension 3:
Dimension 3:
Einstein’s equations are “locally trivial:”
Dimension 3:

Einstein's equations are “locally trivial:”

\[ K(v^\perp) = \frac{s}{2} - r(v, v). \]
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$$(M^3, g) \text{ Einstein } \implies M^3 = S^3/\Gamma, \mathbb{R}^3/\Gamma, \text{ or } \mathcal{H}^3/\Gamma.$$
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

\[ \Rightarrow \text{If } M^3 \text{ carries Einstein metric, } \pi_2(M) = 0, \]
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

\[ \implies \text{If } M^3 \text{ carries Einstein metric, } \pi_2(M) = 0, \]

because \( M \) has universal cover \( S^3, \mathbb{R}^3, \) or \( \mathcal{H}^3 \).
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

⇒ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$. 

Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

\[ \Rightarrow \text{If } M^3 \text{ carries Einstein metric, } \pi_2(M) = 0. \]

\[ \Rightarrow \text{Existence obstructed for connect sums } M^3 \# N^3. \]
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

⇒ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

⇒ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum #: 

\[ \text{Diagrams:} \]
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

\[ \Rightarrow \text{If } M^3 \text{ carries Einstein metric, } \pi_2(M) = 0. \]

\[ \Rightarrow \text{Existence obstructed for connect sums } M^3 \# N^3. \]

Connected sum #:
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

\[ \Rightarrow \text{If } M^3 \text{ carries Einstein metric, } \pi_2(M) = 0. \]

\[ \Rightarrow \text{Existence obstructed for connect sums } M^3 \# N^3. \]

Connected sum \#: 

\[ \text{Connected sum } \\]
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\Rightarrow$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\Rightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum $\#$:
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

⇒ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

⇒ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum #:
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

\[ \implies \text{If } M^3 \text{ carries Einstein metric, } \pi_2(M) = 0. \]

\[ \implies \text{Existence obstructed for connect sums } M^3 \# N^3. \]

Connected sum \#:
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

\[ \Rightarrow \text{If } M^3 \text{ carries Einstein metric, } \pi_2(M) = 0. \]

\[ \Rightarrow \text{Existence obstructed for connect sums } M^3 \# N^3. \]

Connected sum \#: 

\[ \begin{array}{ccc}
   & & \\
   & & \\
   & & \\
   & & \\
\end{array} \]
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

⇒ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

⇒ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum #:
Dimension 3:

Einstein’s equations are “locally trivial:”
Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$. 
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

⇒ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

⇒ Existence obstructed for connect sums $M^3 \# N^3$.

Ricci flow

\[
\frac{\partial}{\partial t} g_{jk} = -2r_{jk}
\]
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

$\implies$ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

$\implies$ Existence obstructed for connect sums $M^3 \# N^3$.

Ricci flow pinches off $S^2$ necks.
Dimension 3:

Einstein’s equations are “locally trivial:”
Einstein metrics have constant sectional curvature.

⇒ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

⇒ Existence obstructed for connect sums $M^3 \# N^3$.

Ricci flow pinches off $S^2$ necks.

First step in geometrization:
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

⇒ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

⇒ Existence obstructed for connect sums $M^3 \# N^3$.

Ricci flow pinches off $S^2$ necks.

First step in geometrization:

Prime Decomposition.
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

⇒ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

⇒ Existence obstructed for connect sums $M^3 \# N^3$.

Ricci flow leads to geometrization:
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

⇒ If $M^3$ carries Einstein metric, $\pi_2(M) = 0$.

⇒ Existence obstructed for connect sums $M^3 \# N^3$.

Ricci flow leads to geometrization:

Decomposes $M$ into Einstein and collapsed pieces.
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

\((M^3, g)\) Einstein \implies M^3 = S^3/\Gamma, \mathbb{R}^3/\Gamma, \text{ or } \mathcal{H}^3/\Gamma.\)
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

\((M^3, g)\ \text{Einstein} \implies M^3 = S^3/\Gamma, \mathbb{R}^3/\Gamma, \text{or } \mathcal{H}^3/\Gamma.\)

Moduli space of Einstein metrics:

\(\mathcal{E}(M) = \{\text{Einstein } g\}/(\text{Diffeos } \times \mathbb{R}^+)\)
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

\((M^3, g)\) Einstein \implies M^3 = S^3/\Gamma, \mathbb{R}^3/\Gamma, \text{ or } \mathcal{H}^3/\Gamma.\)

Moduli space of Einstein metrics:

\(\mathcal{E}(M^3)\) is connected (if \(\neq \emptyset\)).
Dimension 3:

Einstein’s equations are “locally trivial:”

Einstein metrics have constant sectional curvature.

\((M^3, g)\) Einstein \implies M^3 = S^3/\Gamma, \mathbb{R}^3/\Gamma, \text{ or } \mathcal{H}^3/\Gamma.

Moduli space of Einstein metrics:

\(\mathcal{E}(M^3)\) is connected (if \(\neq \emptyset\)).

In fact, typically a point! (Mostow rigidity)
Dimension 5:

By contrast...
Dimension 5:

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components.
Dimension 5:

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components. Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$.  

(Wang-Ziller)
Dimension 5:

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components. Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$.  

(Wang-Ziller)

The moduli space of Einstein metrics on $S^5$ has infinitely many connected components.
Dimension 5:

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components. Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$.  

(Wang-Ziller)

The moduli space of Einstein metrics on $S^5$ has infinitely many connected components.

(B"ohm, Collins-Székelyhidi)
Dimension 5:

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components. Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$. 

(Wang-Ziller)

The moduli space of Einstein metrics on $S^5$ has infinitely many connected components.

(Böhm, Collins-Székelyhidi)

Connected sums $(S^2 \times S^3) \# \cdots \# (S^2 \times S^3)$ admit Einstein metrics for arbitrarily many summands. Moduli space always has infinitely many connected components.
Dimension 5:

The moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many connected components. Unit-volume Einstein metrics exist for sequence of $\lambda \to 0^+$.  

(Wang-Ziller)

The moduli space of Einstein metrics on $S^5$ has infinitely many connected components.

(Böhm, Collins-Székelyhidi)

Connected sums $(S^2 \times S^3) \# \cdots \# (S^2 \times S^3)$ admit Einstein metrics for arbitrarily many summands. Moduli space always has infinitely many connected components.

Similarly: “most” simply-connected spin 5-manifolds.

(Boyer-Galicki, Kollár, Van Coevering)
Dimension > 5:
Dimension $> 5$:

Einstein metrics again seem common.
Dimension $> 5$:

Einstein metrics again seem common. Moduli spaces typically disconnected.
Dimension $> 5$:

Einstein metrics again seem common. Moduli spaces typically disconnected.

Non-standard Einstein metrics exist on $S^n$
Dimension $> 5$:

Einstein metrics again seem common. Moduli spaces typically disconnected.

Non-standard Einstein metrics exist on $S^n$ if $n < 12$ or if e.g. $n \equiv 1 \mod 4$. 
Dimension > 5:

Einstein metrics again seem common. Moduli spaces typically disconnected.

Non-standard Einstein metrics exist on $S^n$ if $n < 12$ or if e.g. $n \equiv 1 \mod 4$.

