Mass in

Kähler Geometry

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Stony Brook University

Trends in Modern Geometry
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Joint work with

Joint work with

Hans-Joachim Hein
University of Maryland

Joint work with

Hans-Joachim Hein
Fordham University

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e-print: arXiv:1507.08885 [math.DG]

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Now on-line in Comm. Math. Phys.

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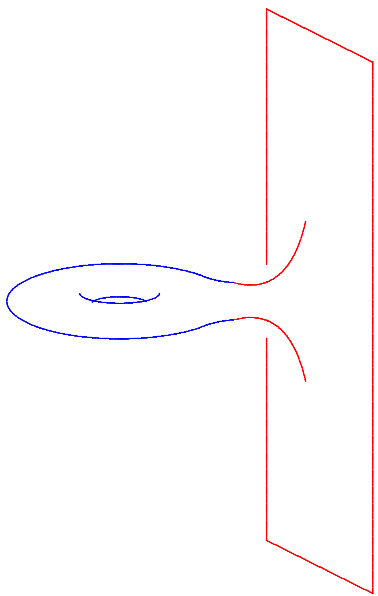
On-line first: Comm. Math. Phys.
doi: 10.1007/200220-016-2661-4

Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g)



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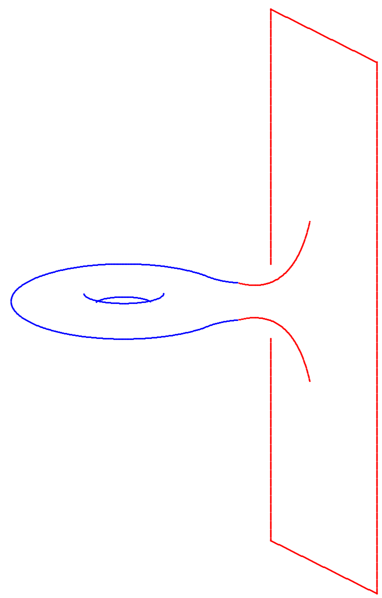

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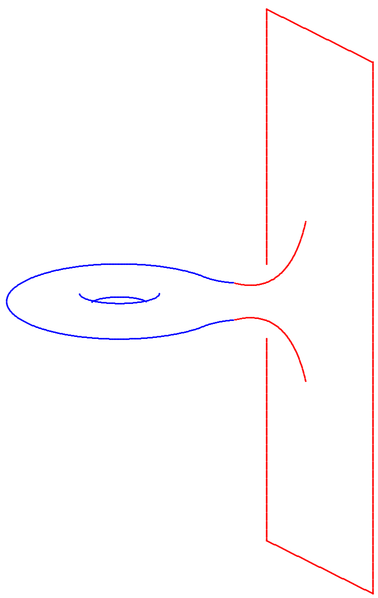

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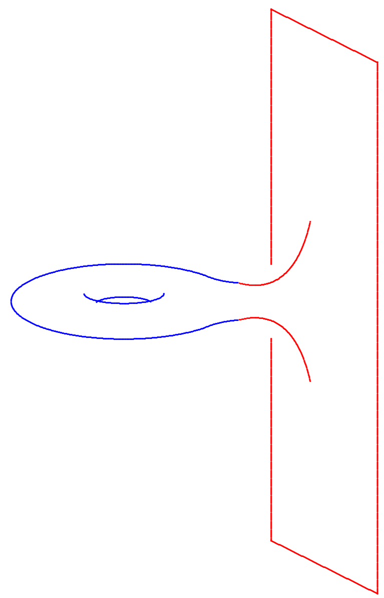

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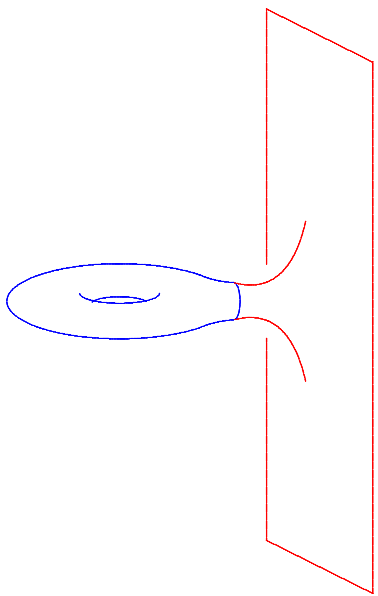
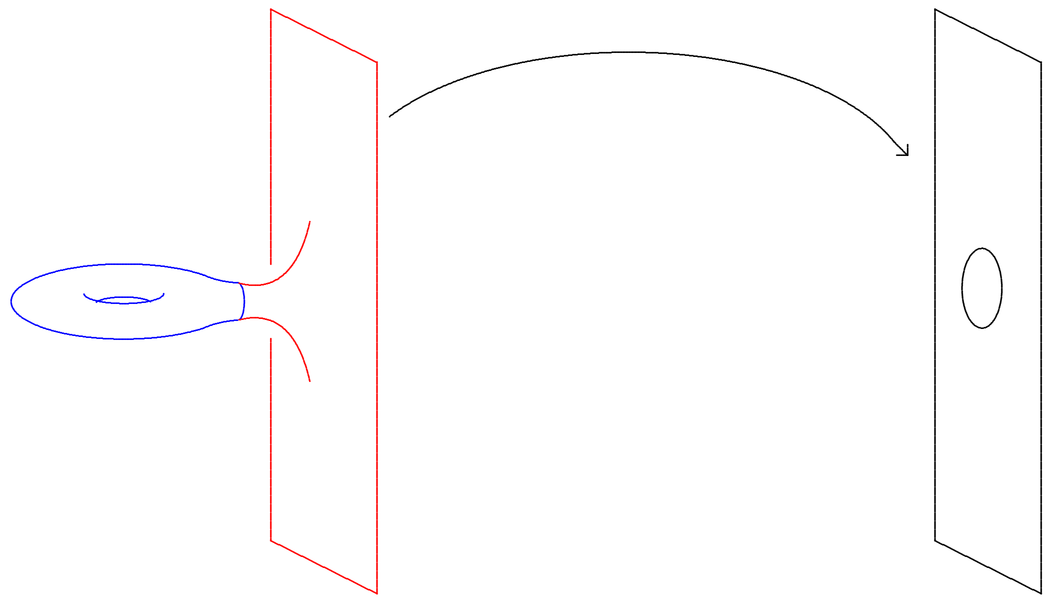


Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g) *is called* asymptotically Euclidean *(AE) if there is a compact set* $K \subset M$ *such that* $M - K$ *is diffeomorphic to* $\mathbb{R}^n - D^n$



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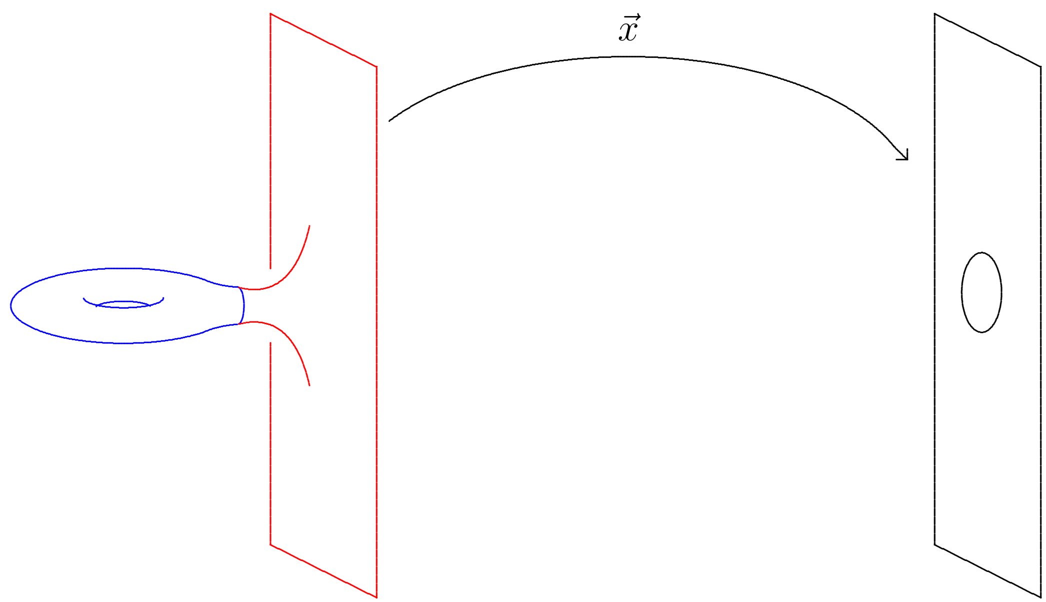

