Einstein Metrics,

Complex Surfaces, &

Symplectic 4-Manifolds

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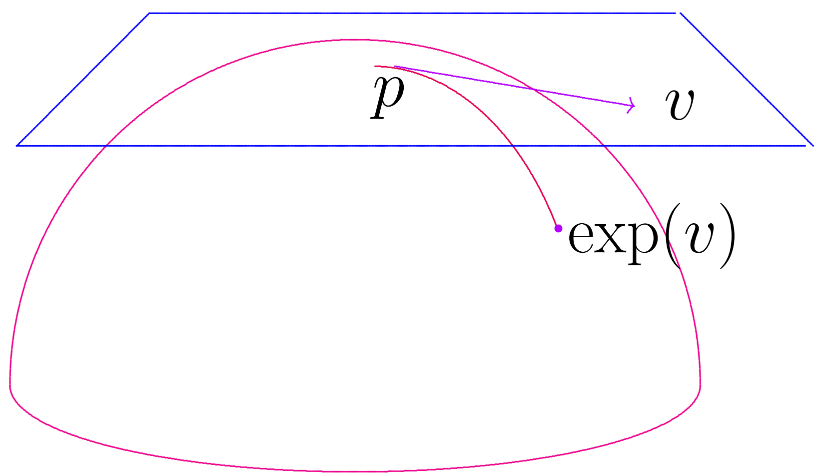

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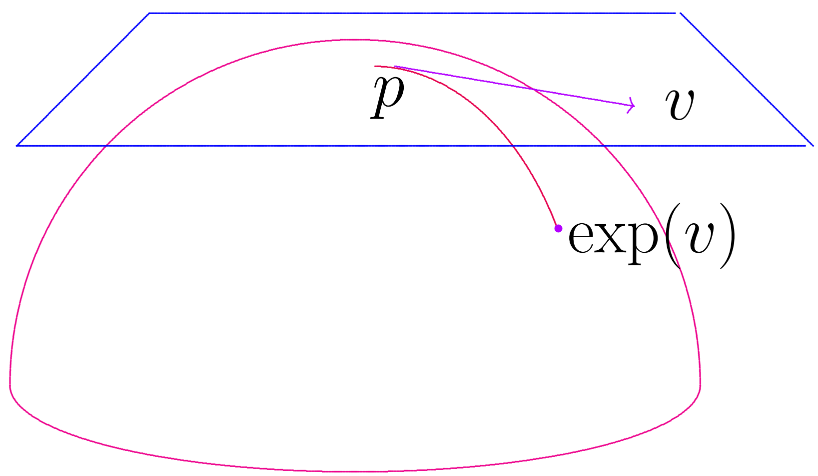

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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Components like $\mathcal{R}_{1212}$ are "sectional curvatures"...

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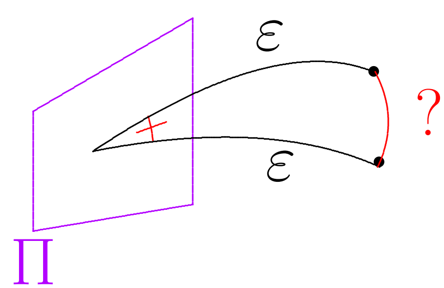

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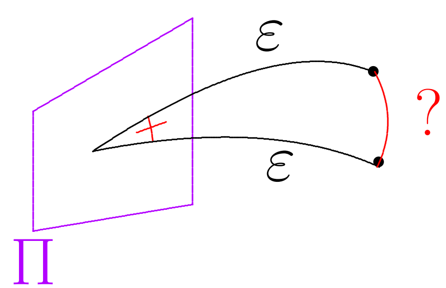

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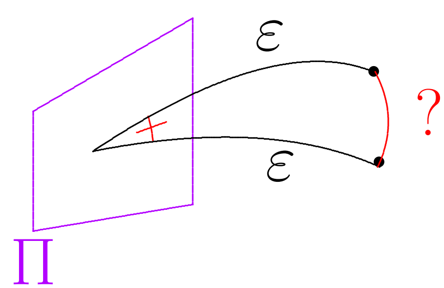

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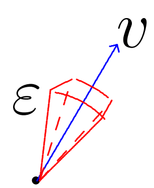

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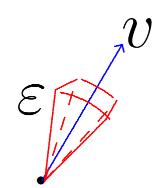

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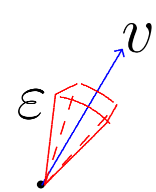

For example, the unit n-sphere $S^n \subset \mathbb{R}^{n+1}$ has Ricci curvature $\equiv +(n-1)$, but this does not locally characterize the standard metric when $n \geq 4$.

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"...the greatest blunder of my life!"

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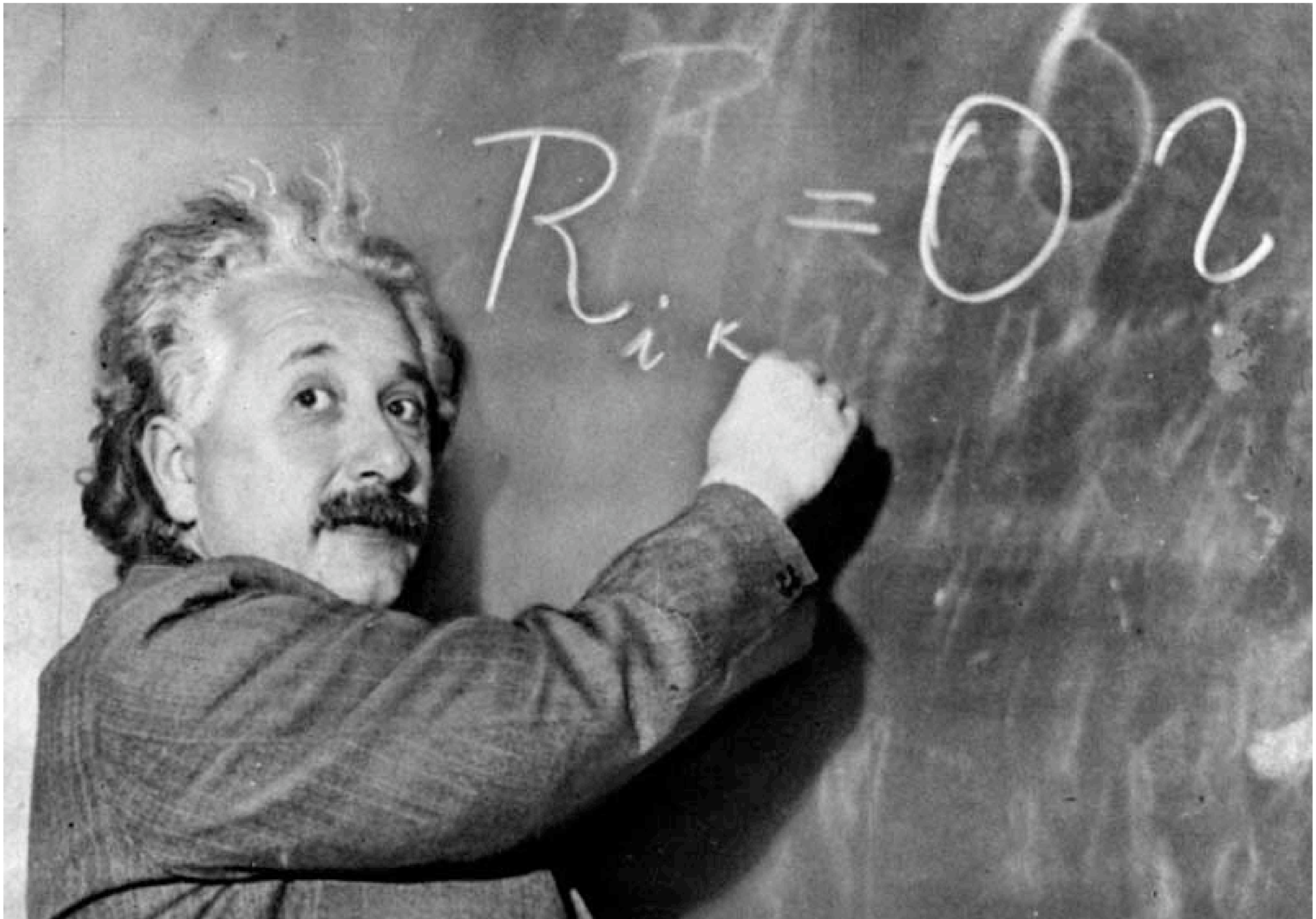

