The Einstein-Maxwell Equations

and

Conformally Kähler Geometry

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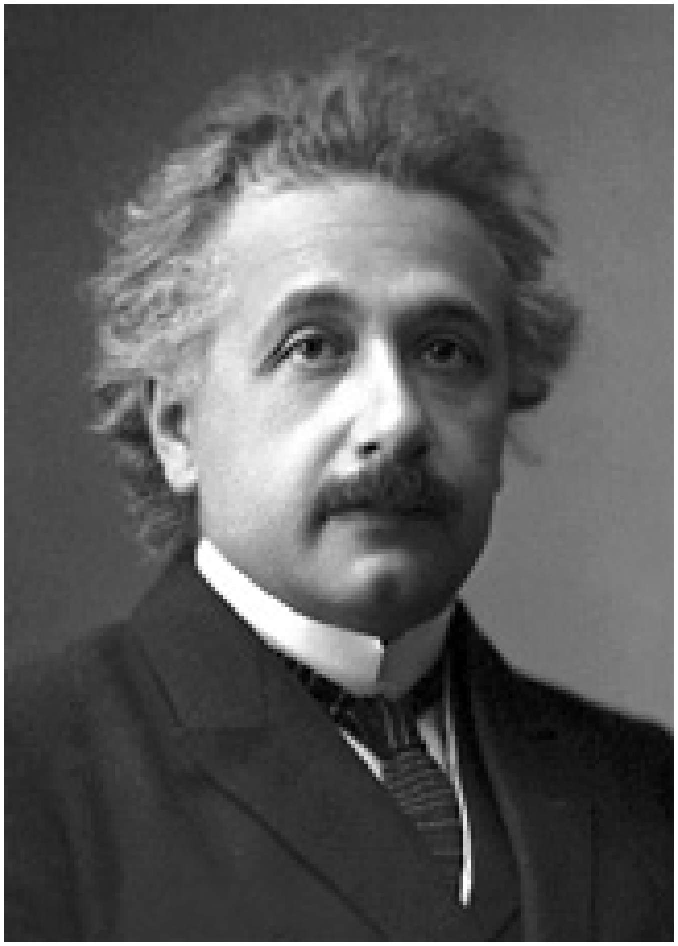

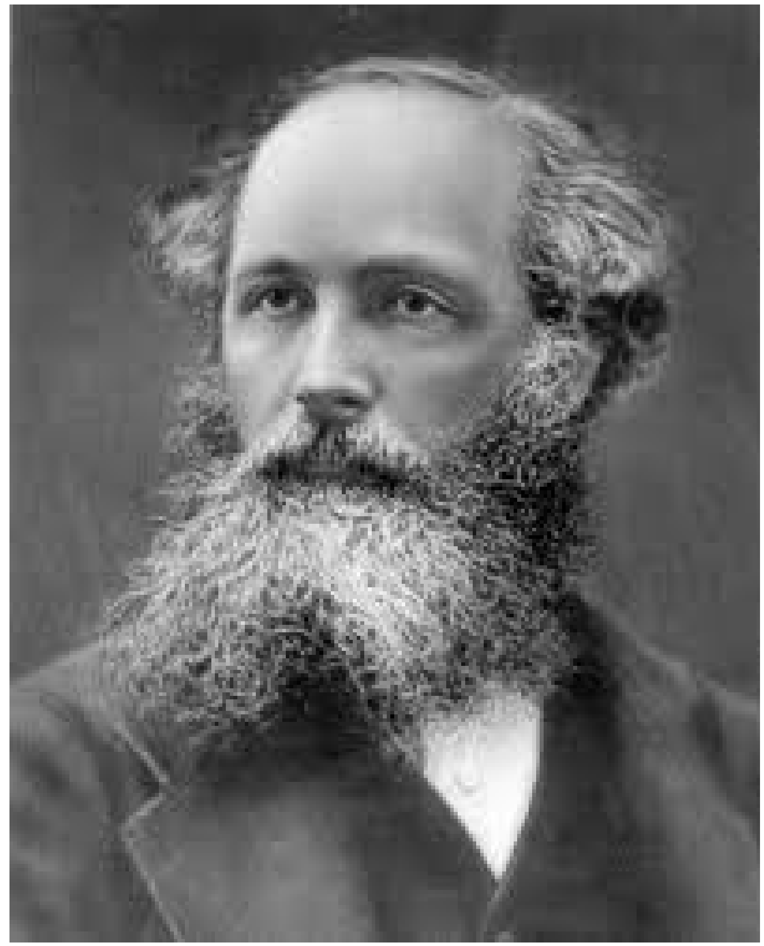

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Purely 4-dimensional phenomenon.