(Jensen, Böhm, Boyer-Galicki-Kollár, ...)
Dimension $> 5$:

Einstein metrics again seem common. Moduli spaces typically disconnected.

Non-standard Einstein metrics exist on $S^n$ if $n < 12$ or if e.g. $n \equiv 1 \mod 4$.

(Jensen, Böhm, Boyer-Galicki-Kollár, ...)

All exotic 7-spheres $\Sigma^7$ admit Einstein metrics.
Dimension > 5:

Einstein metrics again seem common. Moduli spaces typically disconnected.

Non-standard Einstein metrics exist on $S^n$ if $n < 12$ or if e.g. $n \equiv 1 \mod 4$.

(Jensen, Böhm, Boyer-Galicki-Kollár, ...)

All exotic 7-spheres $\Sigma^7$ admit Einstein metrics. Einstein moduli space always disconnected.
Dimension > 5:

Einstein metrics again seem common.
Moduli spaces typically disconnected.

Non-standard Einstein metrics exist on $S^n$ if $n < 12$ or if e.g. $n \equiv 1 \mod 4$.

(Jensen, Böhm, Boyer-Galicki-Kollár, \ldots)

All exotic 7-spheres $\Sigma^7$ admit Einstein metrics.
Einstein moduli space always disconnected.

(Boyer-Galicki-Kollár)
Dimension > 5:

Einstein metrics again seem common. Moduli spaces typically disconnected.

Non-standard Einstein metrics exist on $S^n$ if $n < 12$ or if e.g. $n \equiv 1 \mod 4$.

(Jensen, Böhm, Boyer-Galicki-Kollár, . . .)

All exotic 7-spheres $\Sigma^7$ admit Einstein metrics. Einstein moduli space always disconnected.

(Boyer-Galicki-Kollár)

There’s a smooth compact 8-manifold admitting both $\lambda > 0$ and $\lambda < 0$ Einstein metrics.
Dimension $> 5$:

Einstein metrics again seem common. Moduli spaces typically disconnected.

Non-standard Einstein metrics exist on $S^n$ if $n < 12$ or if e.g. $n \equiv 1 \mod 4$.

(Jensen, Böhm, Boyer-Galicki-Kollár, \ldots)

All exotic 7-spheres $\Sigma^7$ admit Einstein metrics. Einstein moduli space always disconnected.

(Boyer-Galicki-Kollár)

There’s a smooth compact 8-manifold admitting both $\lambda > 0$ and $\lambda < 0$ Einstein metrics. Similarly for all larger $n = 4k$. 
Dimension > 5:

Einstein metrics again seem common. Moduli spaces typically disconnected.

Non-standard Einstein metrics exist on $S^n$ if $n < 12$ or if e.g. $n \equiv 1 \mod 4$.

(Jensen, Böhm, Boyer-Galicki-Kollár, ...)

All exotic 7-spheres $\Sigma^7$ admit Einstein metrics. Einstein moduli space always disconnected.

(Boyer-Galicki-Kollár)

There’s a smooth compact 8-manifold admitting both $\lambda > 0$ and $\lambda < 0$ Einstein metrics. Similarly for all larger $n = 4k$.

(Catanese-L.)
Dimension $\geq 5$: 
Dimension $\geq 5$:

All very interesting, but we’re left with...
Dimension $\geq 5$:

All very interesting, but we’re left with... 

Recognition Problem!
Dimension $\geq 5$:

All very interesting, but we’re left with... Recognition Problem!

In high dimensions, an Einstein metric does not help reveal the identity of $M^n$. 
Dimension $\geq 5$:

All very interesting, but we’re left with. . .

Recognition Problem!

In high dimensions, an Einstein metric does not help reveal the identity of $M^n$.

$\therefore$ Not a meaningful geometrization of manifold!
Dimension 4:

By contrast...
Dimension 4:

**Theorem** (Berger). *Any Einstein metric on 4-torus $T^4$ is flat.*
Dimension 4:

**Theorem** (Berger). *Any Einstein metric on 4-torus $T^4$ is flat.*

$\implies$ Moduli space of Einstein metrics is connected.
Dimension 4:

**Theorem** (Berger). *Any Einstein metric on 4-torus $T^4$ is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

**Theorem** (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*
Dimension 4:

**Theorem (Berger).** Any Einstein metric on 4-torus $T^4$ is flat.

$\implies$ Moduli space of Einstein metrics is connected.

**Theorem (Hitchin).** Any Einstein metric on $K3$ is Ricci-flat Kähler.

(Terminology to be explained later!)
Dimension 4:

**Theorem** (Berger). *Any Einstein metric on 4-torus $T^4$ is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

**Theorem** (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*
Dimension 4:

**Theorem (Berger).** Any Einstein metric on 4-torus $T^4$ is flat.

$\implies$ Moduli space of Einstein metrics is connected.

**Theorem (Hitchin).** Any Einstein metric on K3 is Ricci-flat Kähler.

$\implies$ Moduli space of Einstein metrics is connected.
Dimension 4:

**Theorem** (Berger). *Any Einstein metric on 4-torus $T^4$ is flat.*

$\implies$ Moduli space of Einstein metrics is connected.

**Theorem** (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\implies$ Moduli space of Einstein metrics is connected.

(Kodaira, Yau, Siu, et al.)
Dimension 4:

**Theorem** (Berger). *Any Einstein metric on 4-torus $T^4$ is flat.*

$\Rightarrow$ Moduli space of Einstein metrics is connected.

**Theorem** (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\Rightarrow$ Moduli space of Einstein metrics is connected.

(Kodaira, Yau, Siu, et al.)

**Theorem** (Besson-Courtois-Gallot). *There is only one Einstein metric on compact hyperbolic 4-manifold $\mathbb{H}^4/\Gamma$, up to scale and diffeos.*
Dimension 4:

**Theorem (Berger).** Any Einstein metric on 4-torus $T^4$ is flat.

$\implies$ Moduli space of Einstein metrics is connected.

**Theorem (Hitchin).** Any Einstein metric on $K3$ is Ricci-flat Kähler.

$\implies$ Moduli space of Einstein metrics is connected.  
(Kodaira, Yau, Siu, et al.)

**Theorem (Besson-Courtois-Gallot).** There is only one Einstein metric on compact hyperbolic 4-manifold $\mathcal{H}^4/\Gamma$, up to scale and diffeos.

**Theorem (L).** There is only one Einstein metric on compact complex-hyperbolic 4-manifold $\mathcal{CH}_2/\Gamma$, up to scale and diffeos.
Four Dimensions is Exceptional
Four Dimensions is Exceptional

When \( n = 4 \), existence for Einstein depends delicately on smooth structure.
Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.
Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.

But might allow for geometrization of 4-manifolds by decomposition into Einstein and collapsed pieces.
Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.

But might allow for geometrization of 4-manifolds by decomposition into Einstein and collapsed pieces.

Enough rigidity apparently still holds in dimension four to perhaps call this a geometrization.
Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.

But might allow for geometrization of 4-manifolds by decomposition into Einstein and collapsed pieces.

Enough rigidity apparently still holds in dimension four to perhaps call this a geometrization.

By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.
What’s so special about dimension 4?
What’s so special about dimension 4?

The Lie group $SO(4)$ is not simple
What’s so special about dimension 4?