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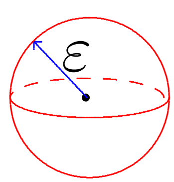

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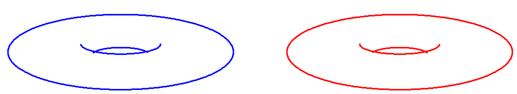

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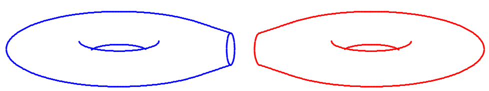

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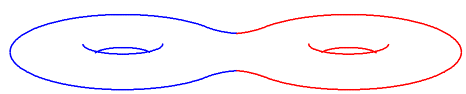

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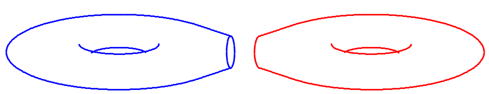

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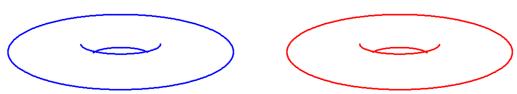

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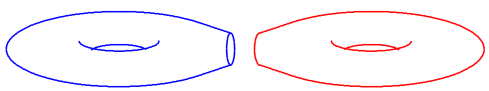

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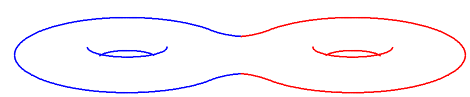

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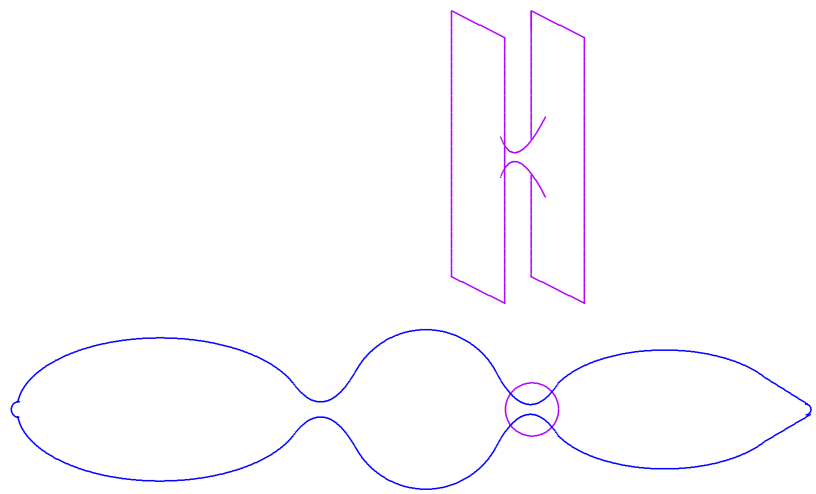

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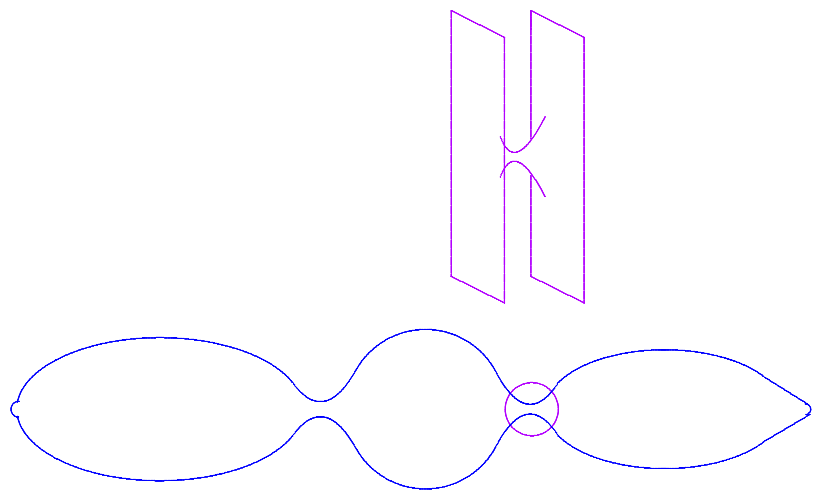

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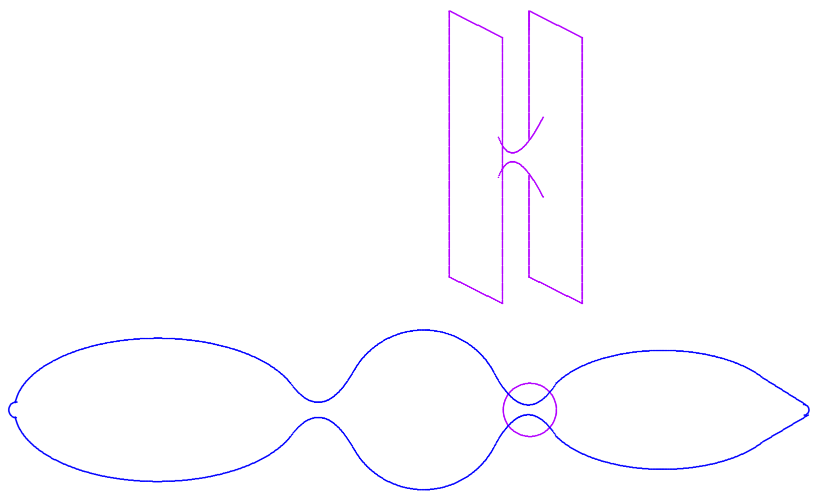

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Similar results for most simply connected spin 5-manifolds. (Boyer, Galicki, Kollár, et al.)

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(Terminology to be explained in a moment!)

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T^4 = Picard torus of curve of genus 2.

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Kähler?