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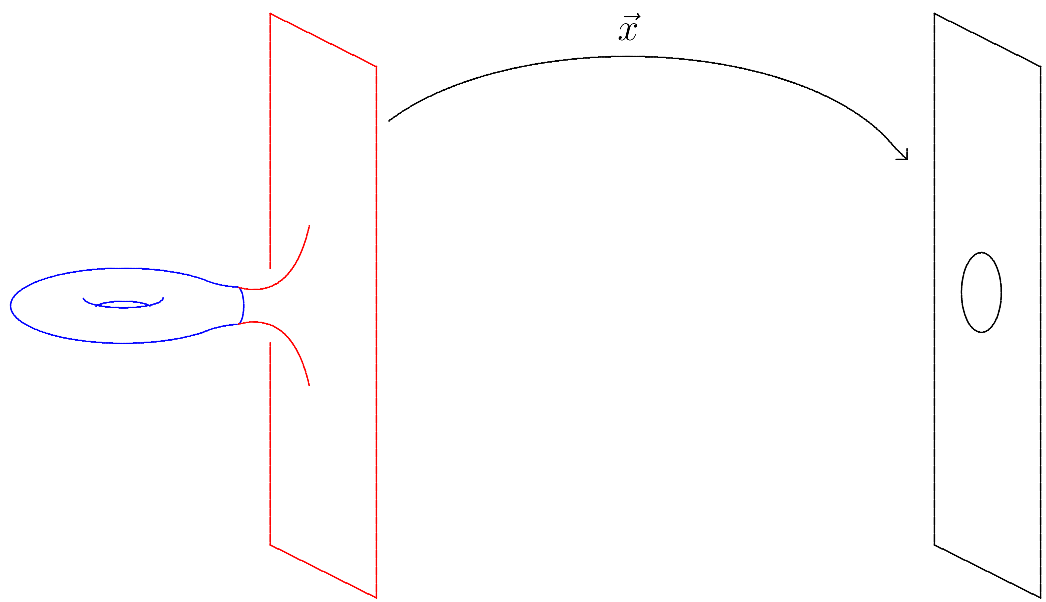

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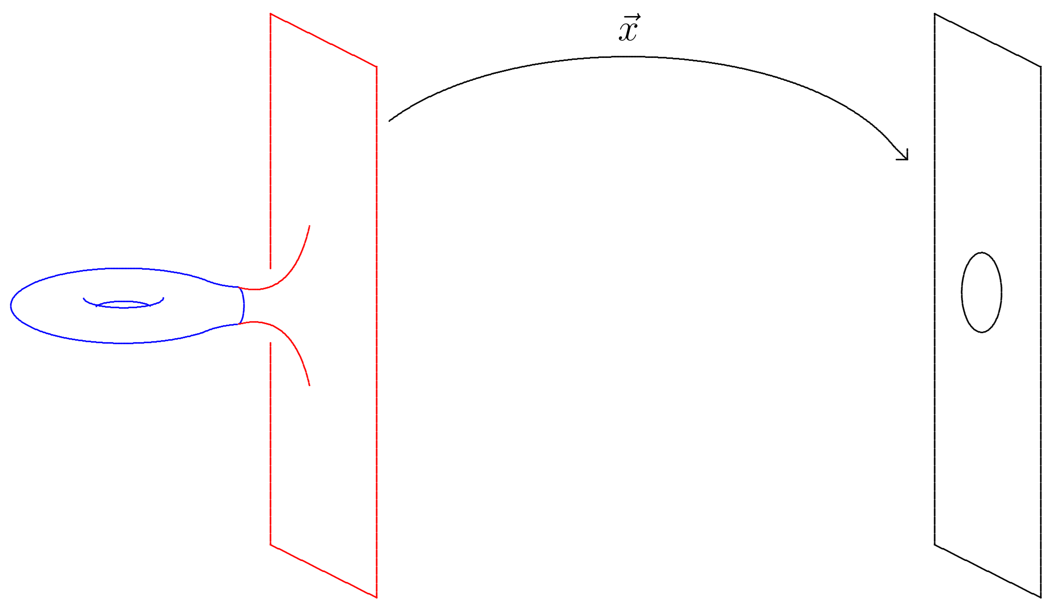

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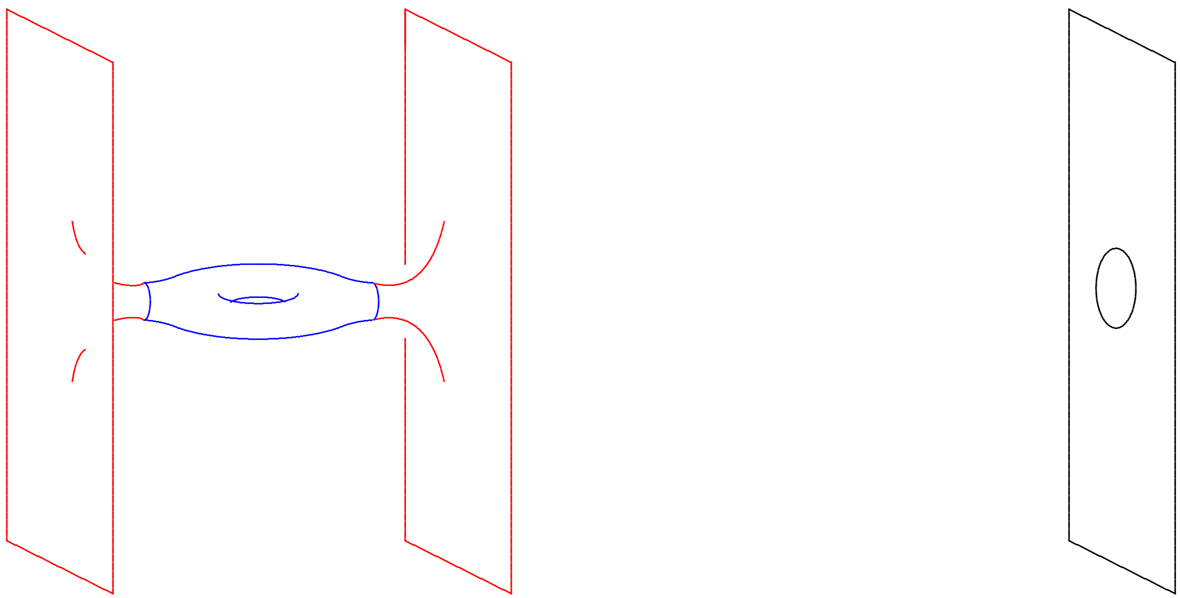

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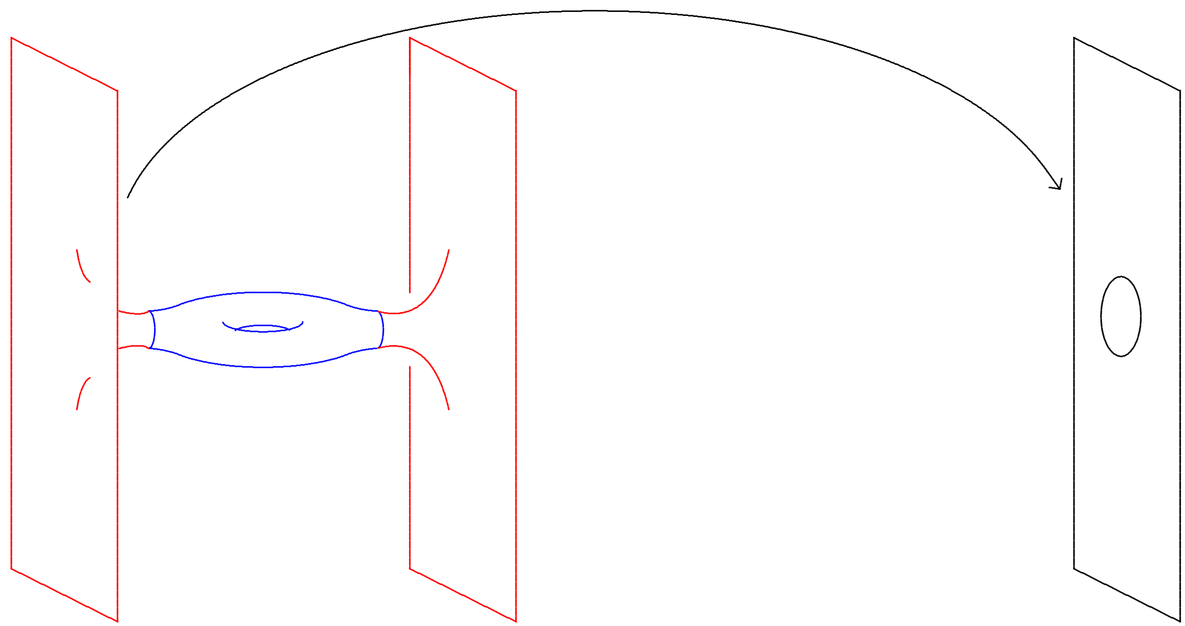