The Lie group $SO(4)$ is \textit{not simple}:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$
What’s so special about dimension 4?

The Lie group $SO(4)$ is not simple:

$$so(4) \cong so(3) \oplus so(3).$$

On oriented $(M^4, g)$,
What’s so special about dimension 4?

The Lie group $SO(4)$ is not simple:

$$so(4) \cong so(3) \oplus so(3).$$

On oriented $(M^4, g)$, $\implies$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$
What’s so special about dimension 4?

The Lie group $SO(4)$ is not simple:

$$so(4) \cong so(3) \oplus so(3).$$

On oriented $(M^4, g)$, $\Rightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are $(\pm 1)$-eigenspaces of

$$\ast : \Lambda^2 \to \Lambda^2,$$
What’s so special about dimension 4?

The Lie group $SO(4)$ is not simple:

$$so(4) \cong so(3) \oplus so(3).$$

On oriented $(M^4, g)$, $\implies$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are $(\pm 1)$-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

$\Lambda^+$ self-dual 2-forms.

$\Lambda^-$ anti-self-dual 2-forms.
Riemann curvature of $g$

\[ \mathcal{R} : \Lambda^2 \rightarrow \Lambda^2 \]
Riemann curvature of $g$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:
Riemann curvature of $g$

\[ \mathcal{R} : \Lambda^2 \to \Lambda^2 \]

splits into 4 irreducible pieces:

<table>
<thead>
<tr>
<th></th>
<th>$\Lambda^+$</th>
<th>$W_+ + \frac{s}{12}$</th>
<th>$\hat{r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda^+$</td>
<td>$W_+ + \frac{s}{12}$</td>
<td>$\hat{r}$</td>
<td></td>
</tr>
<tr>
<td>$\Lambda^-$</td>
<td>$\hat{r}$</td>
<td>$W_- + \frac{s}{12}$</td>
<td></td>
</tr>
</tbody>
</table>
Riemann curvature of $g$

\[ \mathcal{R} : \Lambda^2 \to \Lambda^2 \]

splits into 4 irreducible pieces:

\[
\begin{array}{c|c|c}
\Lambda^+ & W_+ + \frac{s}{12} & \hat{r} \\
\hline
\Lambda^- & \hat{r} & W_- + \frac{s}{12} \\
\end{array}
\]

where

- $s$ = scalar curvature
- $\hat{r}$ = trace-free Ricci curvature
- $W_+$ = self-dual Weyl curvature
- $W_-$ = anti-self-dual Weyl curvature
Riemann curvature of $g$

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

splits into 4 irreducible pieces:

$$
\begin{array}{ccc}
\Lambda^+ & \Lambda^{+-} & \Lambda^{-*} \\
W_+ + \frac{s}{12} & \hat{r} & \\
\hat{r} & W_- + \frac{s}{12} & \\
\end{array}
$$

where

- $s = \text{scalar curvature}$
- $\hat{r} = \text{trace-free Ricci curvature}$
- $W_+ = \text{self-dual Weyl curvature}$ (conformally invariant)
- $W_- = \text{anti-self-dual Weyl curvature}$
Thus \((M^4, g)\) Einstein \iff \(\mathcal{R} : \Lambda^2 \to \Lambda^2\) commutes with \(\star : \Lambda^2 \to \Lambda^2 :\)

\[
\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & \hat{r} \\
\hat{r} & W_- + \frac{s}{12}
\end{pmatrix}.
\]
Thus \((M^4, g)\) Einstein \iff \(R : \Lambda^2 \to \Lambda^2\) commutes with \(\star : \Lambda^2 \to \Lambda^2\):

\[
\star = \begin{pmatrix}
I & \\
- & -I
\end{pmatrix}.
\]
Thus \((M^4, g)\) Einstein ⇐⇒

\[ \mathcal{R} : \Lambda^2 \to \Lambda^2 \]

commutes with

\[ \star : \Lambda^2 \to \Lambda^2 : \]

\[ \mathcal{R} = \begin{pmatrix}
W_{++} + \frac{s}{12} & \hat{r} \\
\hat{r} & W_{--} + \frac{s}{12}
\end{pmatrix} \]
Thus \((M^4, g)\) Einstein \iff \(R : \Lambda^2 \to \Lambda^2\)
commutes with
\[\star : \Lambda^2 \to \Lambda^2:\]
\[
\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & 0 \\
0 & W_- + \frac{s}{12}
\end{pmatrix}.
\]
Thus $(M^4, g)$ Einstein $\iff$

\[ \mathcal{R} : \Lambda^2 \to \Lambda^2 \]

commutes with

\[ \star : \Lambda^2 \to \Lambda^2 : \]

\[
\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{\sqrt{2}} & \dot{r} \\
\dot{r} & W_- + \frac{s}{\sqrt{2}}
\end{pmatrix}.
\]
Thus \((M^4, g)\) Einstein \iff

\[\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2\]

commutes with

\[\star : \Lambda^2 \rightarrow \Lambda^2:\]

\[
\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & 0 \\
0 & W_- + \frac{s}{12}
\end{pmatrix}.
\]
Corollary. A Riemannian 4-manifold \((M, g)\) is Einstein \iff\ sectional curvatures are equal for any pair of perpendicular 2-planes.
Corollary. A Riemannian 4-manifold $(M, g)$ is Einstein $\iff$ sectional curvatures are equal for any pair of perpendicular 2-planes.
Corollary. A Riemannian 4-manifold \((M, g)\) is Einstein \(\iff\) sectional curvatures are equal for any pair of perpendicular 2-planes.
Corollary. A Riemannian 4-manifold \((M, g)\) is Einstein \(\iff\) sectional curvatures are equal for any pair of perpendicular 2-planes.

\[ K(P) = K(P^\perp) \]
\((M, g)\) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

\[
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + \right) d\mu
\]
\((M, g)\) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

\[
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 \right) \, d\mu
\]
\((M, g)\) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

\[
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 \right) \, d\mu
\]
$(M, g)$ compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$
\((M, g)\) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

\[
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\dddot r|^2}{2} \right) d\mu
\]

for Euler-characteristic \(\chi(M) = \sum_j (-1)^j b_j(M)\).
\((M, g)\) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

\[
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{\|\hat{\kappa}\|^2}{2} \right) d\mu
\]

for Euler-characteristic \(\chi(M) = 2 - 2b_1 + b_2\).
4-dimensional Hirzebruch signature formula

\[ \tau(M) = \frac{1}{12\pi^2} \int_M \left( |W_+|^2 \right) d\mu \]
4-dimensional Hirzebruch signature formula

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left( |W_+|^2 - |W_-|^2 \right) d\mu$$
4-dimensional Hirzebruch signature formula

\[ \tau(M) = \frac{1}{12\pi^2} \int_M \left( |W_+|^2 - |W_-|^2 \right) d\mu \]

for signature \( \tau(M) = b_+(M) - b_-(M) \).
Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing.
Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$( [\varphi], [\psi] ) \quad \longmapsto \int_M \varphi \wedge \psi$$
Here \( \tau(M) = b_+(M) - b_-(M) \) defined in terms of intersection pairing

\[
H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \rightarrow \mathbb{R}
\]

\[
( [\varphi], [\psi] ) \mapsto \int_M \varphi \wedge \psi
\]