(M^n, g):

holonomy

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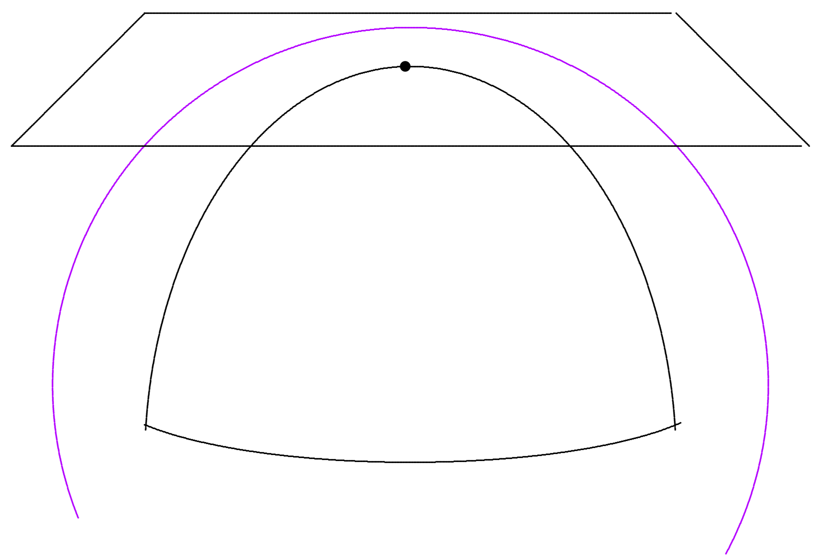

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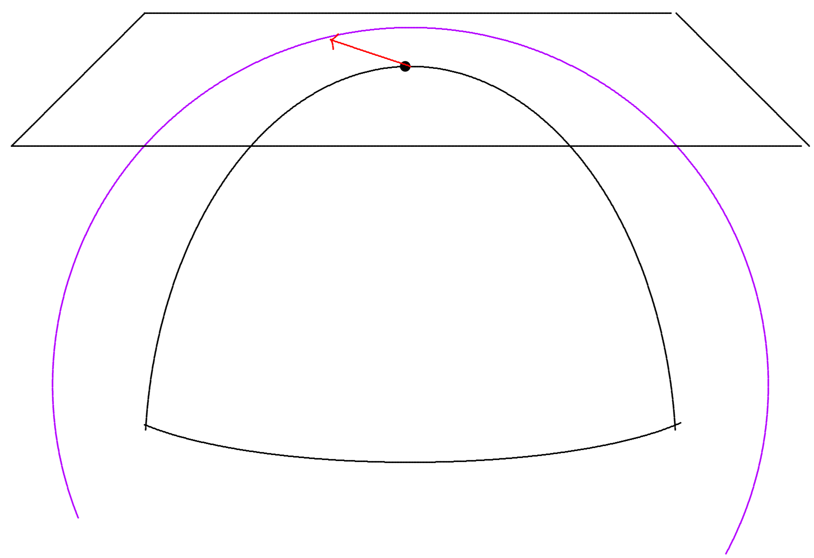

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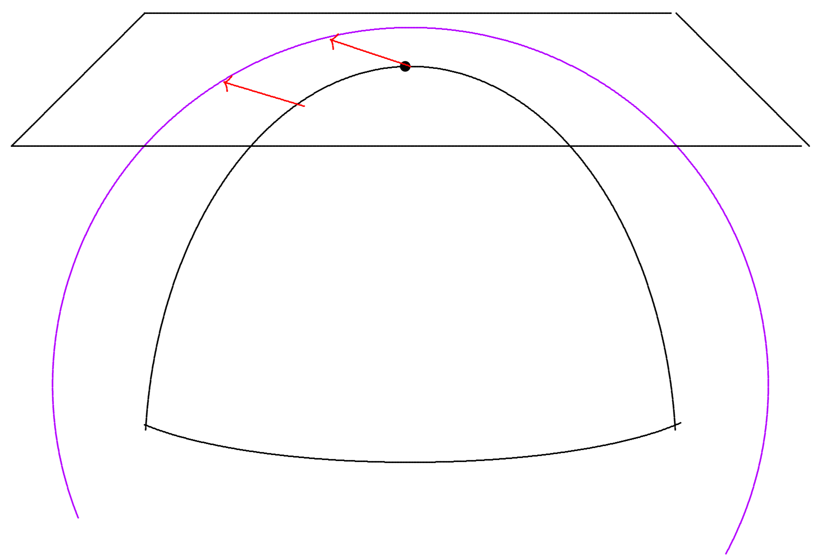

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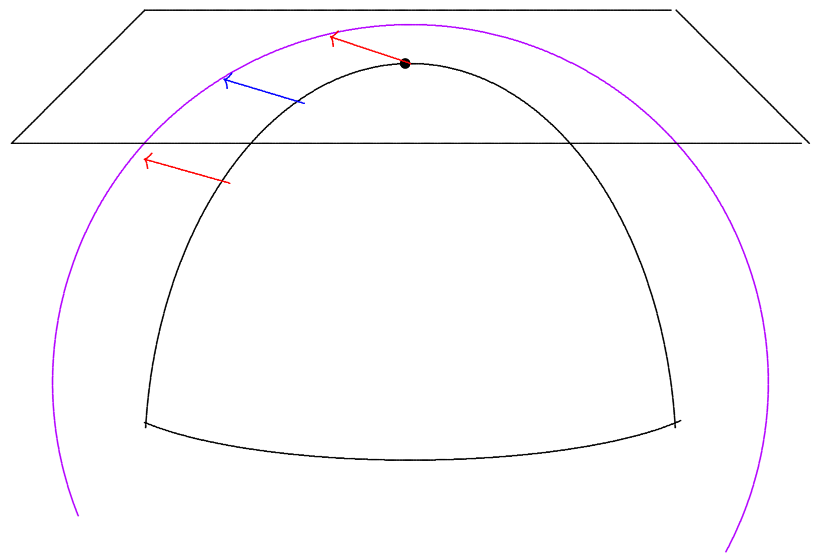

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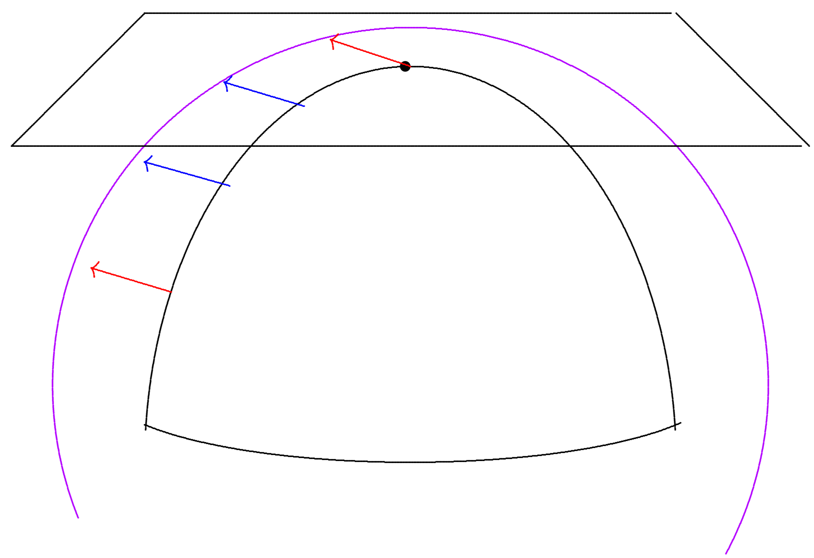

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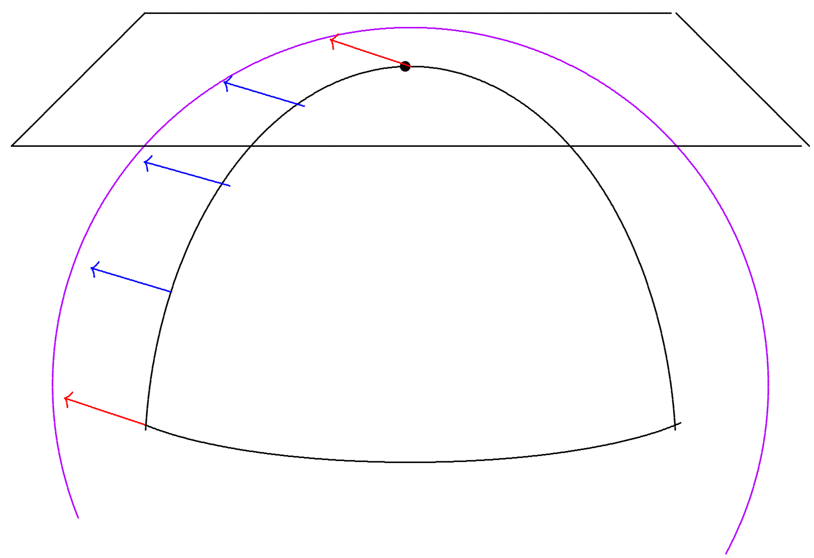