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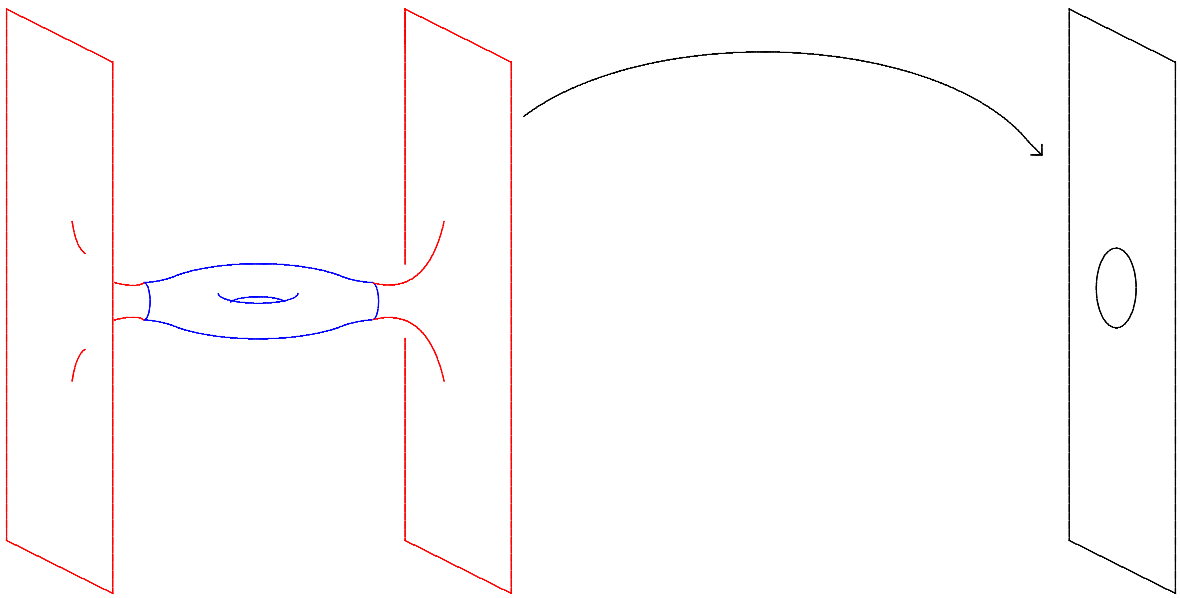

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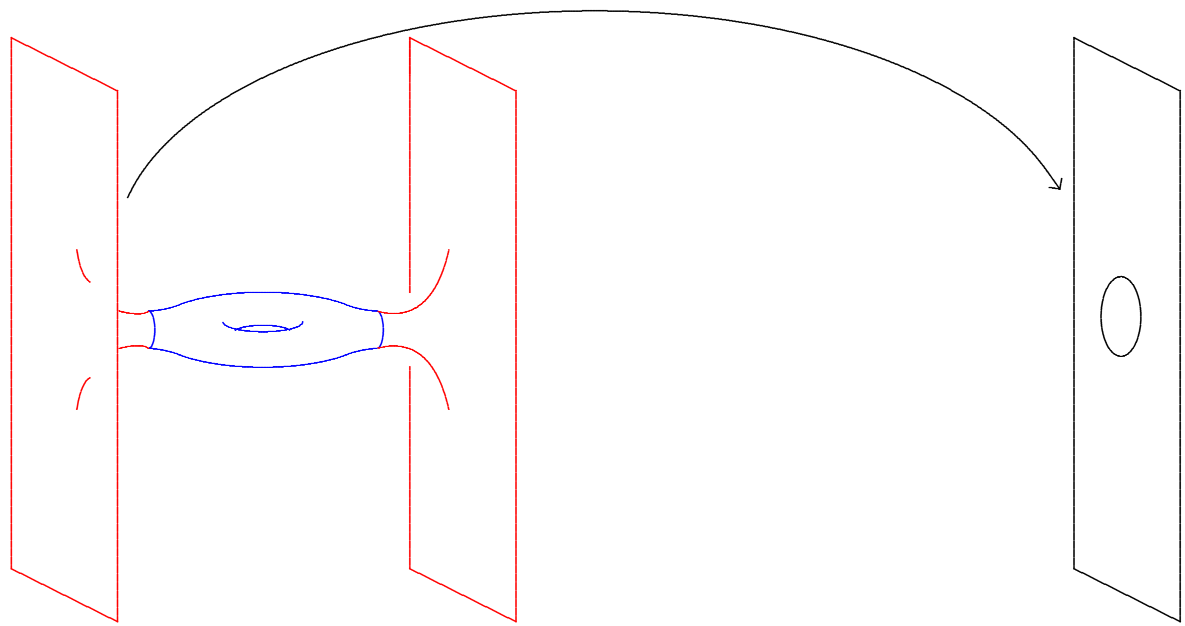

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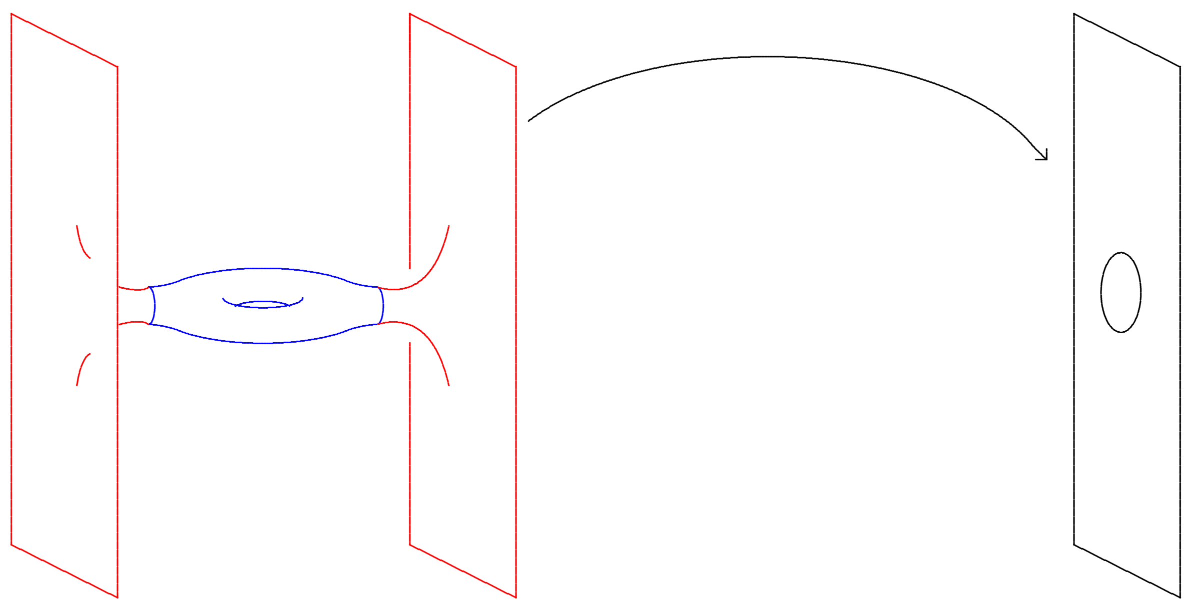

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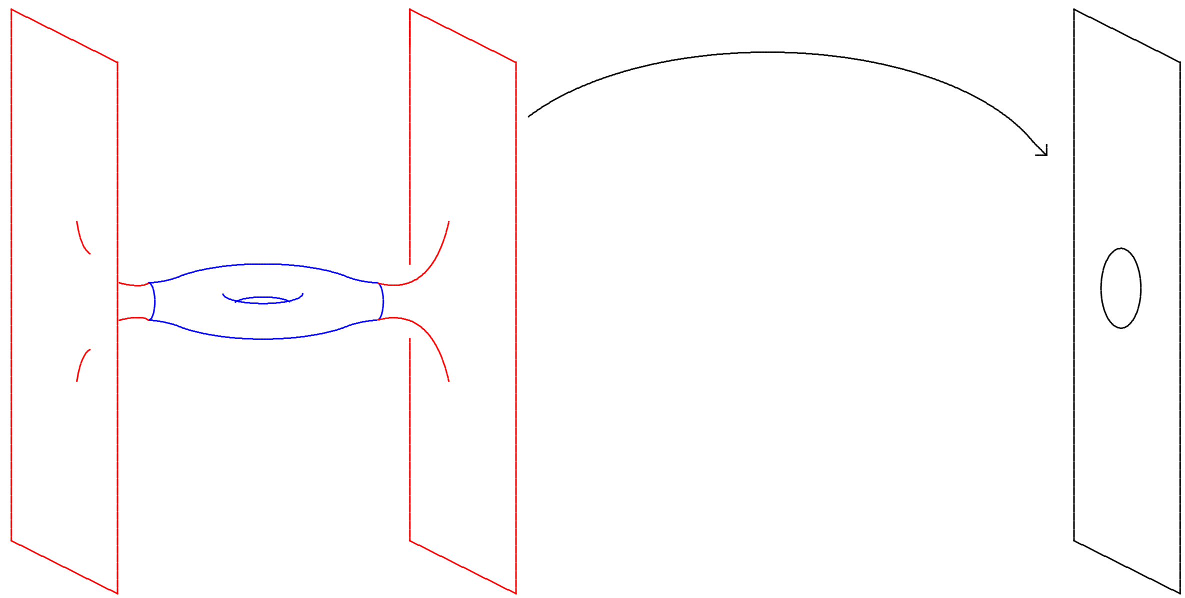