Diagonalize:
Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$( [\varphi], [\psi] ) \mapsto \int_M \varphi \wedge \psi$$

Diagonalize:

$$\begin{bmatrix}
+1 & \cdots & +1 \\
\cdots & \ddots & \cdots \\
+1 & \cdots & -1
\end{bmatrix}.$$
Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$( [\varphi], [\psi] ) \mapsto \int_M \varphi \wedge \psi$$

Diagonalize:

$$
\begin{bmatrix}
+1 & & \\
& \ddots & \\
& & +1 \\
\end{bmatrix}
$$

DIAG: $b_+(M)$

$$
\begin{bmatrix}
-1 & & \\
& \ddots & \\
& & -1 \\
\end{bmatrix}
$$

DIAG: $b_-(M)$
For $(M^4, g)$ compact oriented Riemannian,

**Euler characteristic**

\[
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{\varkappa^2}{2} \right) d\mu
\]

**Signature**

\[
\tau(M) = \frac{1}{12\pi^2} \int_M \left( |W_+|^2 - |W_-|^2 \right) d\mu
\]
**Theorem** (Freedman/Donaldson). *Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if*
Theorem (Freedman/Donaldson). Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if

- they have the same Euler characteristic $\chi$;
Theorem (Freedman/Donaldson). Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if

- they have the same Euler characteristic $\chi$;
- they have the same signature $\tau$;
Theorem (Freedman/Donaldson). Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if

- they have the same Euler characteristic $\chi$;
- they have the same signature $\tau$; and
- both are spin, or both are non-spin.
Theorem (Freedman/Donaldson). Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if

- they have the same Euler characteristic $\chi$;
- they have the same signature $\tau$; and
- both are spin, or both are non-spin.

$$w_2 = 0 \quad w_2 \neq 0$$
Theorem (Freedman/Donaldson). Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if

- they have the same Euler characteristic $\chi$;
- they have the same signature $\tau$; and
- both are spin, or both are non-spin.

$w_2 = 0 \quad w_2 \neq 0$

Warning: “Exotic differentiable structures!”
Theorem (Freedman/Donaldson). Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if

- they have the same Euler characteristic $\chi$;
- they have the same signature $\tau$; and
- both are spin, or both are non-spin.

$$w_2 = 0 \quad w_2 \neq 0$$

Warning: “Exotic differentiable structures!”

No diffeomorphism classification currently known!
**Theorem** (Freedman/Donaldson). *Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if*

- they have the same Euler characteristic $\chi$;
- they have the same signature $\tau$; and
- both are spin, or both are non-spin.

$$w_2 = 0 \quad \text{or} \quad w_2 \neq 0$$

Warning: “Exotic differentiable structures!”

No diffeomorphism classification currently known!

Typically, one homeotype $\leftrightarrow \infty$ many diffeotypes.
Theorem (Freedman/Donaldson). Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if

- they have the same Euler characteristic $\chi$;
- they have the same signature $\tau$; and
- both are spin, or both are non-spin.
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to
Corollary. *Any smooth compact simply connected non-spin 4-manifold* \( M \) *is homeomorphic to a connect sum*

\[
j \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2} = \underbrace{\mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2}_j \underbrace{\# \overline{\mathbb{CP}^2} \# \cdots \# \overline{\mathbb{CP}^2}}_k
\]
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum

$$j\mathbb{CP}_2 \# k\overline{\mathbb{CP}_2} = \mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2 \# \overline{\mathbb{CP}_2} \# \cdots \# \overline{\mathbb{CP}_2}$$

where $j = b_+(M)$ and $k = b_-(M)$. 
Convention:

\( \overline{\mathbb{CP}_2} = \text{reverse oriented } \mathbb{CP}_2. \)
Convention:

$\mathbb{CP}_2 = \text{reverse oriented } \mathbb{CP}_2.$

Connected sum #: 

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1} \\
\includegraphics[width=0.2\textwidth]{diagram2}
\end{array}
\]
Convention:

\( \mathbb{CP}_2 = \) reverse oriented \( \mathbb{CP}_2 \).

Connected sum #: 

![Diagram of connected sum of two projective planes]
Convention:

\( \overline{\mathbb{CP}_2} = \) reverse oriented \( \mathbb{CP}_2 \).

Connected sum \( \# \):
Convention:

\[ \overline{\mathbb{CP}_2} = \text{reverse oriented } \mathbb{CP}_2. \]

Connected sum \#:
Convention:

\(\overline{\mathbb{CP}_2} = \text{reverse oriented } \mathbb{CP}_2.\)

Connected sum \#: 

\[\begin{array}{cc}
\text{\textbullet} & \text{\textbullet} \\
\end{array}\]
Convention:

\( \overline{\mathbb{C}P_2} = \text{reverse oriented } \mathbb{C}P_2. \)

Connected sum \(\#\):
Convention:

\( \overline{\mathbb{CP}_2} = \text{reverse oriented } \mathbb{CP}_2. \)

Connected sum \#:
Convention:

$\overline{\mathbb{CP}_2} = \text{reverse oriented } \mathbb{CP}_2$.

Connected sum $\#$:
**Corollary.** *Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum*

$$j\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2} = \underbrace{\mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2}_{j} \# \underbrace{\overline{\mathbb{CP}^2} \# \cdots \# \overline{\mathbb{CP}^2}}_{k}$$

*where $j = b_+(M)$ and $k = b_-(M)$.***
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$. 
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$.

What about spin case?
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$.

What about spin case?

Need new building block!
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\CP_2 \# k\CP_2$.

What about spin case?

Need new building block!

$K3$ manifold...
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$
\( K3 = \text{Kummer-Kähler-Kodaira manifold} \).

—André Weil, 1958
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Simply connected complex surface with $c_1 = 0$. 
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Simply connected complex surface with $c_1 = 0$.

Only one deformation type.
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Simply connected complex surface with $c_1 = 0$.

Only one diffeomorphism type.
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Simply connected complex surface with $c_1 = 0$.

Only one diffeomorphism type.

Spin, $\chi = 24$, $\tau = -16$. 
$K3 = \text{Kummer-Kähler-Kodaira manifold}.$

Simply connected complex surface with $c_1 = 0$. 
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{CP}_3$. 
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{CP}_3$. 
$K3 = \text{Kummer-Kähler-Kodaira manifold}.$

Kummer construction:
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$: 
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$: 

![Diagram](image-url)
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$: 

![Diagram of Kummer construction](image)
$K3 = $ Kummer-Kähler-Kodaira manifold.

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$: 

![Diagram showing the Kummer construction](image)
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$:
$K3 = $ Kummer-Kähler-Kodaira manifold.

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$: 

![Diagram showing $T^4/\mathbb{Z}_2$ construction](image)
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$: 

![Diagram of Kummer construction](attachment:diagram.png)
$K3 =$ Kummer-Kähler-Kodaira manifold.

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$:
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$:

Replace $\mathbb{R}^4/\mathbb{Z}_2$ neighborhood of each singular point with copy of $T^*S^2$. 
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

**Kummer construction:**

Begin with $T^4/\mathbb{Z}_2$:

Replace $\mathbb{R}^4/\mathbb{Z}_2$ neighborhood of each singular point with copy of $T^*S^2$.