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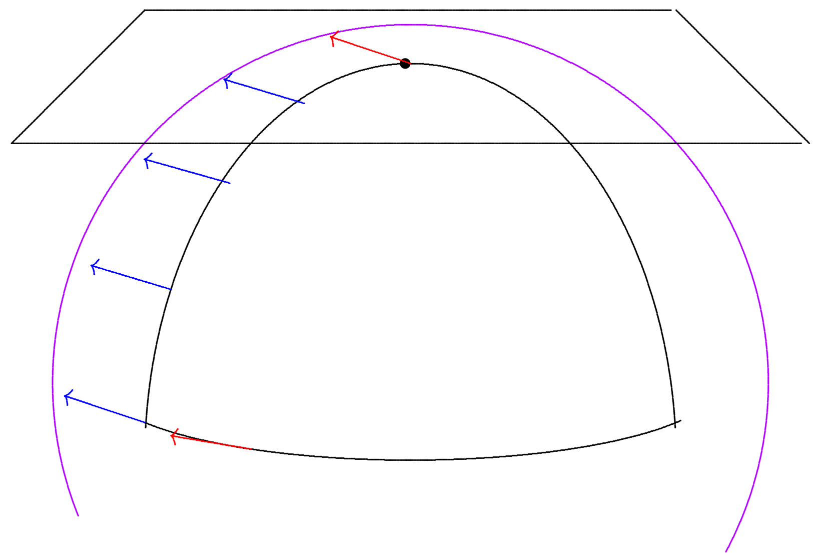

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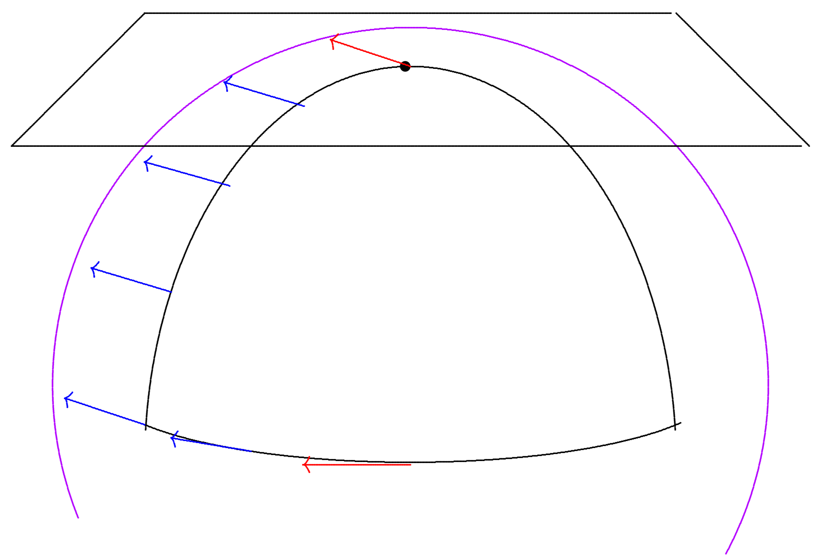

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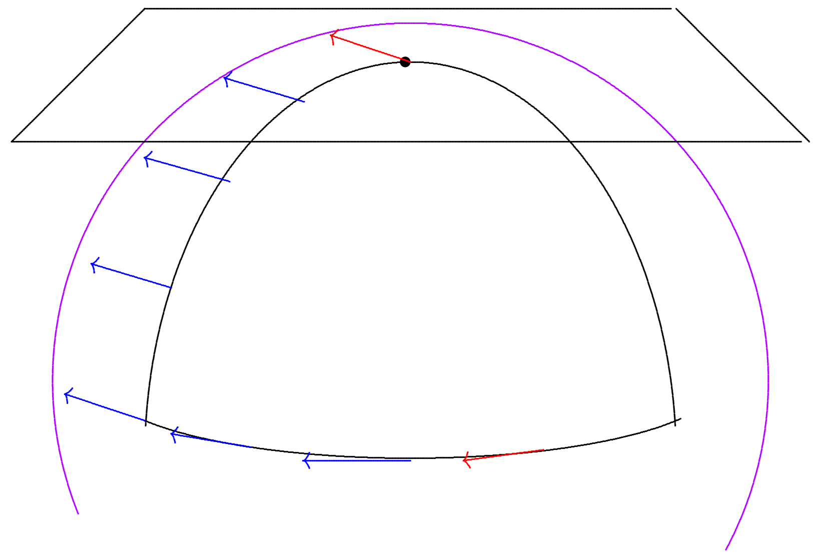

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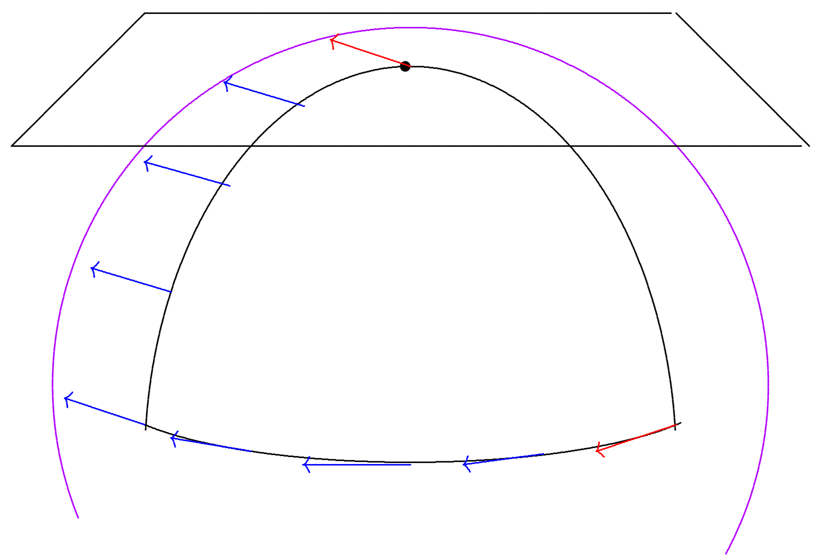

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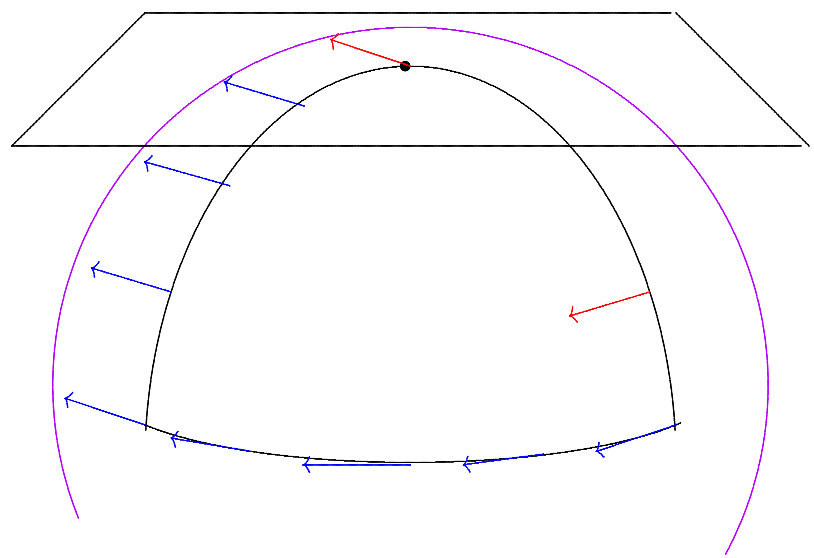