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Idea due to Apostolov-Calderbank-Gauduchon.

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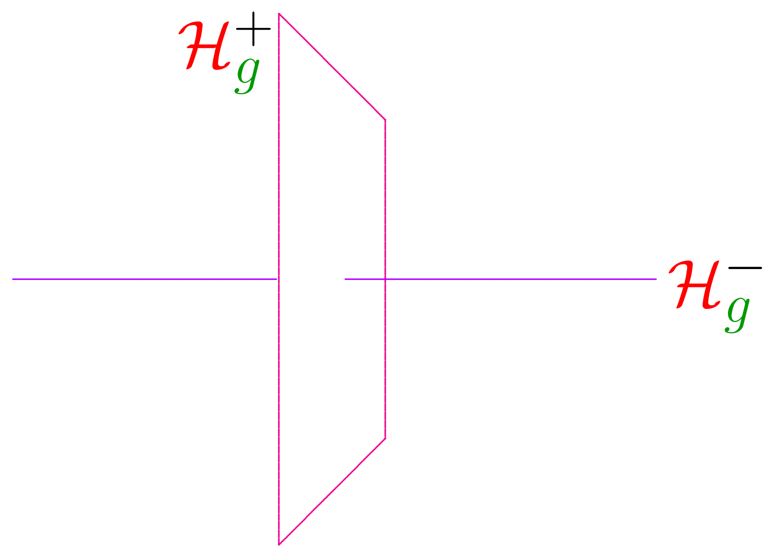
Decomposition is conformally invariant.

The numbers

$$b_\pm(M) = \dim \mathcal{H}^\pm_g.$$

are important homotopy invariants of M.



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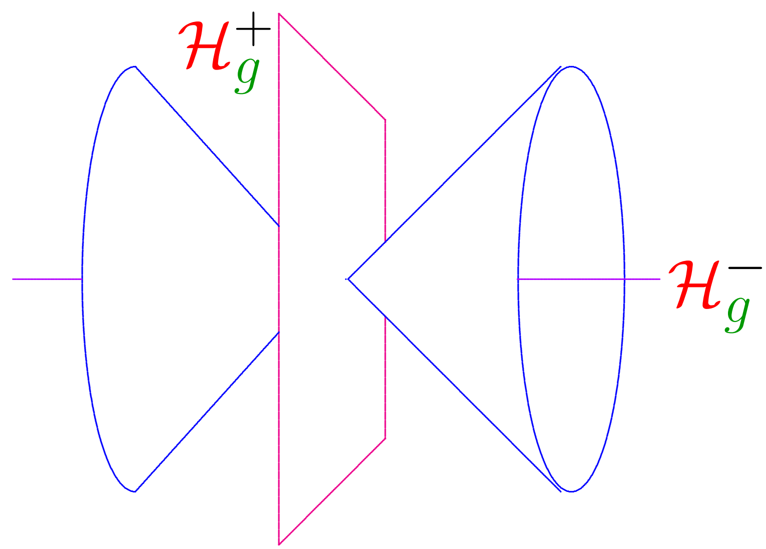