Result is a $K3$ surface.
$K3 = \text{Kummer-Kähler-Kodaira manifold}.$

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$. 
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

$T^4 = \text{Picard torus of curve of genus 2.}$
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Kummer: $\mathbb{T}^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}^3$.

Remove singularities by deforming equation.
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

Remove singularities by deforming equation.
\( K3 = \text{Kummer-Kähler-Kodaira manifold.} \)

Kummer construction:

Kummer: \( T^4 / \mathbb{Z}_2 \): Singular quartic in \( \mathbb{CP}_3 \).

Remove singularities by deforming equation.
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

Remove singularities by deforming equation.
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

Remove singularities by deforming equation.
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

Generic quartic is then a $K3$ surface.
\( K3 = \text{Kummer-Kähler-Kodaira manifold.} \)

Kummer construction:

Kummer: \( T^4/\mathbb{Z}_2 \): Singular quartic in \( \mathbb{CP}_3 \).

Generic quartic is then a \( K3 \) surface. Example:

\[
0 = t^4 + u^4 + v^4 + w^4
\]
$K3 = \text{Kummer-Kähler-Kodaira manifold.}$

Kummer construction:

Begin with $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

Generic quartic is then a $K3$ surface. Example:

$$0 = (t^2 + u^2 + v^2 - w^2)^2 - 8[(1 - v^2)^2 - 2t^2][(1 + v^2)^2 - 2u^2]$$
**Theorem (Freedman/Donaldson).** Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if

- they have the same Euler characteristic $\chi$;
- they have the same signature $\tau$; and
- both are spin, or both are non-spin.
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$. 
**Corollary.** *Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$.*

**Conjecture (11/8 Conjecture).** *Any smooth compact simply connected spin 4-manifold $M$ is (un-orientedly) homeomorphic to either $S^4$ or a connected sum $jK3\#k(S^2 \times S^2)$.*
**Corollary.** Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{CP}^2\# k\overline{\mathbb{CP}}^2$.

**Conjecture (11/8 Conjecture).** Any smooth compact simply connected spin 4-manifold $M$ is (un-orientedly) homeomorphic to either $S^4$ or a connected sum $jK3\# k(S^2 \times S^2)$.

Equivalent to asserting that such manifolds satisfy

$$b_2 \geq \frac{11}{8}|\tau|.$$
**Corollary.** Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$.

**Conjecture (11/8 Conjecture).** Any smooth compact simply connected spin 4-manifold $M$ is (un-orientedly) homeomorphic to either $S^4$ or a connected sum $jK3 \# k(S^2 \times S^2)$.

Equivalent to asserting that such manifolds satisfy

$$b_2 \geq \frac{11}{8} |\tau|.$$ 

**Furuta:**

$$b_2 \geq \frac{10}{8} |\tau| + 2.$$
Corollary. Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j\mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}$.

Conjecture (11/8 Conjecture). Any smooth compact simply connected spin 4-manifold $M$ is (un-orientedly) homeomorphic to either $S^4$ or a connected sum $jK3 \# k(S^2 \times S^2)$.

Equivalent to asserting that such manifolds satisfy

$$b_2 \geq \frac{11}{8} |\tau|.$$
**Corollary.** Any smooth compact simply connected non-spin 4-manifold $M$ is homeomorphic to a connect sum $j \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$.

**Conjecture** (11/8 Conjecture). Any smooth compact simply connected spin 4-manifold $M$ is (un-orientedly) homeomorphic to either $S^4$ or a connected sum $j K3 \# k (S^2 \times S^2)$.

Equivalent to asserting that such manifolds satisfy

$$b_2 \geq \frac{11}{8} |\tau|.$$ 

Certainly true of all examples in these lectures!
Question. Which smooth compact 4-manifolds $M^4$ admit Einstein metrics?
**Question.** Which smooth compact $4$-manifolds $M^4$ admit Einstein metrics?

**Question.** In dimension four, how unique are Einstein metrics, when they exist?
\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathbf{r}|^2}{2} \right) d\mu_g \]
\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{\|\vec{r}\|^2}{2} \right) d\mu_g \]

Einstein \Rightarrow = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 \right) d\mu_g
Berger’s Inequality:

\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{\hat{r}^2}{2} \right) d\mu_g \]

Einstein \Rightarrow \quad = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 \right) d\mu_g

\geq 0.
Berger’s Inequality:

\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{\hat{r}^2}{2} \right) d\mu_g \]

Einstein \(\Rightarrow\) \[ = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 \right) d\mu_g \geq 0. \]

**Theorem (Berger Inequality).** If smooth compact \(M^4\) admits Einstein \(g\), then

\[ \chi(M) \geq 0, \]
Berger’s Inequality:

\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{\lVert \hat{r} \rVert^2}{2} \right) d\mu_g \]

Einstein \Rightarrow\quad = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 \right) d\mu_g

\geq 0.

**Theorem** (Berger Inequality). *If smooth compact* \( M^4 \) *admits Einstein* \( g \), *then*

\[ \chi(M) \geq 0, \]

*with equality only if* \( (M, g) \) *flat,*
Berger’s Inequality:

\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\hat{r}|^2}{2} \right) d\mu_g \]

Einstein \Rightarrow \quad = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 \right) d\mu_g \geq 0.

**Theorem** (Berger Inequality). *If smooth compact \( M^4 \) admits Einstein \( g \), then*

\[ \chi(M) \geq 0, \]

*with equality only if \((M, g)\) flat, and finitely covered by \( T^4 = \mathbb{R}^4/\Lambda \).*
Berger’s Inequality:

\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{\hat{r}^2}{2} \right) d\mu_g \]

Einstein \Rightarrow \quad = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 \right) d\mu_g \geq 0.

**Theorem** (Berger Inequality). *If smooth compact \( M^4 \) admits Einstein \( g \), then*

\[ \chi(M) \geq 0, \]

*with equality only if \((M, g)\) flat, and finitely covered by \( T^4 = \mathbb{R}^4/\Lambda \).*

\( \implies \) No Einstein metric on \( S^3 \times S^1 \).
Berger’s Inequality:

\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{\hat{r}^2}{2} \right) d\mu_g \]

Einstein \implies \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 \right) d\mu_g \geq 0.

**Theorem** (Berger Inequality). *If smooth compact \( M^4 \) admits Einstein \( g \), then \( \chi(M) \geq 0 \), with equality only if \((M, g)\) flat, and finitely covered by \( T^4 = \mathbb{R}^4 / \Lambda \).

\( \implies \) No Einstein metric on \( k(S^3 \times S^1) \# \ell(T^4) \).
Berger’s Inequality:

\[ \chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\hat{r}|^2}{2} \right) d\mu_g \]

Einstein \Rightarrow \chi(M) \geq 0.