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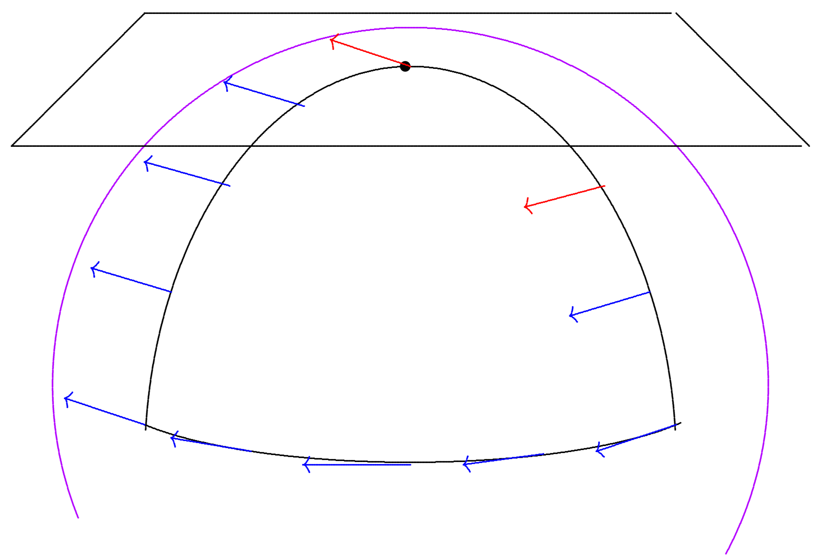

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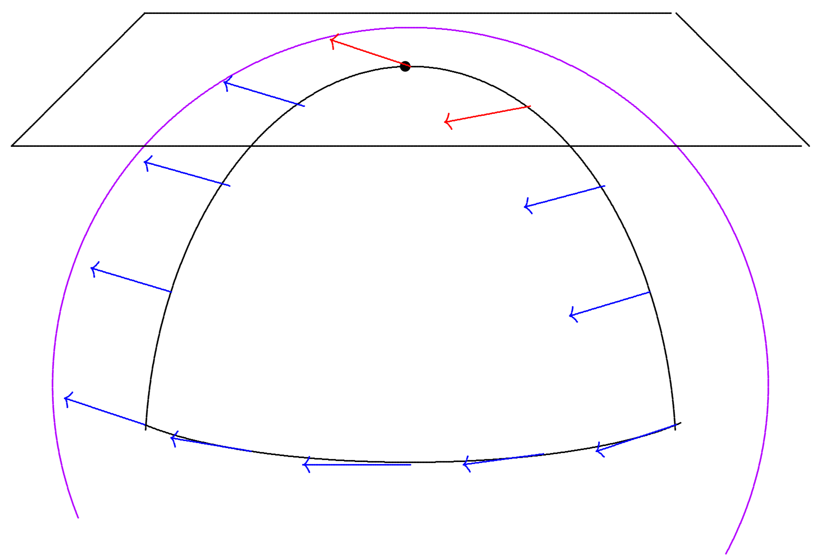

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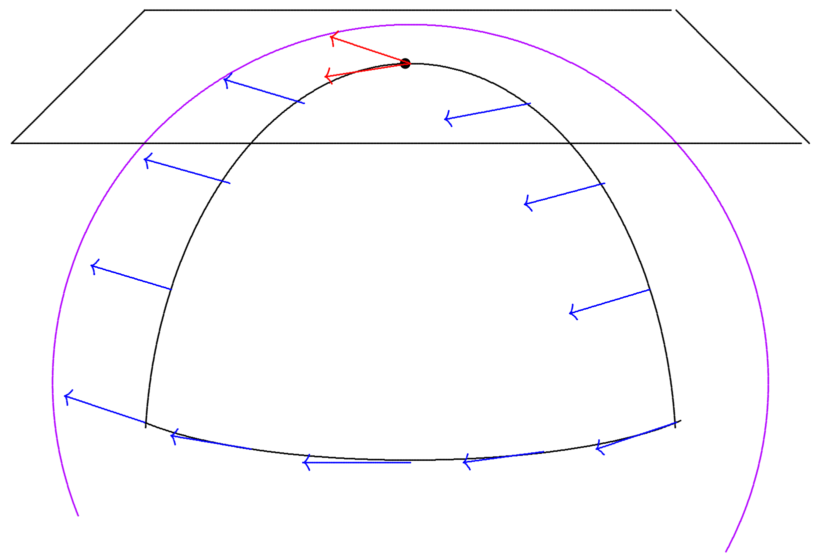

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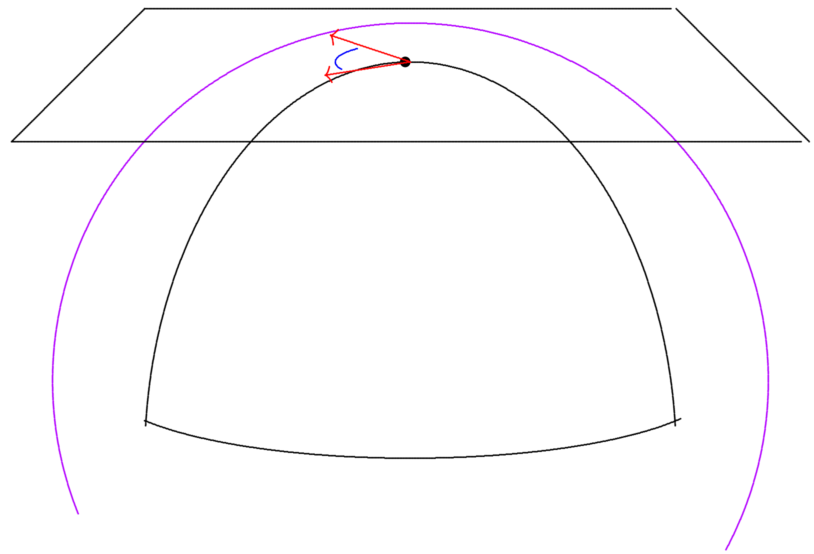
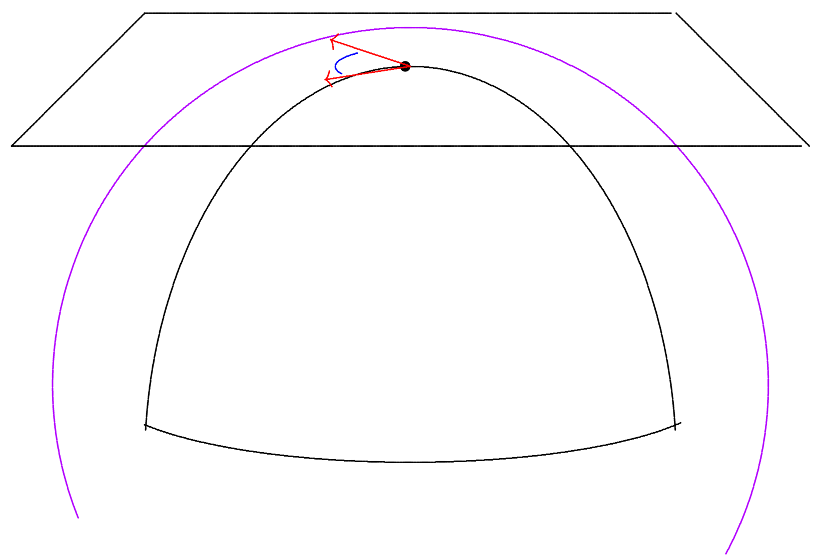
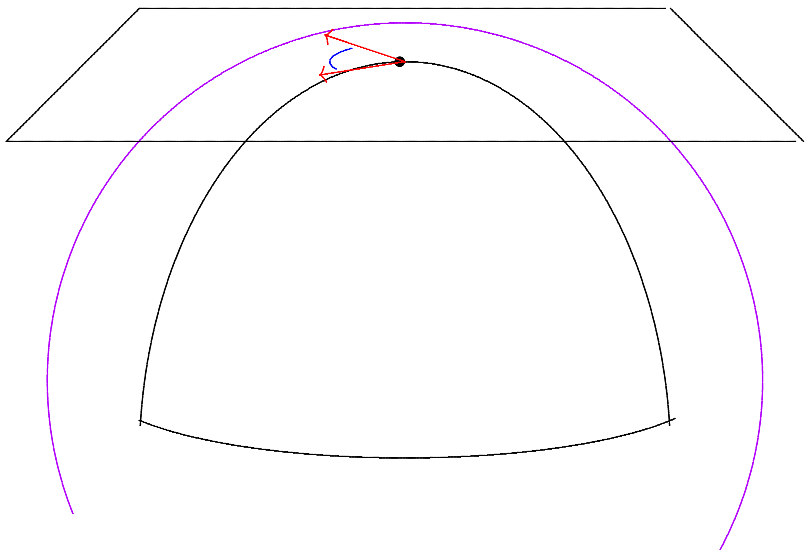