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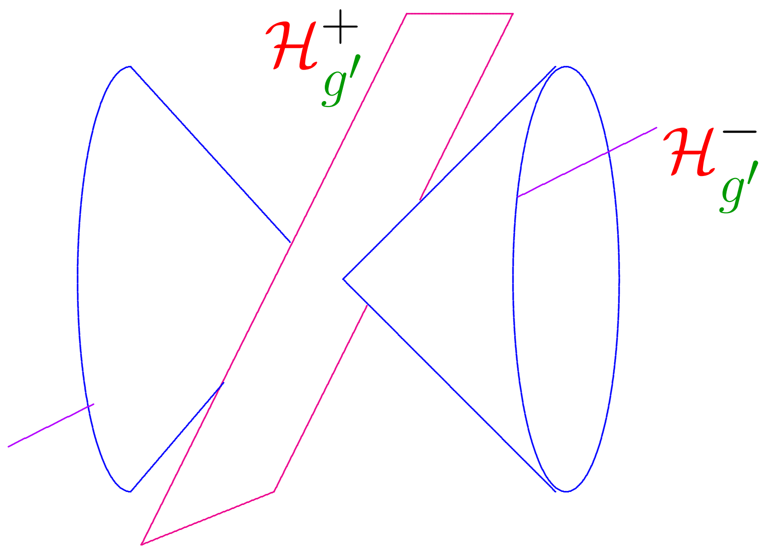

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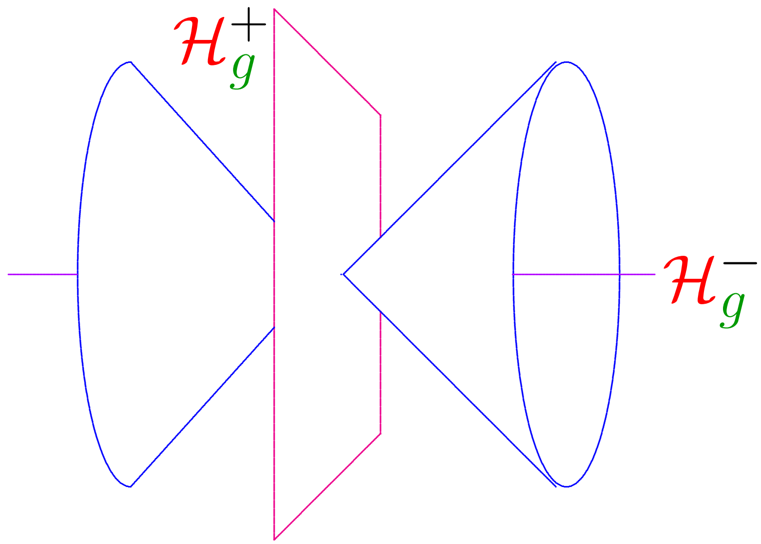

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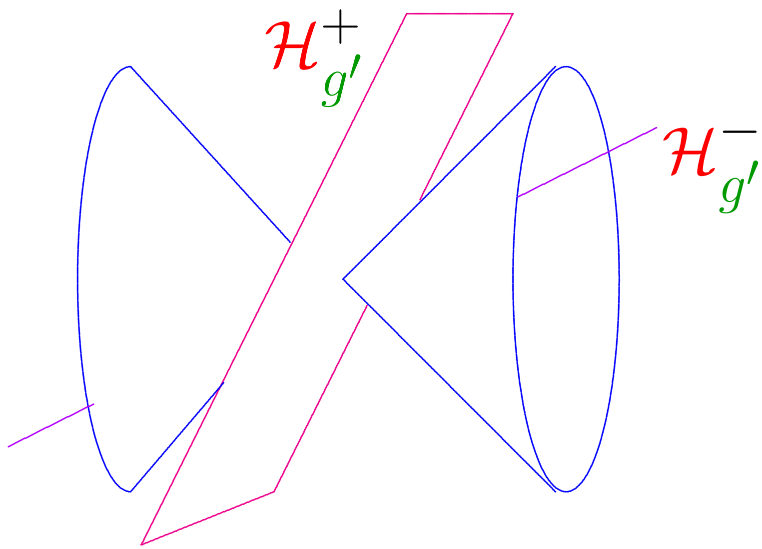

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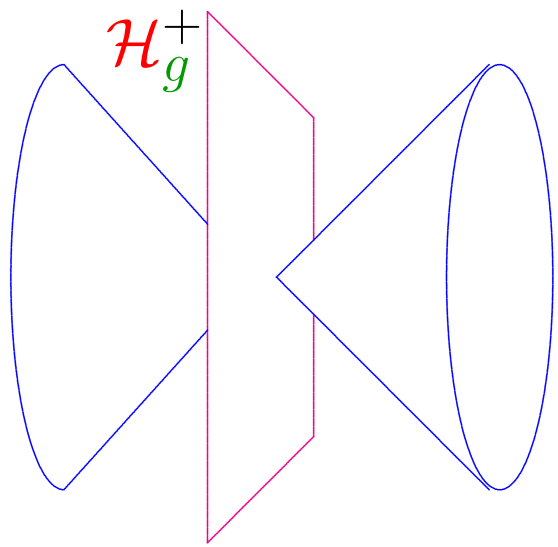