**Theorem** (Berger Inequality). *If smooth compact \(M^4\) admits Einstein \(g\), then*

\[ \chi(M) \geq 0, \]

*with equality only if \((M, g)\) flat, and finitely covered by \(T^4 = \mathbb{R}^4/\Lambda\).*

However, \(\chi(M) > 0\) if \(M^4\) is simply connected...
\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\vec{r}|^2}{2} \right) d\mu_g \]
\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\hat{r}|^2}{2} \right) d\mu_g \]

Einstein \[\Rightarrow\] \[= \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g \]

\[\geq 0.\]
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\vec{r}|^2}{2} \right) d\mu_g\]

**Theorem** (Hitchin-Thorpe Inequality). *If smooth compact oriented \(M^4\) admits Einstein \(g\), then\n
\[(2\chi + 3\tau)(M) \geq 0\]
If $M$ any smooth oriented 4-manifold,
If $M$ any smooth oriented 4-manifold,

$$(2\chi+3\tau)(M \# \overline{\mathbb{CP}^2} \# \cdots \# \overline{\mathbb{CP}^2}) = (2\chi+3\tau)(M) - k$$
If $M$ any smooth oriented 4-manifold,

$$(2\chi + 3\tau)(M \# \overline{\mathbb{C}P^2} \# \cdots \# \overline{\mathbb{C}P^2}) = (2\chi + 3\tau)(M) - k$$

so Hitchin-Thorpe $\implies$ not Einstein if $k \gg 0$. 
If $M$ any smooth oriented 4-manifold,

$$(2\chi+3\tau)(M \# \mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2) = (2\chi+3\tau)(M) - k$$

so Hitchin-Thorpe $\implies$ not Einstein if $k \gg 0$.

$M$ simply connected $\leadsto$ simply connected examples.
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W| + |\bar{\nabla}|^2 - \frac{|\bar{\nabla}|^2}{2} \right) d\mu_g \]

Einstein \[\Rightarrow\] \[= \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W| + |\bar{\nabla}|^2 \right) d\mu_g \]

**Theorem** (Hitchin-Thorpe Inequality). *If smooth compact oriented \(M^4\) admits Einstein \(g\), then\n
\[(2\chi + 3\tau)(M) \geq 0,\]
Hitchin-Thorpe Inequality:

\[ (2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{\dot{r}^2}{2} \right) d\mu_g \]

Einstein \Rightarrow \quad = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g

\[ \mathcal{R} = \begin{pmatrix} W_+ + \frac{s}{12} & \dot{r} \\ \dot{r} & W_- + \frac{s}{12} \end{pmatrix} \]
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{\lvert \dot{\rho} \rvert^2}{2} \right) d\mu_g \]

Einstein \(\Rightarrow\) \[= \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g \]

\[\mathcal{R} = \begin{pmatrix}
W_+ + \frac{s}{12} & \dot{\rho} \\
\dot{\rho} & W_- + \frac{s}{12}
\end{pmatrix}
\]

Curvature \(\Lambda^+\) \quad Curvature \(\Lambda^-\)
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W +|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu_g \]

Einstein \(\Rightarrow\)

\[= \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W +|^2 \right) d\mu_g \]

**Theorem** (Hitchin-Thorpe Inequality). If smooth compact oriented \(M^4\) admits Einstein \(g\), then

\[(2\chi + 3\tau)(M) \geq 0,\]

with equality only if \(\Lambda^+\) is flat on \((M, g)\).
$(M^n, g)$: holonomy
$(M^n, g)$: holonomy
$(M^n, g)$: \hspace{1cm} \text{holonomy}
$(M^n, g)$: holonomy
Kähler metrics: \((M^n, g)\): Kähler \iff \text{holonomy} \subset \text{U}(m)\)
$(M^n, g)$: holonomy
$\left( M^n, g \right)$: holonomy
$(M^n, g)$: \[ \text{holonomy} \]
\((M^n, g)\): holonomy
\((M^n, g)\): \text{holonomy}
$(M^n, g)$: holonomy
$(M^n, g)$: holonomy
\((M^n, g)\): holonomy
(\(M^n, g\)):

holonomy
\((M^n, g)\): holonomy
$(M^n, g)$: holonomy
$(M^n, g)$: \quad \text{holonomy} \subset O(n)$
Kähler metrics:

\((M^{2m}, g)\): \quad \text{holonomy}
Kähler metrics:

\((M^{2m}, g)\) Kähler \iff\text{holonomy } \subset \mathbf{U}(m)\)
Kähler metrics:

$$(M^{2m}, g) \text{ Kähler } \iff \text{ holonomy } \subset U(m)$$

$$U(m) := O(2m) \cap GL(m, \mathbb{C})$$
Kähler metrics:

\[(M^{2m}, g) \text{ Kähler} \iff \text{holonomy } \subset U(m)\]

Makes tangent space a complex vector space!
Kähler metrics:

$$(M^{2m}, g) \text{ Kähler } \iff \text{holonomy } \subset \mathbf{U}(m)$$

Makes tangent space a complex vector space!

$$J : TM \rightarrow TM, \quad J^2 = -\text{identity}$$

“almost-complex structure”
Kähler metrics:

\((M^{2m}, g)\) Kähler \iff \text{holonomy} \subset \mathbf{U}(m)\)

Makes tangent space a complex vector space!

Invariant under parallel transport!
Kähler metrics:

\[(M^{2m}, g) \text{ Kähler } \iff \text{holonomy } \subset U(m)\]

\[\iff \exists \text{ almost complex-structure } J \text{ with } \nabla J = 0 \]

\[\text{and } g(J\cdot, J\cdot) = g.\]
Kähler metrics:

\((M^{2m}, g)\) Kähler ⇔ holonomy ⊂ \(\text{U}(m)\)

⇔ ∃ almost complex-structure \(J\) with \(\nabla J = 0\) and \(g(J\cdot, J\cdot) = g\).

⇔ \((M^{2m}, g)\) is a complex manifold & ∃ \(J\)-invariant closed 2-form \(\omega\) such that \(g = \omega(\cdot, J\cdot)\).
Kähler metrics:

$$(M^{2m}, g) \text{ Kähler } \iff \text{holonomy } \subset U(m)$$

$$\iff \exists \text{ almost complex-structure } J \text{ with } \nabla J = 0 \text{ and } g(J \cdot, J \cdot) = g.$$ 

$$\iff (M^{2m}, g) \text{ is a complex manifold & } \exists J\text{-invariant closed 2-form } \omega \text{ such that } g = \omega(\cdot, J \cdot).$$

$$d\omega = 0$$
Kähler metrics:

\[(M^{2m}, g) \text{ Kähler } \iff \text{holonomy } \subset U(m)\]

\[\iff \exists \text{ almost complex-structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g.\]

\[\iff (M^{2m}, g) \text{ is a complex manifold } \& \exists J\text{-invariant closed 2-form } \omega \text{ such that } g = \omega(\cdot, J\cdot).\]

\[[\omega] \in H^2(M)\]

“Kähler class”
Kähler metrics:

\((M^{2m}, g)\) Kähler \iff\ holonomy \subset \mathbf{U}(m)\)

\iff\ \exists\ \text{almost complex-structure} \ J \ \text{with} \ \nabla J = 0 \ \text{and} \ g(J \cdot, J \cdot) = g.