(M^n, g): holonomy $\subset \mathbf{O}(n)$



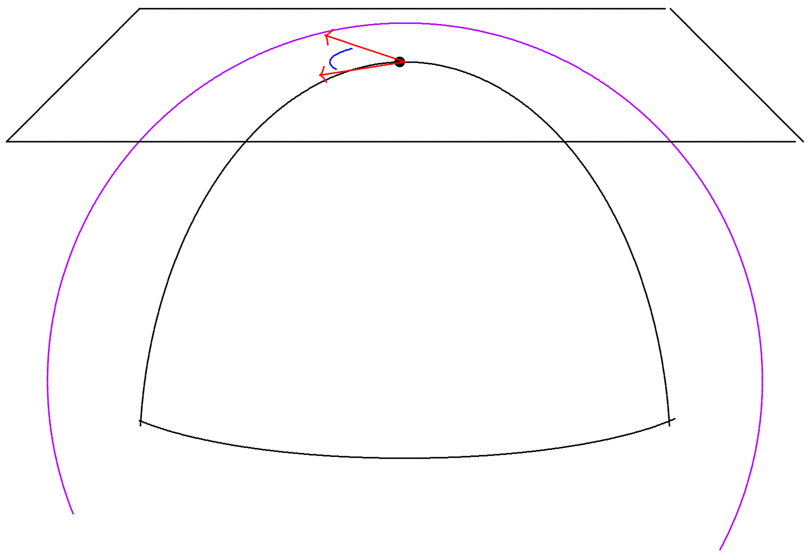
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



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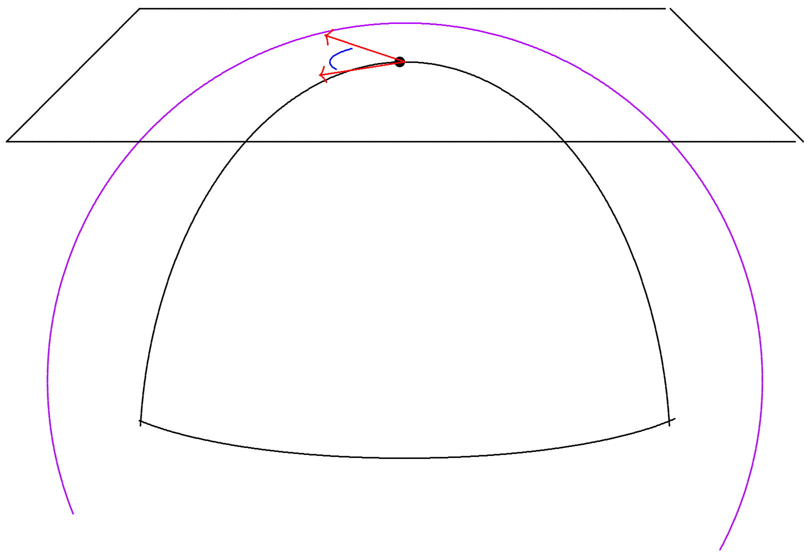

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

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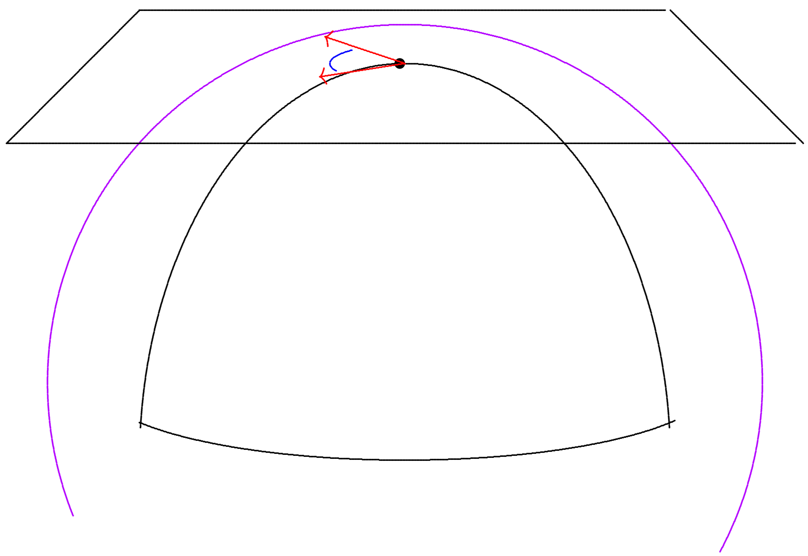

Makes tangent space a complex vector space!

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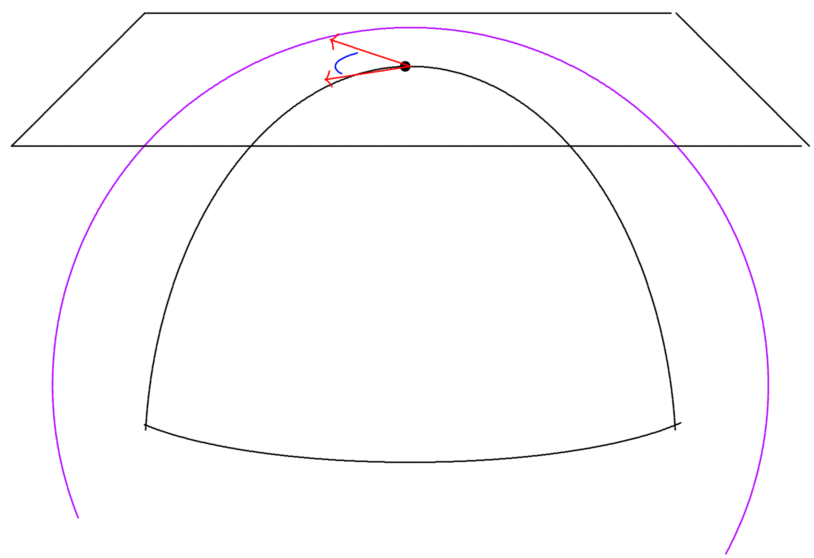

Makes tangent space a complex vector space!

$$J : TM \to TM, \qquad J^2 = -\text{identity}$$

"almost-complex structure"