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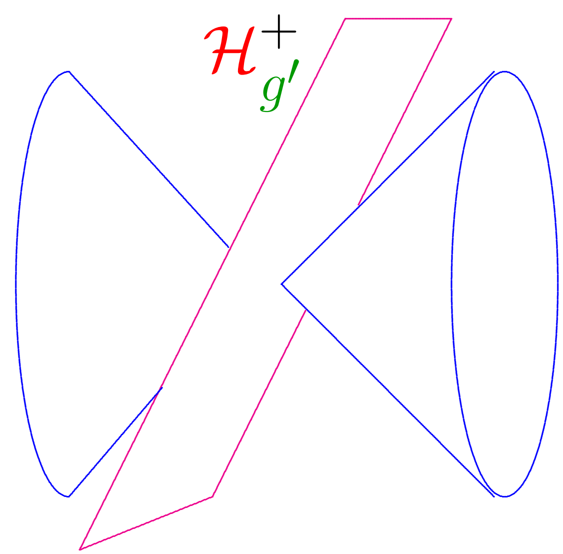

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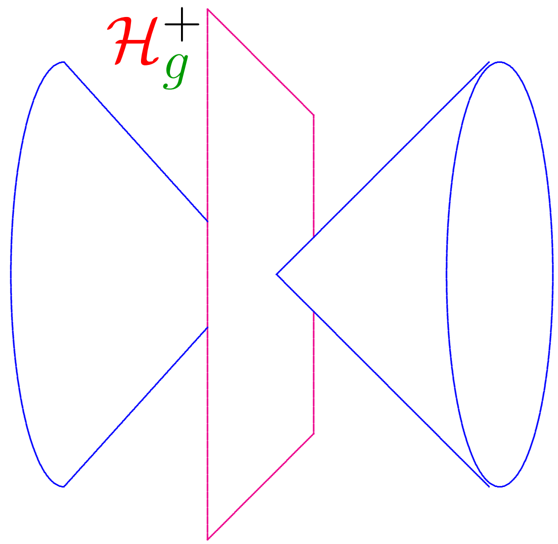

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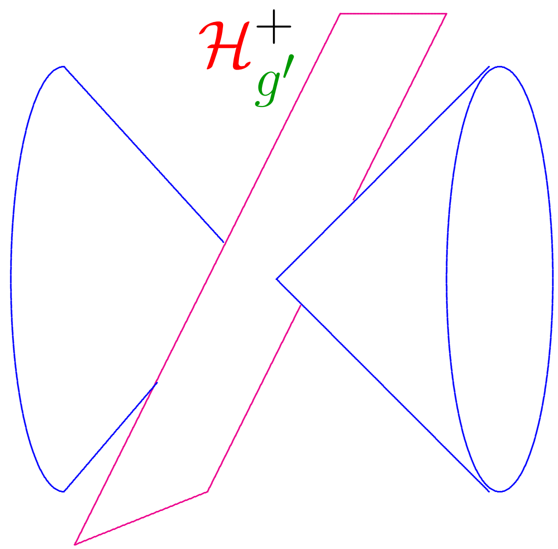

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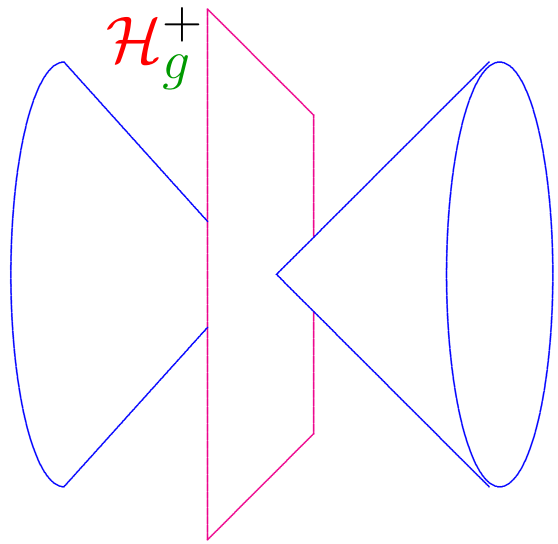