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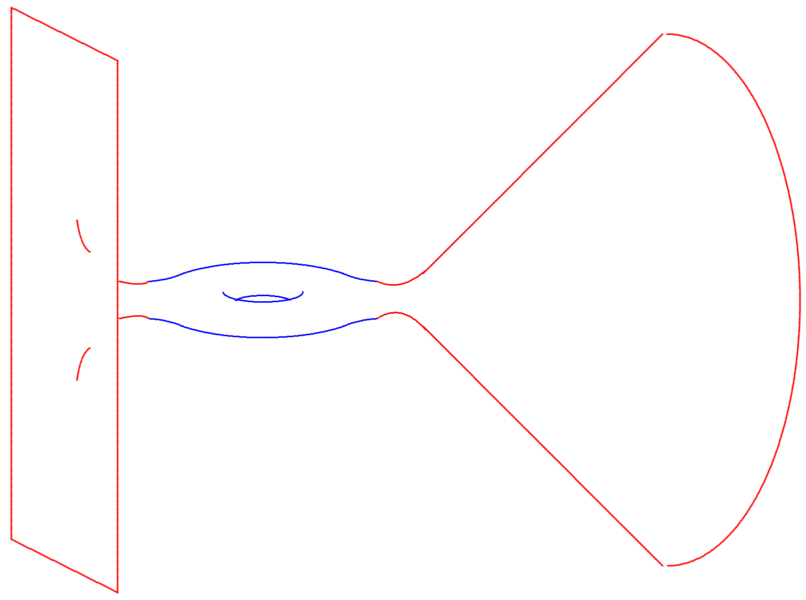

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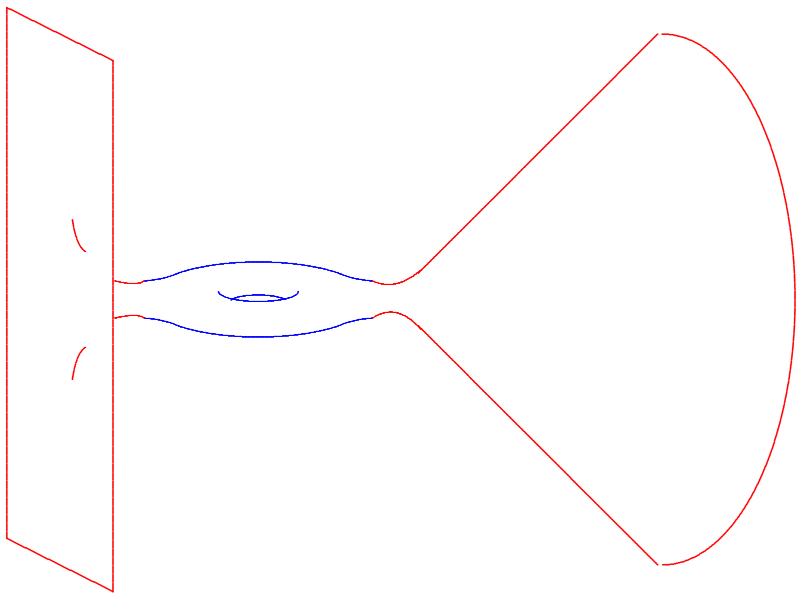

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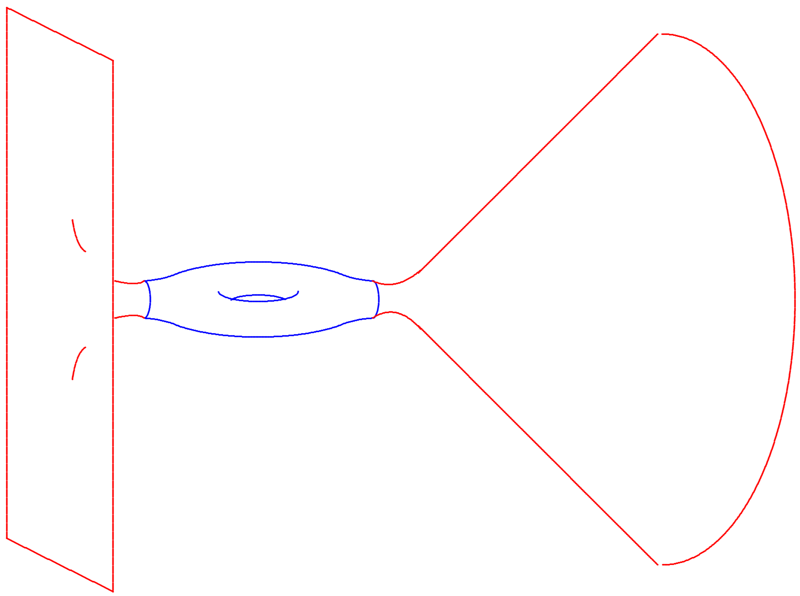

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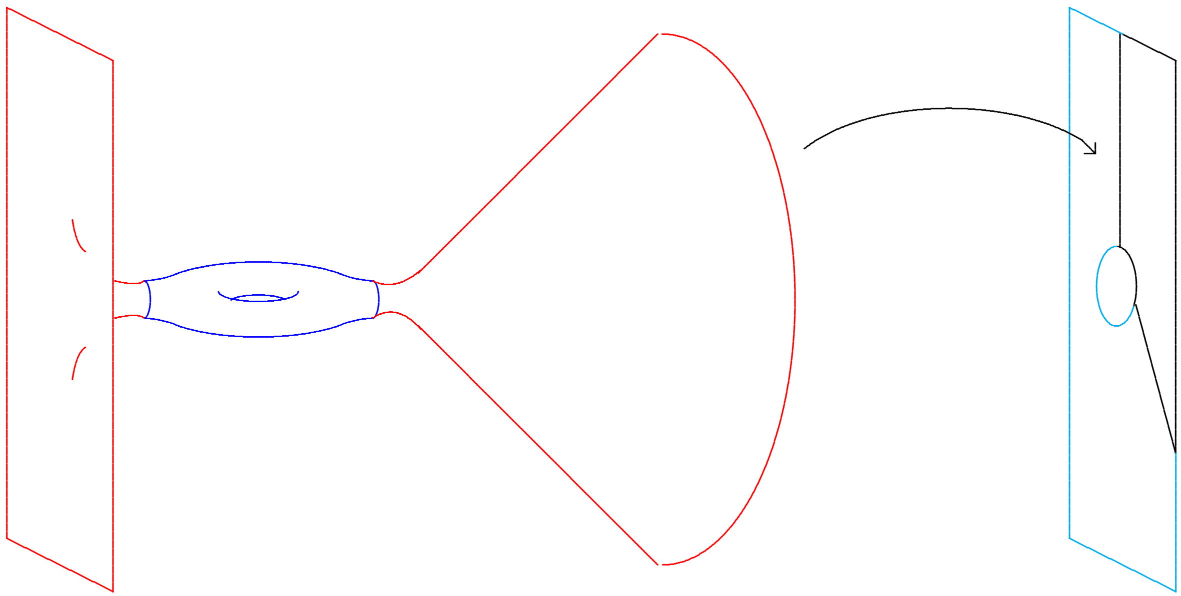

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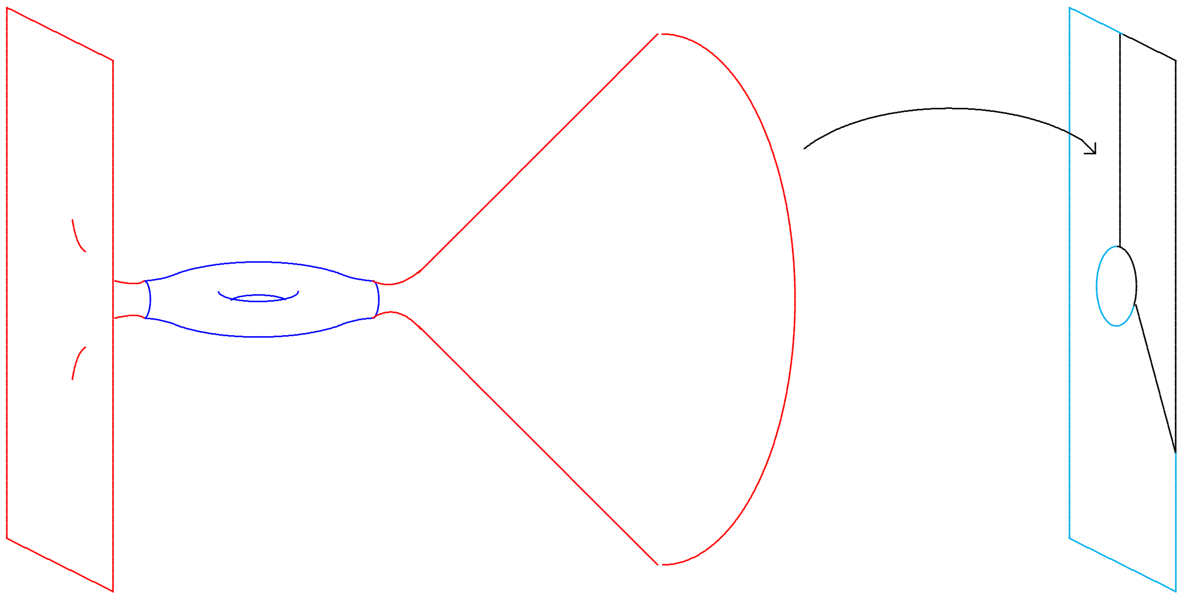