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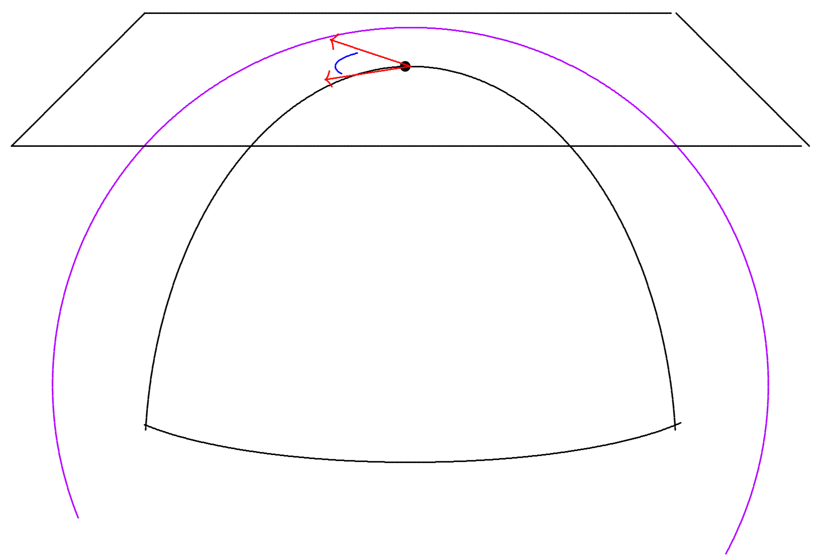

Makes tangent space a complex vector space!

Invariant under parallel transport!

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$\iff \exists$ almost complex-structure J with $\nabla J = 0$
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ω called "Kähler form."

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Kähler magic:

$$r = -\sum_{j,k=1}^{m} \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det[g_{p\bar{q}}] \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j\right]$$

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Kähler magic:

If we define the Ricci form by

$$\rho = r(J\cdot, \cdot)$$

then $i\rho$ is curvature of canonical line bundle $\Lambda^{m,0}$.

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ω non-degenerate closed 2-form:

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ω non-degenerate closed 2-form: symplectic form

$$\omega^{\wedge m} \neq 0$$

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$\Longleftrightarrow$ (M^{2m}, g) is a complex manifold & $\exists$ J-invariant closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

$$d\omega = 0$$

ω non-degenerate closed 2-form: symplectic form

$$\underbrace{\omega \wedge \cdots \wedge \omega}_{m} \neq 0$$

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Simply connected complex surface with $c_1 = 0$.

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By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.

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Λ^+ self-dual 2-forms.
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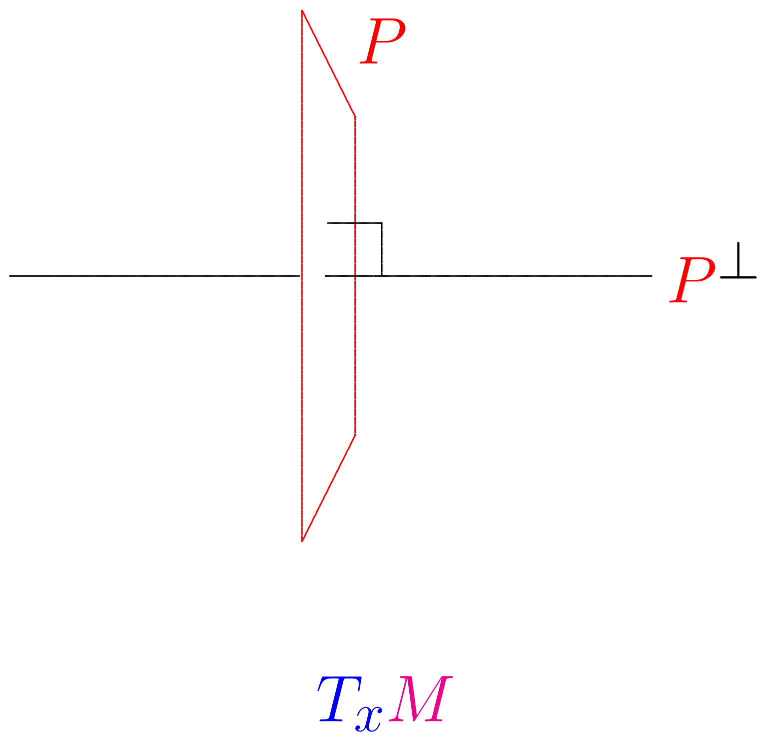

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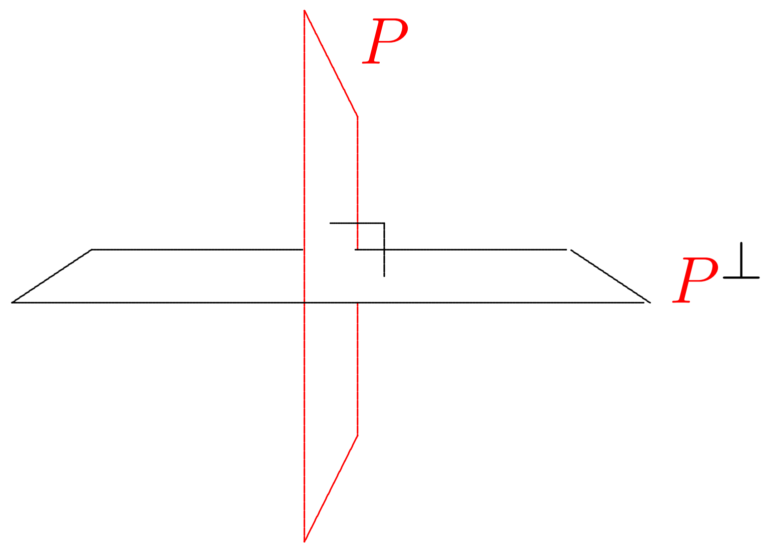

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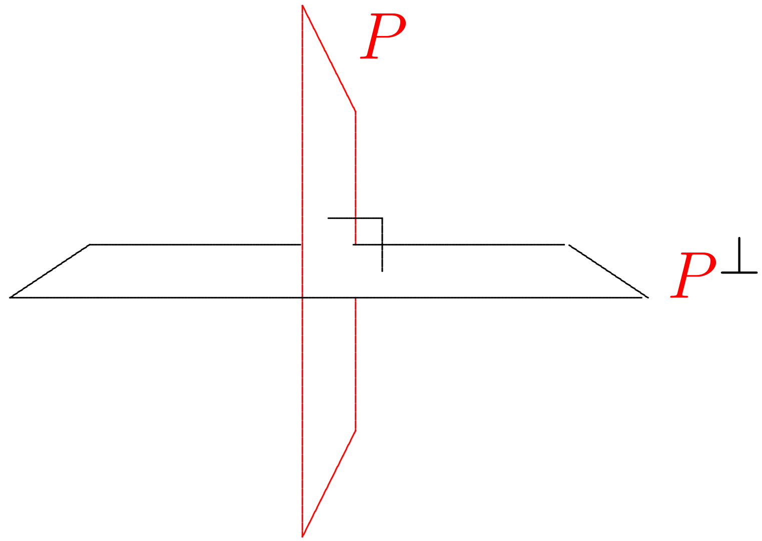

$$K(P) = K(P^{\perp})$$

(M, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + \qquad\qquad\qquad\qquad\qquad\qquad \right) d\mu$$

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Convention:

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$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