\iff\ (M^{2m}, g) \text{ is a complex manifold} & \exists \ J\text{-invariant closed 2-form} \ \omega \ \text{such that} \ g = \omega(\cdot, J \cdot).
Kähler metrics:

\[(M^{2m}, g) \text{ Kähler } \iff \text{holonomy } \subset U(m)\]

\[\iff \exists \text{ almost complex-structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g.\]

\[\iff (M^{2m}, g) \text{ is a complex manifold } & \exists J\text{-invariant closed 2-form } \omega \text{ such that } g = \omega(\cdot, J\cdot).\]

\[\iff \text{In local complex coordinates } (z^1, \ldots, z^m), \]

\[\sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[ dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right] \]
Kähler metrics:

\[(M^{2m}, g) \text{ Kähler } \iff \text{holonomy } \subset \text{U}(m)\]

\[\iff \exists \text{ almost complex-structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g.\]

\[\iff (M^{2m}, g) \text{ is a complex manifold } \& \exists J\text{-invariant closed 2-form } \omega \text{ such that } g = \omega(\cdot, J\cdot).\]

\[\iff \text{In local complex coordinates } (z^1, \ldots, z^m), \exists f(z)\]

\[g = -\sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[ dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right] \]
Kähler metrics:

\((M^{2m}, g)\) Kähler \iff \text{holonomy} \subset \text{U}(m)\)

\iff \exists \text{almost complex-structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g.

\iff (M^{2m}, g) \text{ is a complex manifold & } \exists J\text{-invariant closed 2-form } \omega \text{ such that } g = \omega(\cdot, J\cdot).

\iff \text{In local complex coordinates } (z^1, \ldots, z^m), \exists f(z)

\omega = i \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \, dz^j \wedge d\bar{z}^k
Kähler metrics:

\[(M^{2m}, g) \text{ Kähler } \iff \text{holonomy } \subset \mathbf{U}(m)\]

\[\iff \exists \text{ almost complex-structure } J \text{ with } \nabla J = 0 \text{ and } g(J \cdot, J \cdot) = g.\]

\[\iff (M^{2m}, g) \text{ is a complex manifold } & \exists J\text{-invariant closed 2-form } \omega \text{ such that } g = \omega(\cdot, J \cdot).\]

\[\iff \text{In local complex coordinates } (z^1, \ldots, z^m), \exists f(z)\]

\[g = -\sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[ dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right] \]
Kähler metrics:

\((M^{2m}, g)\) Kähler \iff\ holonomy \subset U(m) \iff\exists\ \text{almost complex-structure } J \text{ with } \nabla J = 0 \text{ and } g(J\cdot, J\cdot) = g. \iff\ (M^{2m}, g) \text{ is a complex manifold } \& \exists J\text{-invariant closed 2-form } \omega \text{ such that } g = \omega(\cdot, J\cdot). \)

Kähler magic:

\[ r = - \sum_{j,k=1}^{m} \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det[g_{p\bar{q}}] \left[ dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right] \]
Kähler metrics:

\((M^{2m}, g)\) Kähler ⇔ holonomy \(\subset U(m)\)

⇔ ∃ almost complex-structure \(J\) with \(∇J = 0\) and \(g(J\cdot, J\cdot) = g\).

⇔ \((M^{2m}, g)\) is a complex manifold & ∃ \(J\)-invariant closed 2-form \(\omega\) such that \(g = \omega(\cdot, J\cdot)\).

Kähler magic:

If we define the Ricci form by

\[ \rho = r(J\cdot, \cdot) \]

then \(i\rho\) is curvature of canonical line bundle \(\Lambda^{m,0}\).
Kähler metrics:

\((M^{2m}, g)\): holonomy
Kähler metrics:

$$(M^{2m}, g): \text{Ricci-flat Kähler} \iff \text{holonomy} \subset SU(m)$$
Kähler metrics:

\((M^{2m}, g)\): Ricci-flat Kähler ⇐ holonomy \(\subset \text{SU}(m)\)

\[\text{SU}(m) \subset \text{U}(m) : \{ A \mid \det A = 1 \}\]
Kähler metrics:

\((M^{2m}, g)\): Ricci-flat Kähler \iff holonomy \subset SU(m)
Kähler metrics:

$$(M^{2m}, g): \text{Ricci-flat Kähler} \iff \text{holonomy} \subset SU(m)$$

if $M$ is simply connected.
Hyper-Kähler metrics:

\((M^{4\ell}, g)\)
Hyper-Kähler metrics:

\((M^{4\ell}, g)\) hyper-Kähler \iff \text{holonomy} \subset \text{Sp}(\ell)\)
Hyper-Kähler metrics:

\[(M^{4\ell}, g) \text{ hyper-Kähler } \iff \text{holonomy } \subset \text{Sp}(\ell)\]

\[\text{Sp}(\ell) := \text{O}(4\ell) \cap \text{GL}(\ell, \mathbb{H})\]
Hyper-Kähler metrics:

\[(M^{4\ell}, g) \text{ hyper-Kähler } \iff \text{holonomy } \subset \text{Sp}(\ell)\]

\[\text{Sp}(\ell) \subset \text{SU}(2\ell)\]
Hyper-Kähler metrics:

\[(M^{4\ell}, g) \text{ hyper-Kähler} \iff \text{holonomy } \subset \text{Sp}(\ell)\]

\[\text{Sp}(\ell) \subset \text{SU}(2\ell)\]

in many ways!
Hyper-Kähler metrics:

\((M^{4\ell}, g)\) hyper-Kähler \iff \text{holonomy} \subset \text{Sp}(\ell)

\[
\text{Sp}(\ell) \subset \text{SU}(2\ell)
\]

in many ways! (For example, permute \(i, j, k, \ldots\))
Hyper-Kähler metrics:

\[(M^{4\ell}, g) \text{ hyper-Kähler } \iff \text{holonomy } \subset \text{Sp}(\ell)\]

\[\text{Sp}(\ell) \subset \text{SU}(2\ell)\]

Ricci-flat and Kähler,

for many different complex structures!
Hyper-Kähler metrics:

\[(M^4, g) \text{ hyper-Kähler } \iff \text{holonomy } \subset \text{Sp}(\ell)\]
Hyper-Kähler metrics:

\((M^4, g)\) hyper-Kähler \iff \text{holonomy} \subset \text{Sp}(1)

\[ \text{Sp}(1) = \text{SU}(2) \]
Hyper-Kähler metrics:

\begin{align*}
(M^4, g) \text{ hyper-Kähler} & \iff \text{holonomy } \subset \text{Sp}(1) \\
\text{Sp}(1) = \text{SU}(2)
\end{align*}

When \((M^4, g)\) simply connected:

hyper-Kähler \iff \text{Ricci-flat Kähler} \iff \Lambda^+ \text{ flat.}
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\hat{r}|^2}{2} \right) d\mu_g \]

Einstein ⇒ \[= \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g \]

**Theorem (Hitchin-Thorpe Inequality).** If smooth compact oriented \(M^4\) admits Einstein \(g\), then

\[(2\chi + 3\tau)(M) \geq 0,\]

with equality only if \(\Lambda^+\) is flat on \((M, g)\).
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W|^2 - \frac{\left|\bar{r}\right|^2}{2} \right) d\mu_g \]

Einstein \(\Rightarrow\) Einstein \(\Rightarrow\) Einstein \(\Rightarrow\) Einstein

\[
\int_M \left( \frac{s^2}{24} + 2|W|^2 \right) d\mu_g
\]

**Theorem** (Hitchin-Thorpe Inequality). *If smooth compact oriented* \(M^4\) *admits Einstein* \(g\), *then*

\[(2\chi + 3\tau)(M) \geq 0,\]