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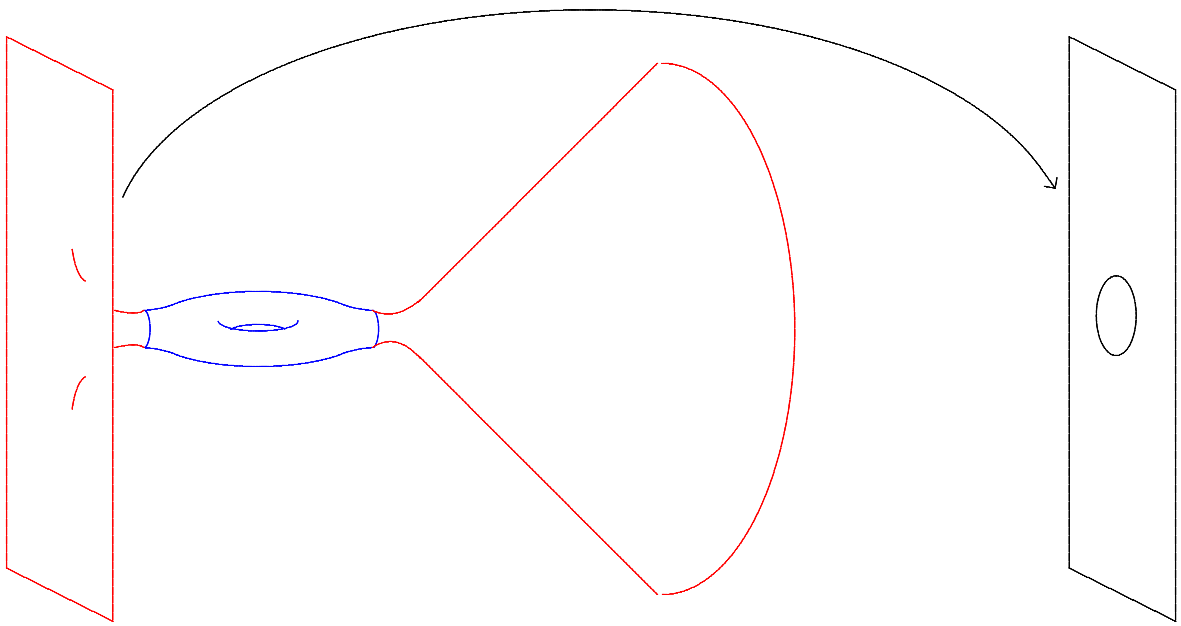

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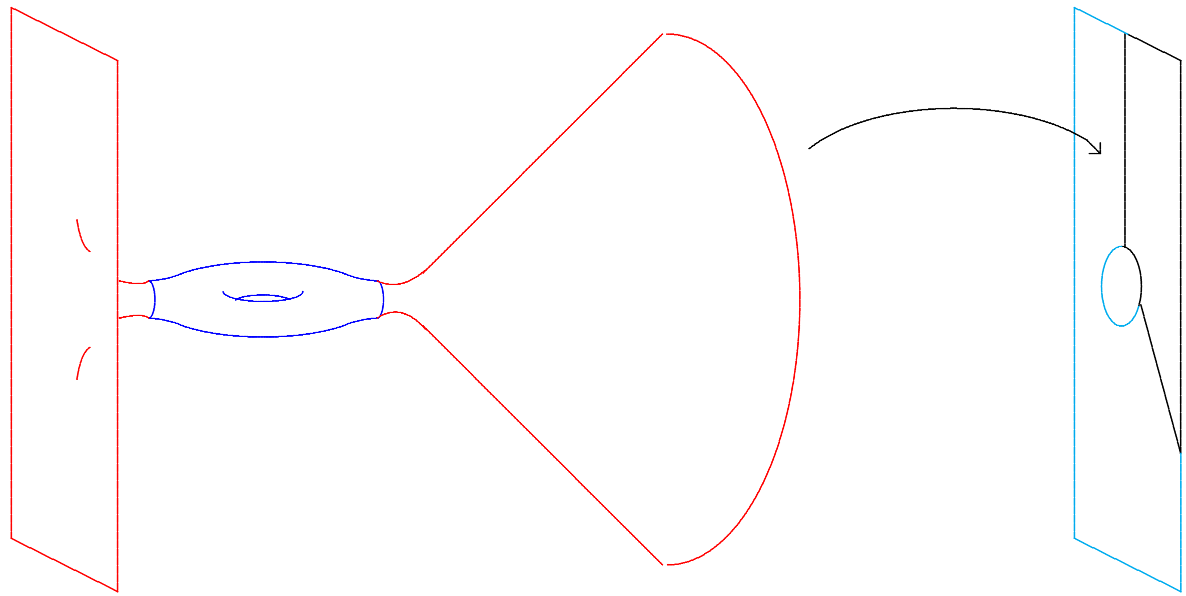

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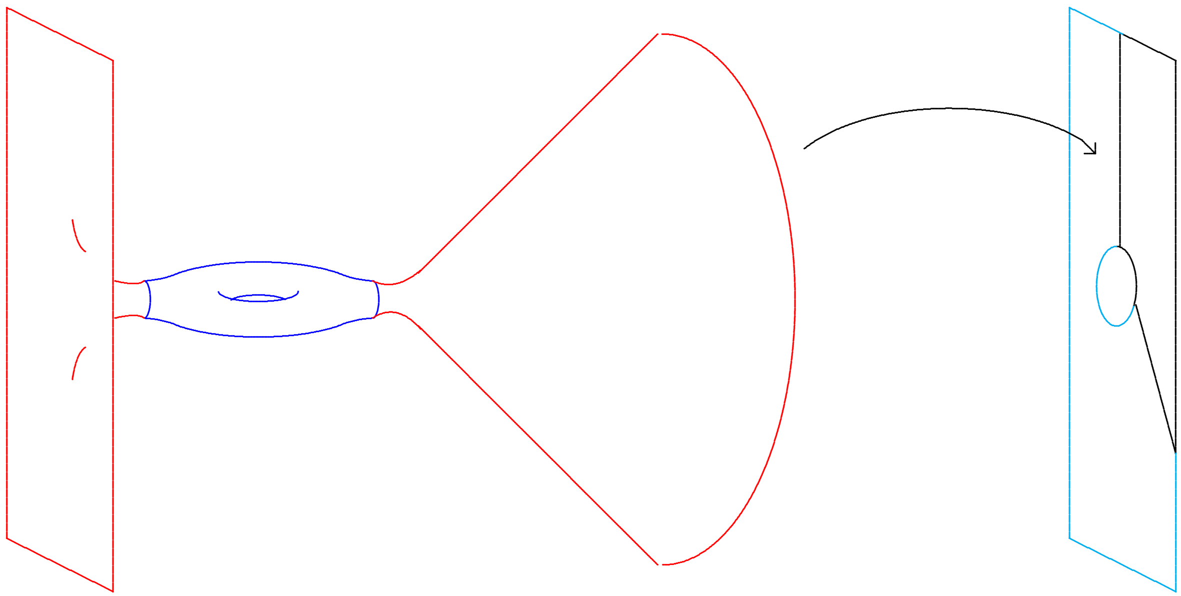

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$$\mathfrak{m}(M, g) := \qquad\qquad \left[g_{ij,i} - g_{ii,j}\right]$$

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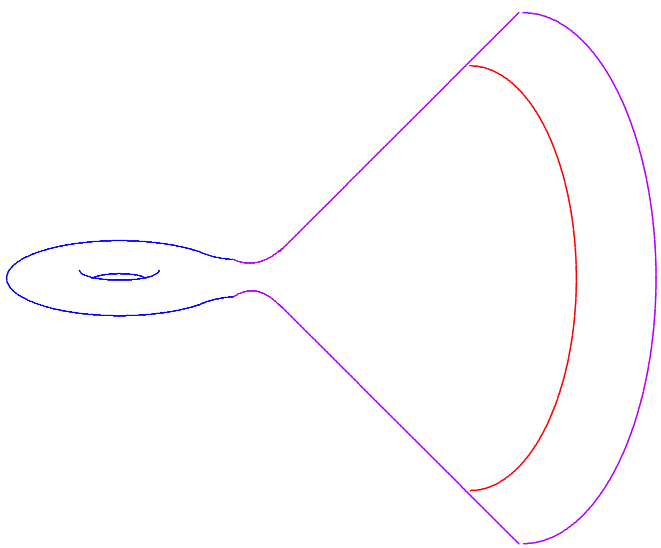

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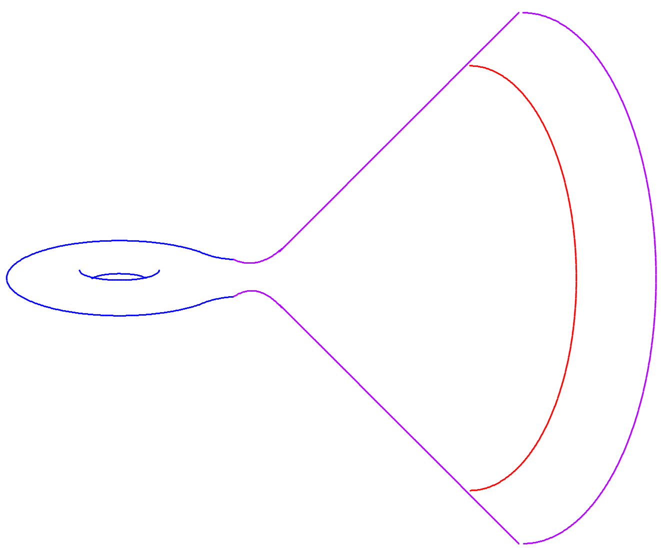

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Seems to depend on choice of coordinates!