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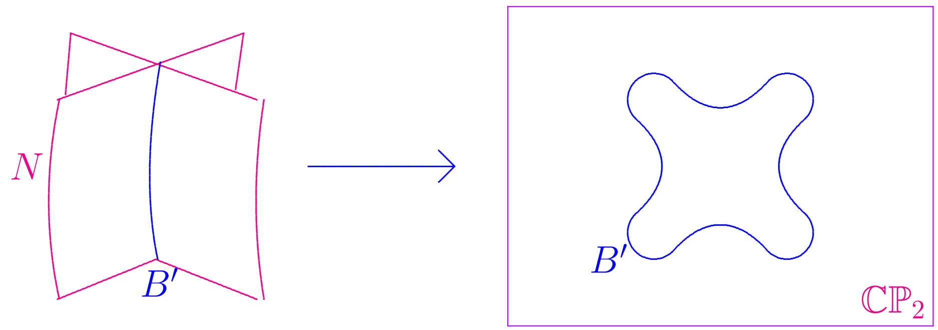

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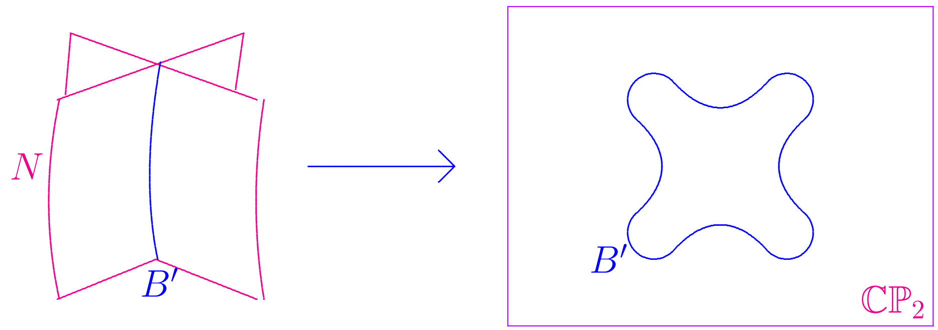

Aubin/Yau $\Longrightarrow$ N carries Einstein metric.

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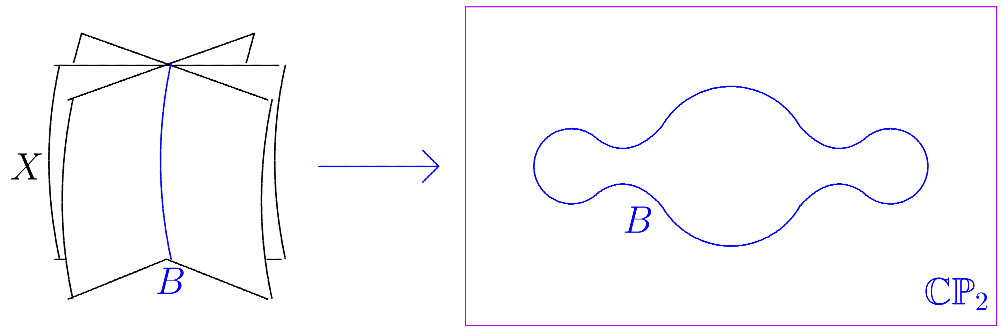

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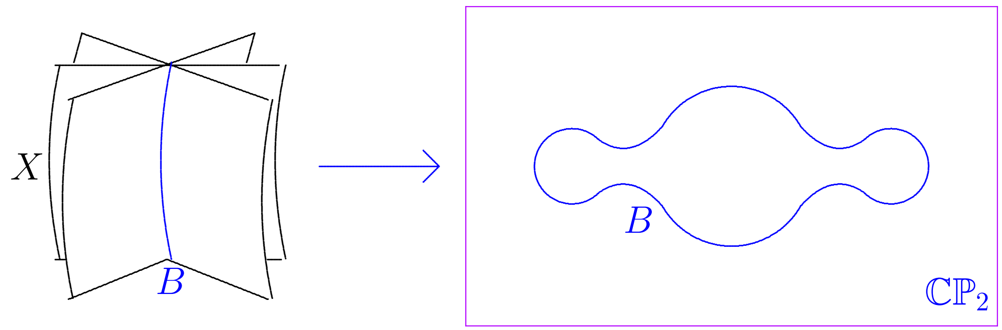

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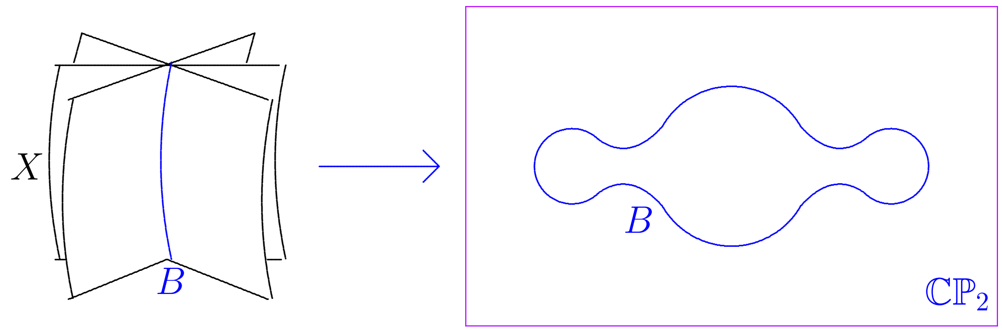

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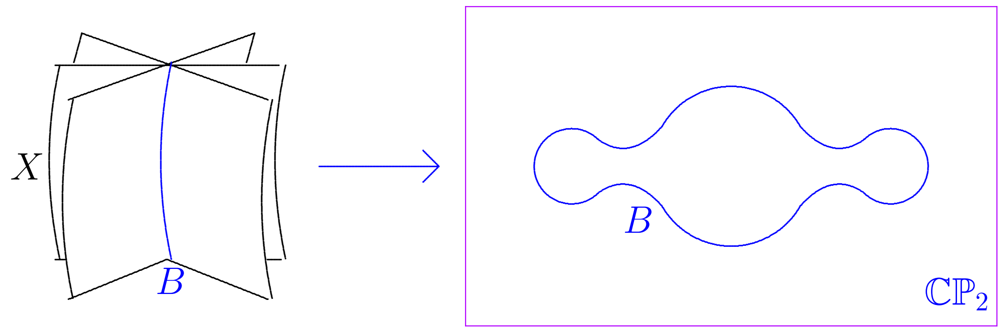

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$$b_+(X) = 7 > 1.$$

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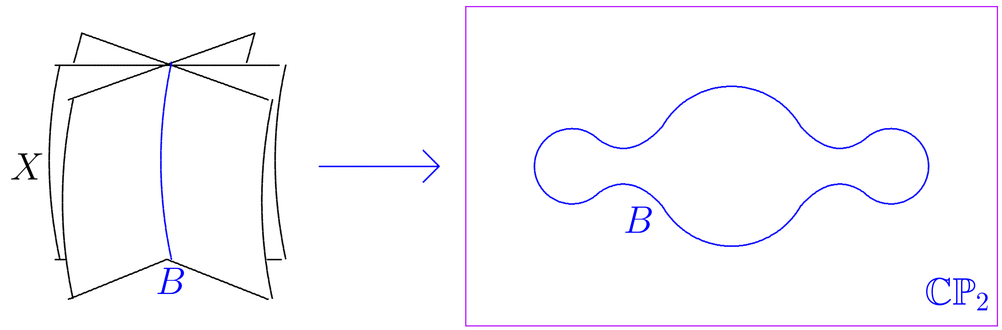

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Theorem $\Longrightarrow$ *no* Einstein metric on M.

But M and N are both
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Hence Freedman $\Longrightarrow$ M homeomorphic to N!

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Moral: Existence depends on diffeotype!

Merci aux organisateurs,

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