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Remark Notice, however, that

$$\mathcal{G}_\Omega = \mathcal{G}_{\lambda\Omega}$$

for any $\lambda \in \mathbb{R}^\times$. Moreover, $\mathcal{G}_\Omega$ invariant under $\mathrm{Diff}_0(M)$ and conformal rescalings.

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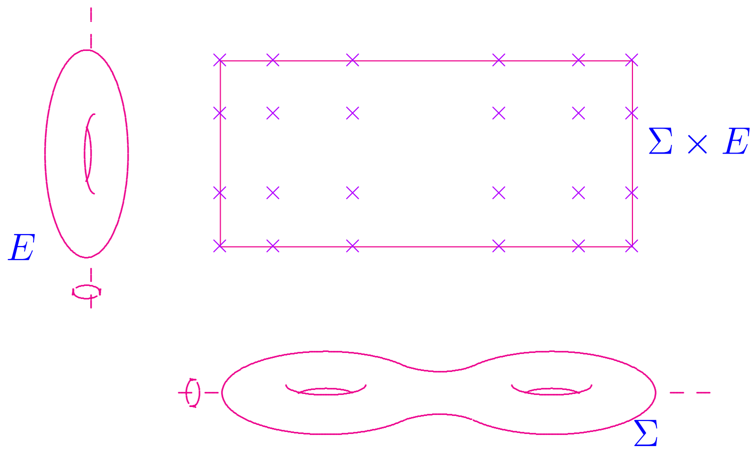

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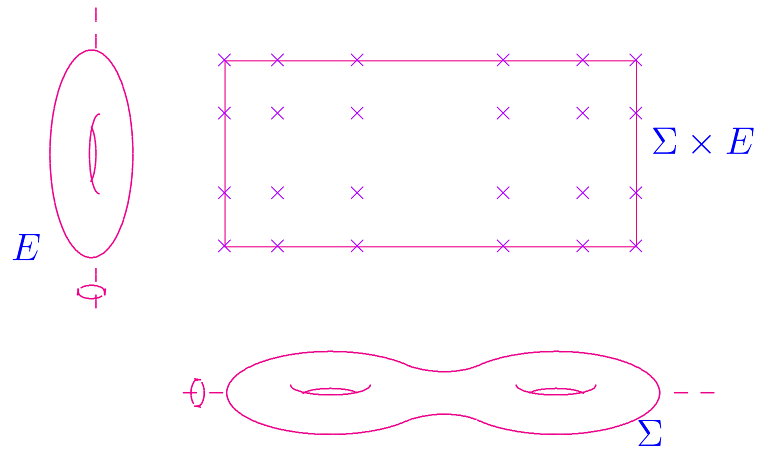

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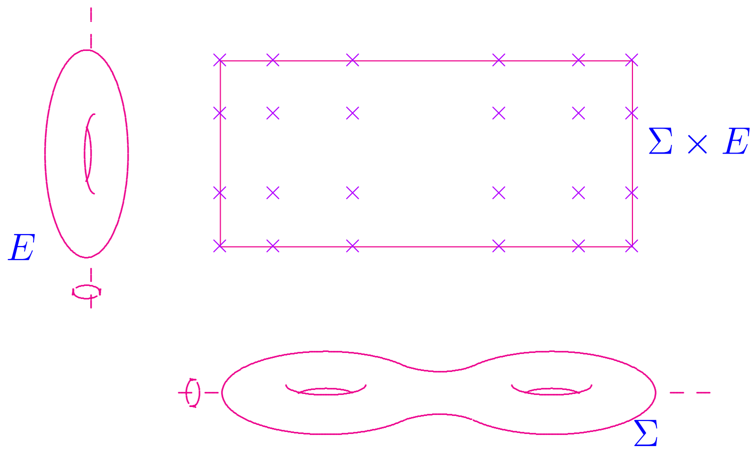