*with equality only if \(\Lambda^+\) is flat on* \((M, g)\).

\[\iff\text{universal cover Ricci-flat Kähler.}\]
Theorem (Yau). A compact complex manifold $(M^{2m}, J)$ admits $J$-compatible Ricci-flat Kähler metrics iff
Theorem (Yau). A compact complex manifold \((M^{2m}, J)\) admits \(J\)-compatible Ricci-flat Kähler metrics.
Theorem (Yau). A compact complex manifold \((M^{2m}, J)\) admits \(J\)-compatible Ricci-flat Kähler metrics iff
Theorem (Yau). A compact complex manifold $(M^{2m}, J)$ admits $J$-compatible Ricci-flat Kähler metrics iff

- it admits Kähler metrics; and
Theorem (Yau). A compact complex manifold \((M^{2m}, J)\) admits \(J\)-compatible Ricci-flat Kähler metrics iff

- it admits Kähler metrics; and
- \(c_1(M^{2m}, J) = 0 \in H^2(M, \mathbb{R})\).
**Theorem (Yau).** A compact complex manifold $(M^{2m}, J)$ admits $J$-compatible Ricci-flat Kähler metrics iff

- it admits Kähler metrics; and
- $c_1(M^{2m}, J) = 0 \in H^2(M, \mathbb{R})$.

*When this happens, there is a unique such metric in every Kähler class $[\omega]$.***
Theorem (Yau). A compact complex manifold \((M^{2m}, J)\) admits \(J\)-compatible Ricci-flat Kähler metrics iff

- it admits Kähler metrics; and
- \(c_1(M^{2m}, J) = 0 \in H^2(M, \mathbb{R}).\)

When this happens, there is a unique such metric in every Kähler class \([\omega]\).

Conjectured by Calabi (1954)
Theorem (Yau). A compact complex manifold \((M^{2m}, J)\) admits \(J\)-compatible Ricci-flat Kähler metrics iff

- it admits Kähler metrics; and
- \(c_1(M^{2m}, J) = 0 \in H^2(M, \mathbb{R})\).

When this happens, there is a unique such metric in every Kähler class \([\omega]\).

Conjectured by Calabi (1954)

who proved necessity & uniqueness.
**Theorem (Yau).** A compact complex manifold \((M^{2m}, J)\) admits \(J\)-compatible Ricci-flat Kähler metrics iff

- it admits Kähler metrics; and
- \(c_1(M^{2m}, J) = 0 \in H^2(M, \mathbb{R})\).

When this happens, there is a unique such metric in every Kähler class \([\omega]\).

Conjectured by Calabi (1954)

who proved necessity & uniqueness.

“Calabi-Yau metrics.”
Corollary. $\exists \lambda = 0$ Einstein metrics on $K3$. 
Corollary. $\exists \lambda = 0$ Einstein metrics on $K3$. 
Corollary. \( \exists \lambda = 0 \) Einstein metrics on \( K3 \).
Corollary. \( \exists \lambda = 0 \) Einstein metrics on \( K3 \).
Corollary. $\exists \lambda = 0$ Einstein metrics on $K3$.

Indeed, $\exists$ sequences of these $\rightarrow$ flat orbifold $T^4/\mathbb{Z}_2$. 
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\tilde{r}|^2}{2} \right) d\mu_g \]

Einstein \(\Rightarrow\) \[= \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g \]

**Theorem** (Hitchin-Thorpe Inequality). *If smooth compact oriented \(M^4\) admits Einstein \(g\), then*

\[(2\chi + 3\tau)(M) \geq 0,\]

*with equality only if \(\Lambda^+\) is flat on \((M, g)\).*
Hitchin-Thorpe Inequality:

\[
(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{\rho}|^2}{2} \right) d\mu_g
\]

Theorem (Hitchin-Thorpe Inequality). If smooth compact oriented \( M^4 \) admits Einstein \( g \), then

\[
(2\chi + 3\tau)(M) \geq 0,
\]

with equality only if \( \Lambda^+ \) is flat on \((M, g)\). The latter happens only if \((M, g)\) finitely covered by a flat \( T^4 \) or by a Calabi-Yau \( K3 \).
Hitchin-Thorpe Inequality:

\[(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{\Omega}{2} \right) d\mu_g\]

Einstein \(\Rightarrow\) 

\[= \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g\]

**Theorem** (Hitchin-Thorpe Inequality). *If smooth compact oriented* \(M^4\) *admits Einstein* \(g\), *then*

\[(2\chi + 3\tau)(M) \geq 0,\]

*with equality only if \(\Lambda^+\) is flat on* \((M, g)\). *The latter happens only if* \((M, g)\) *finitely covered by a flat* \(T^4\) *or by a Calabi-Yau K3.*

Cheeger-Gromoll splitting theorem & Bieberbach
Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\hat{\rho}|^2}{2} \right) d\mu_g$$

Einstein $\Rightarrow$ $\quad = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g$

**Theorem (Hitchin-Thorpe Inequality).** If smooth compact oriented $M^4$ admits Einstein $g$, then

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if $\Lambda^+$ is flat on $(M, g)$. The latter happens only if $(M, g)$ finitely covered by a flat $T^4$ or by a Calabi-Yau $K3$. 
Corollary. Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. 
Corollary. Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. Then $M$ does not admit Einstein metrics.
Corollary. Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. Then $M$ does not admit Einstein metrics.

Kodaira: $\exists$ complex surfaces that are homotopy equivalent to $K3$, but which have $c_1 \neq 0$. 
Corollary. Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. Then $M$ does not admit Einstein metrics.

Kodaira: $\exists$ complex surfaces that are homotopy equivalent to $K3$, but which have $c_1 \neq 0$.

(Of course, still have $c_1^2 = 2\chi + 3\tau = 0$.)
**Corollary.** Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. Then $M$ does not admit Einstein metrics.

**Kodaira:** $\exists$ complex surfaces that are homotopy equivalent to $K3$, but which have $c_1 \neq 0$.

(Of course, still have $c_1^2 = 2\chi + 3\tau = 0$.)

For any integer $n$, $\exists$ examples where $2n|c_1$. 
**Corollary.** Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. Then $M$ does not admit Einstein metrics.

**Kodaira:** $\exists$ complex surfaces that are homotopy equivalent to $K3$, but which have $c_1 \neq 0$.

(Of course, still have $c_1^2 = 2\chi + 3\tau = 0$.)

For any integer $n$, $\exists$ examples where $2n | c_1$.

Tomorrow: These are pairwise non-diffeomorphic, even though all are homeomorphic to $K3$. 
Corollary. Suppose that $M^4$ is homeomorphic, but not diffeomorphic, to $K3$. Then $M$ does not admit Einstein metrics.

Kodaira: $\exists$ complex surfaces that are homotopy equivalent to $K3$, but which have $c_1 \neq 0$.

(Of course, still have $c_1^2 = 2\chi + 3\tau = 0$.)

For any integer $n$, $\exists$ examples where $2n | c_1$.

Tomorrow: These are pairwise non-diffeomorphic, even though all are homeomorphic to $K3$.

$\therefore$ Topological manifold $|K3|$ has infinitely many smooth structures, but only one of these admits Einstein metrics.