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Chruściel-type fall-off:

$$g_{jk} - \delta_{jk} \in C^1_{-\tau}, \quad \tau > \frac{n-2}{2}$$

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- α_E *is the volume* $(n-1)$*-from induced by the Euclidean metric.*

Bartnik/Chruściel (1986): With weak fall-off conditions, the mass is well-defined & coordinate independent.

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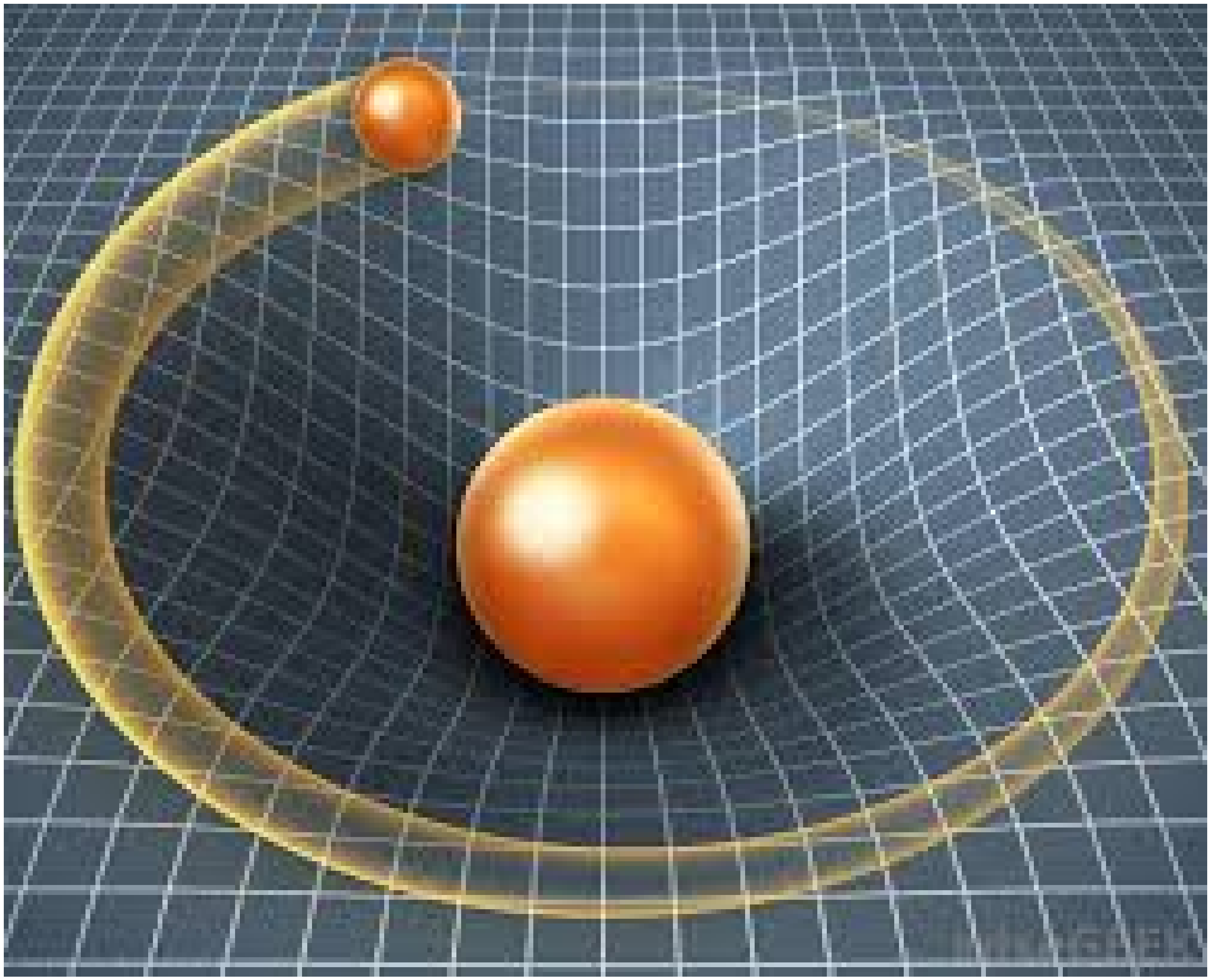

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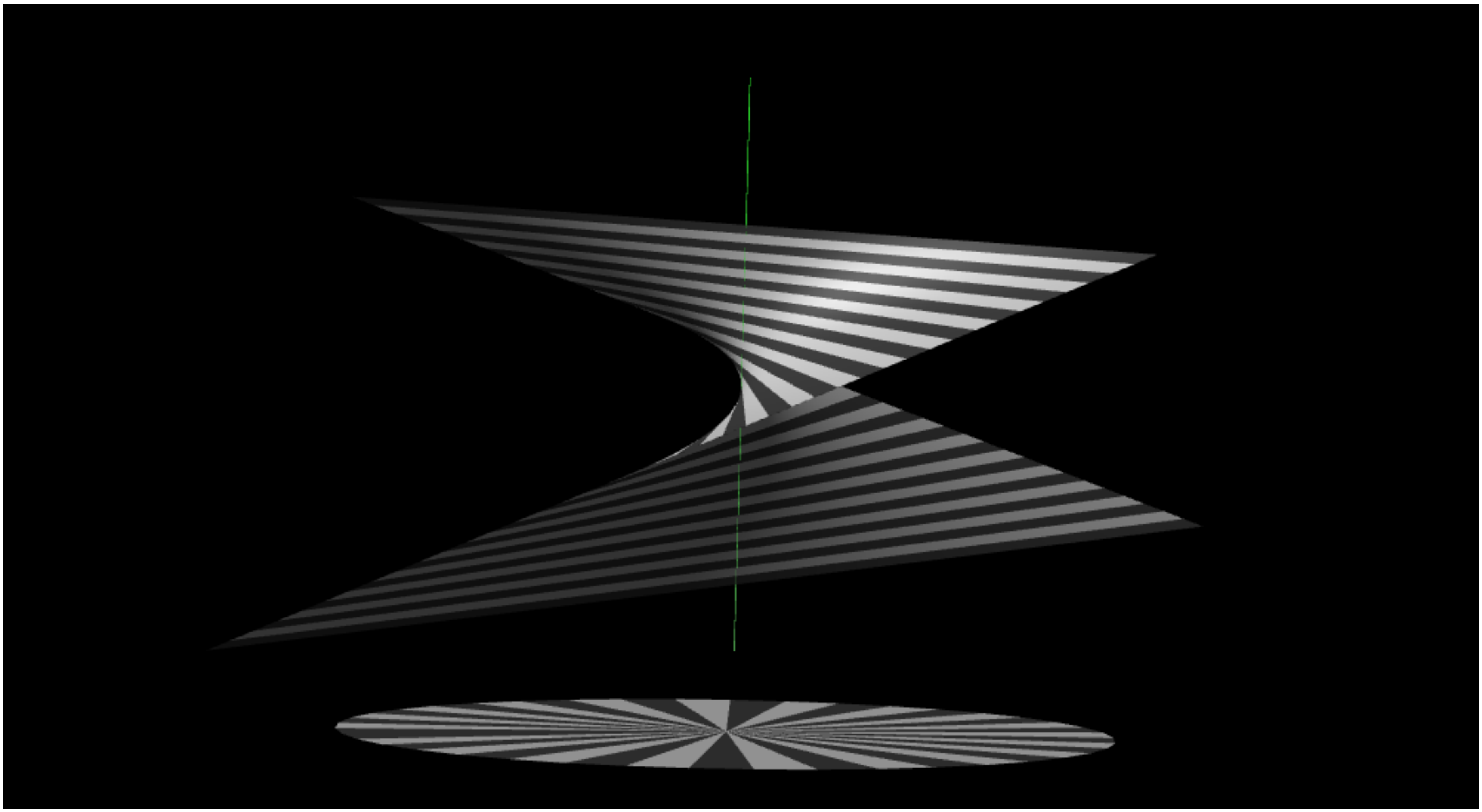

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on line bundles $L \to \mathbb{CP}_1$ of Chern-class ≤ -3.