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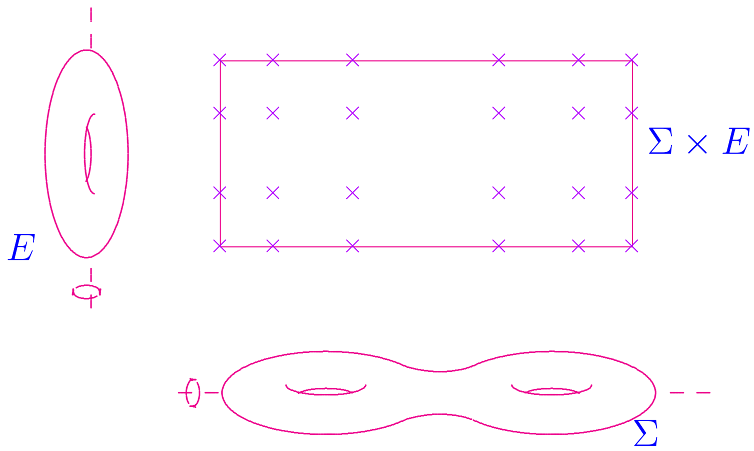

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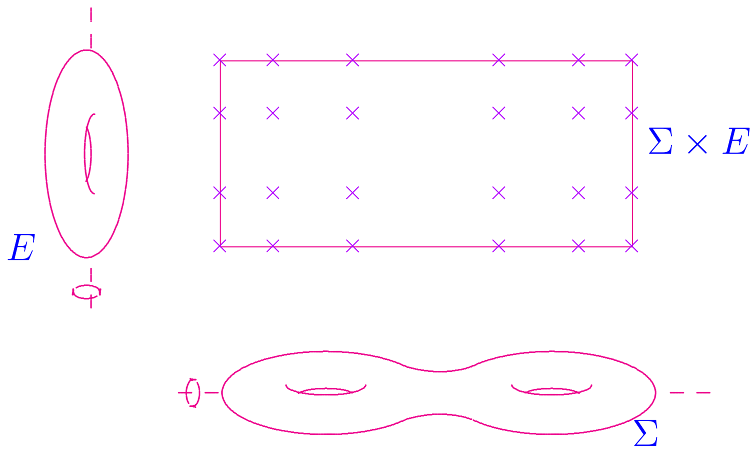

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Systematic study: Yujen Shu's thesis.

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$$F = \omega + \frac{\left[f\rho + 2i\,\partial\bar{\partial}f\right]_0}{2f^3}$$

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Constructions & Proofs

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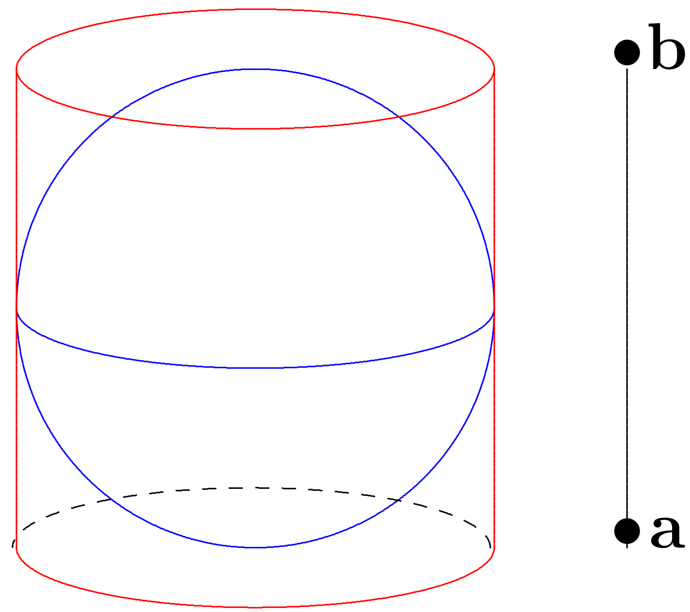

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Global solution:

$$\Phi(\mathbf{a}) = \Phi(\mathbf{b}) = 0, \quad \Phi'(\mathbf{a}) = -\Phi'(\mathbf{b}) = 2, \quad \Phi'(0) = 0.$$