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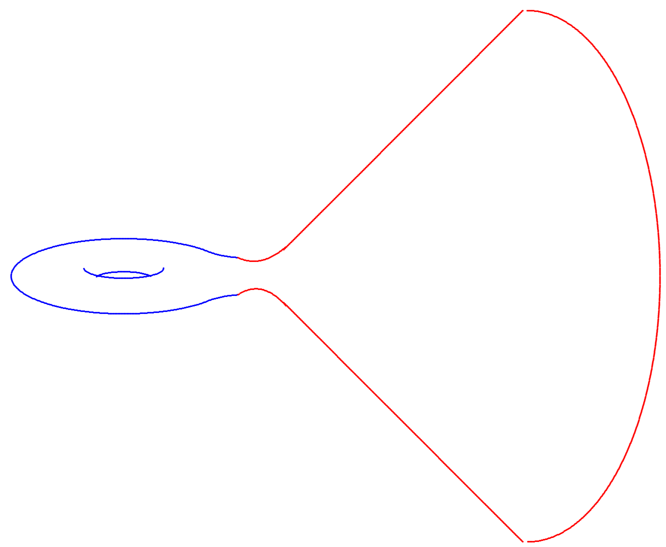

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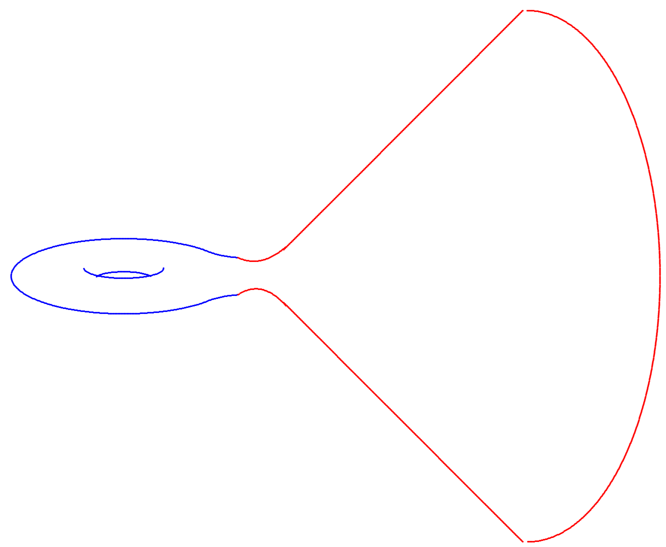

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Mass of an ALE Kähler manifold is unambiguous.

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Mass of an ALE Kähler manifold is unambiguous.

Does not depend on the choice of an end!

We begin with the scalar-flat Kähler case.

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Theorem A.

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Theorem A. *The mass*

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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Applies, for examples, to new examples discussed at this conference by **Jeff Viaclovsky**.

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$\mathcal{E}^p_c(M) :=$ {Smooth, compactly supported p-forms on M}.

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to denote the **inverse** *of the natural map*

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induced by the inclusion of compactly supported smooth forms into all forms.

We can now state our mass formula:

Theorem C.

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$$\frac{4\pi^m(2m-1)}{(m-1)!}\mathfrak{m}(M,g) = -\frac{4\pi}{(m-1)!}\langle \clubsuit(c_1), [\omega]^{m-1}\rangle + \int_M s_g d\mu_g$$

For a compact Kähler manifold (M^{2m}, g, J),

$$\int_M s_g d\mu_g = \frac{4\pi}{(m-1)!} \langle c_1, [\omega]^{m-1} \rangle$$

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$$0 = -\frac{4\pi}{(m-1)!}\langle c_1, [\omega]^{m-1}\rangle + \int_M s_g d\mu_g$$

For an ALE Kähler manifold (M^{2m}, g, J),

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$

Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}}.$$

Corollary. *Any ALE scalar-flat Kähler manifold (M, g, J) of complex dimension m has mass given by*

$$m(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}}.$$

So **Theorem A** is an immediate consequence!

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$$g^{jk}\left(g_{j\ell,k} - g_{jk,\ell}\right)\nu^\ell \alpha_E = -\star d \log\left(\sqrt{\det g}\right) + O(\varrho^{-3-\varepsilon}).$$

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However, since $s = 0$,

$$d(\theta\wedge\omega) = \rho\wedge\omega = \frac{s}{4}\omega^2 = 0.$$

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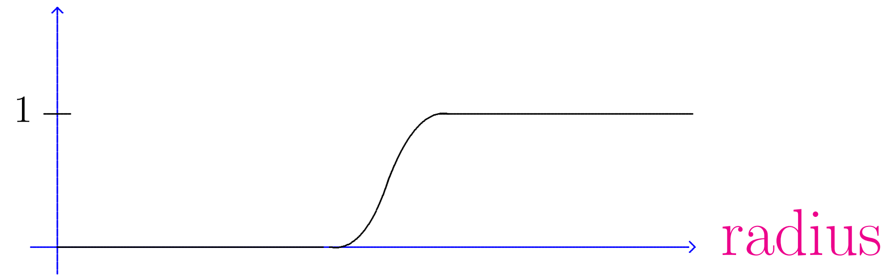

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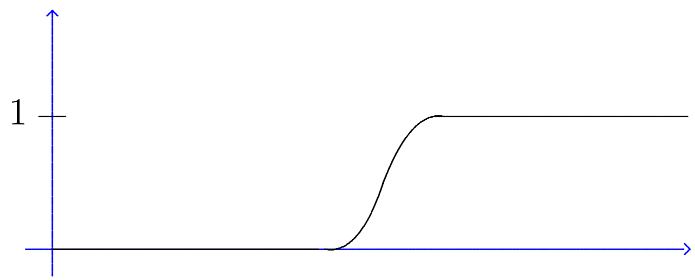

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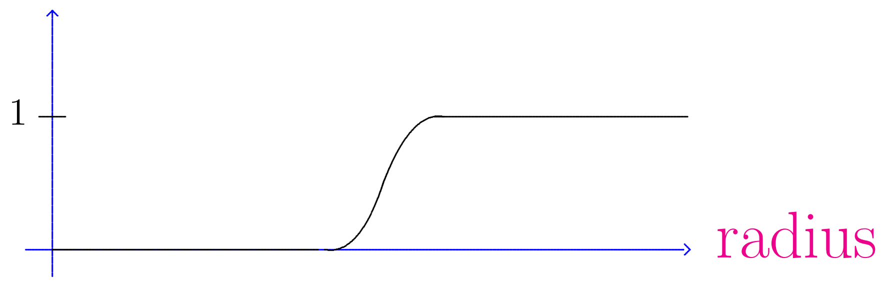

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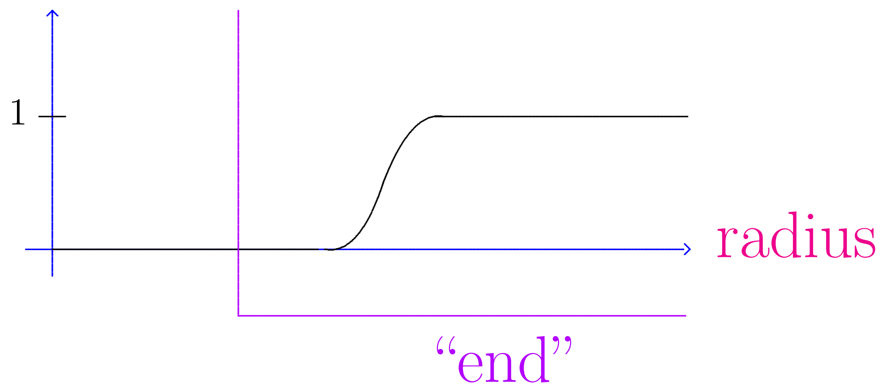

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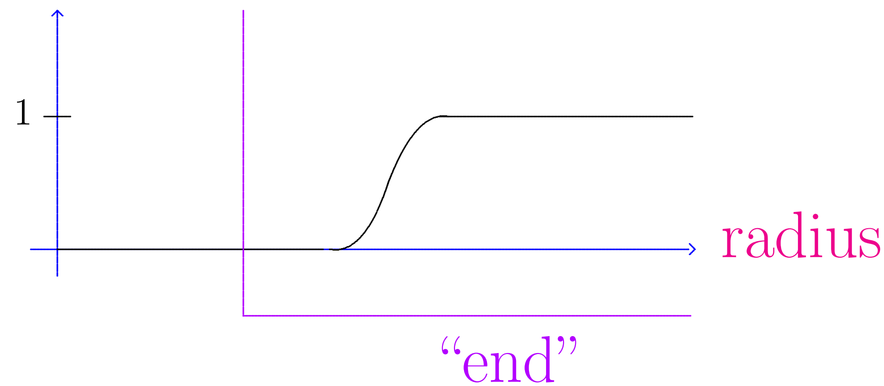

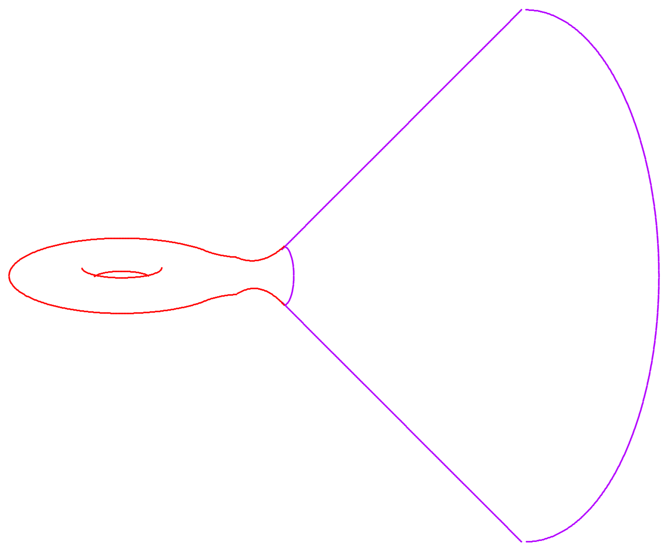

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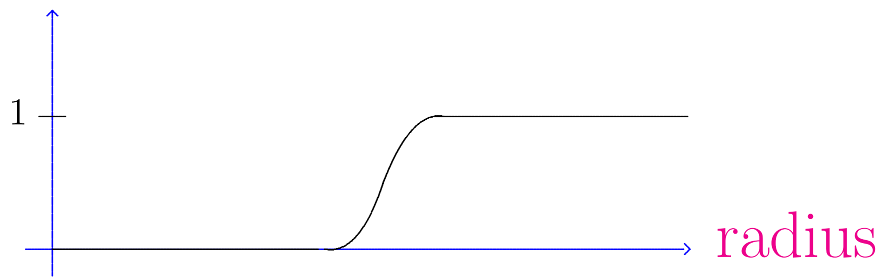

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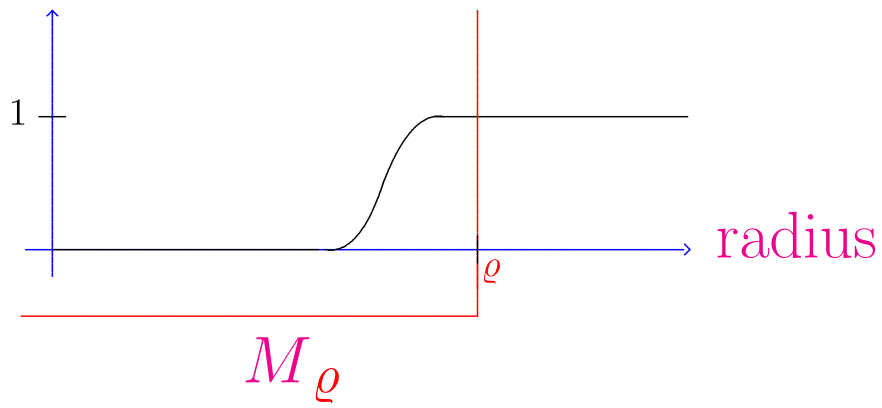

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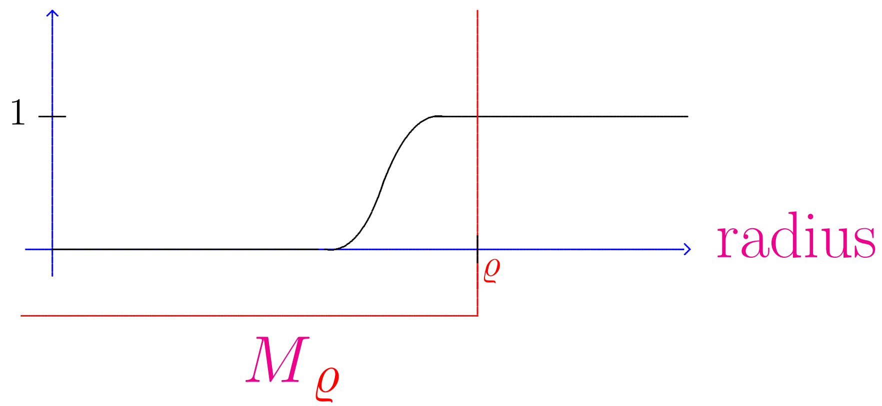

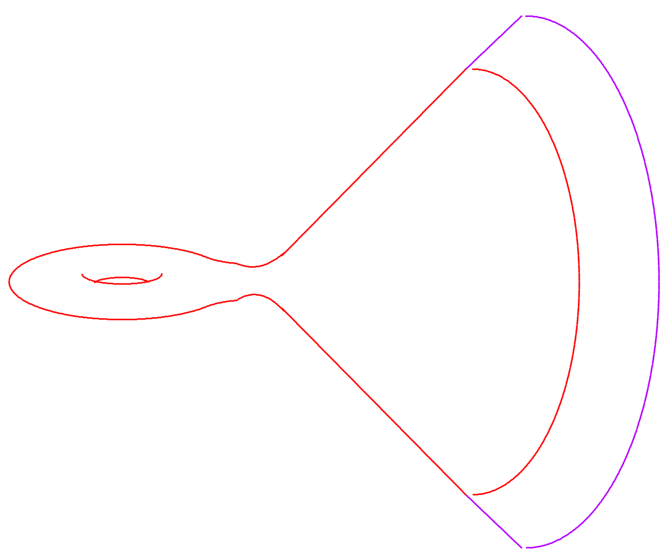

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Compactly supported, because $d\theta = \rho$ near infinity.

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where M_ϱ defined by radius $\leq \varrho$.

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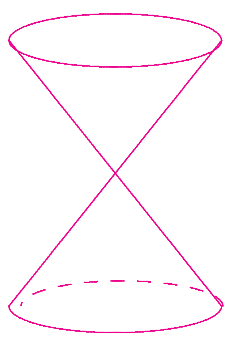

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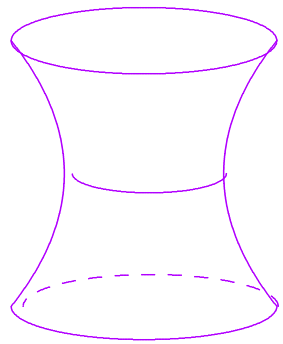

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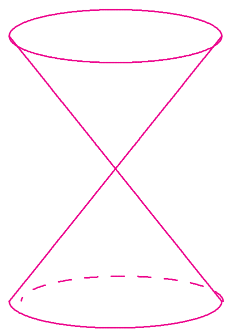

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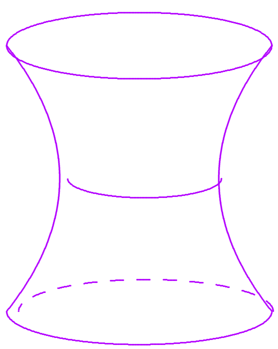

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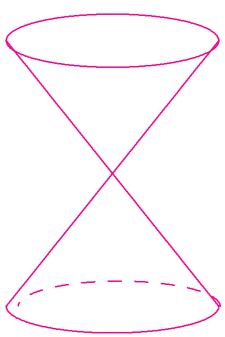

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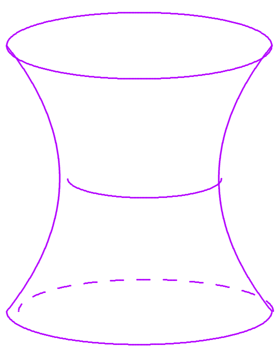

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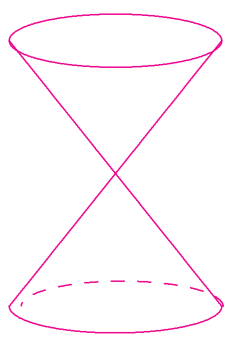
Example: Honda metrics. Non-hyper-Kähler.



General case:

- General $m \geq 2$: straightforward...
- $s \not\equiv 0$, compensate by adding $\int s \, d\mu$...
- If $m > 2$, J is always standard at infinity.
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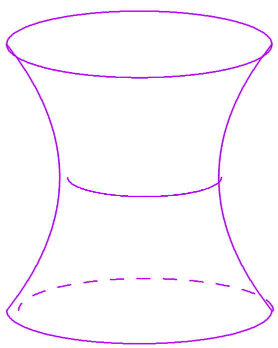
Example: Honda metrics. Deform $\mathcal{O}(-3) \to \mathbb{CP}_1$.



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$$J = J_0 + O(\varrho^{-3}), \qquad \nabla J = O(\varrho^{-4})$$

in suitable asymptotic coordinates adapted to g.

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This has some interesting consequences…

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Proof actually shows something stronger!

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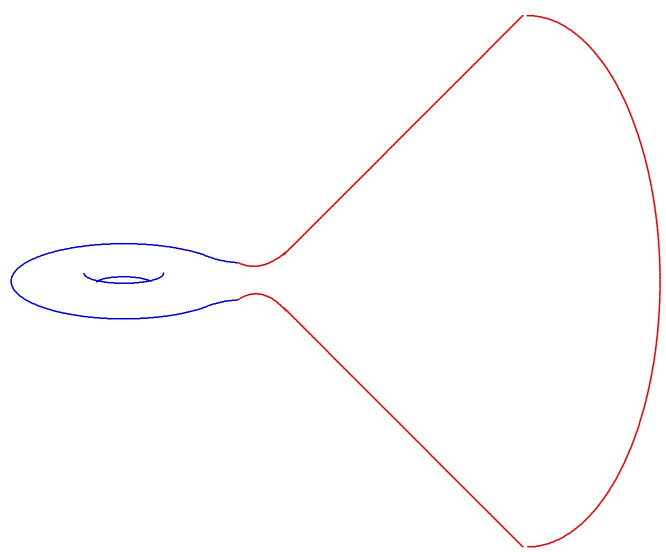
so the mass formula implies the claim.

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$



Before ending,

Before ending, let me now say one more thing. . .

かいぎ　さそ

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いじょう
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