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$$\Phi(t) = \frac{(t-\mathbf{a})(t-\mathbf{b})}{\mathbf{a}-\mathbf{b}}\left[2 - \frac{(t-\mathbf{a})(t-\mathbf{b})}{\mathbf{a}b}\right]$$

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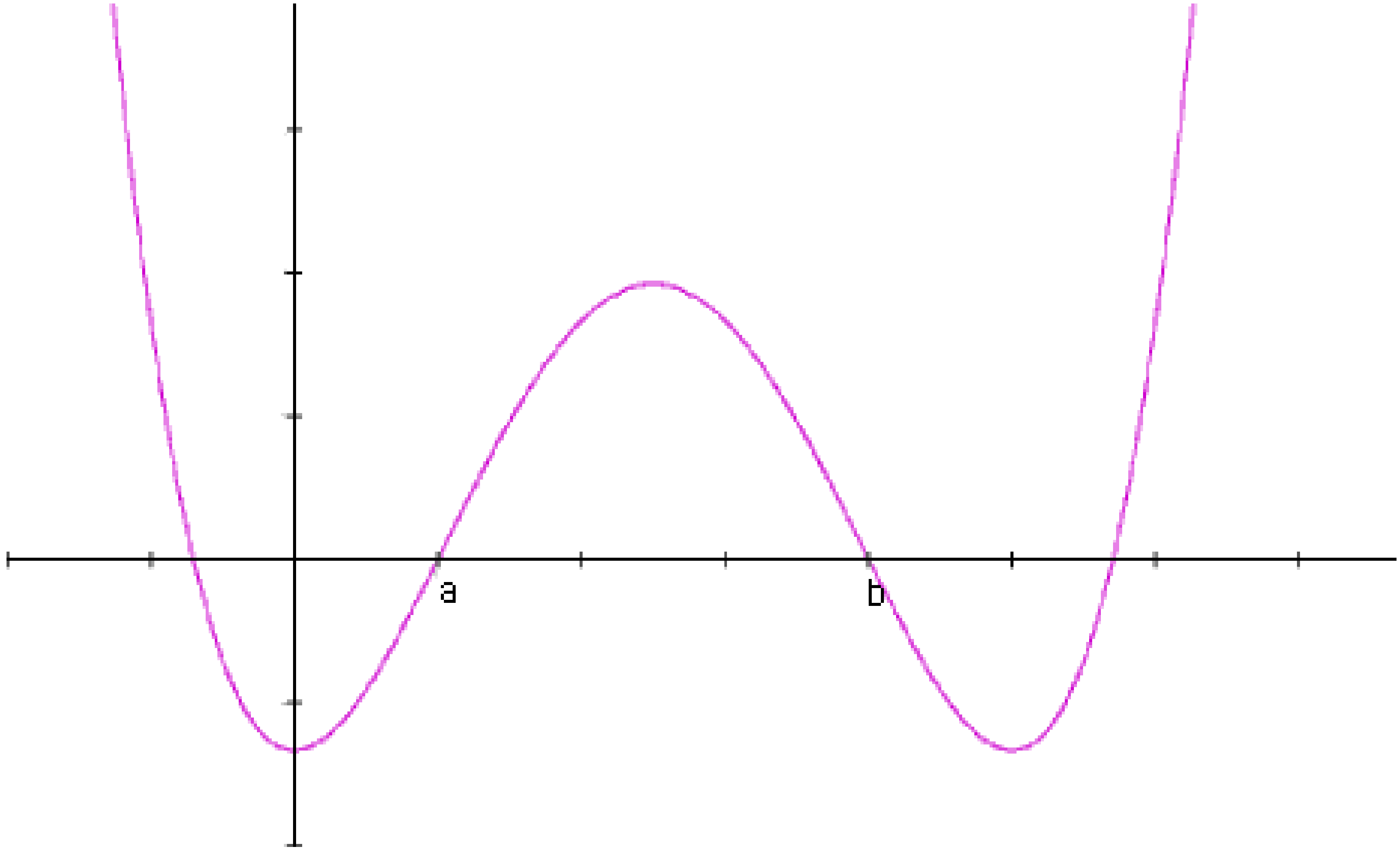

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