

*Mass, Scalar Curvature,
Kähler Geometry, and All That*

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Dipartimento di Matematica e Fisica
22 maggio, 2019

Core results joint with

Core results joint with

Hans-Joachim Hein
Fordham University

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Mass in Kähler Geometry
Comm. Math. Phys. 347 (2016) 621–653.

Recent technical improvements:

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Mass, Kähler Manifolds,
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arXiv: 1810.11417 [math.DG]

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Ann. Glob. An. Geom. to appear

doi: 10.1007/s10455-019-09658-9

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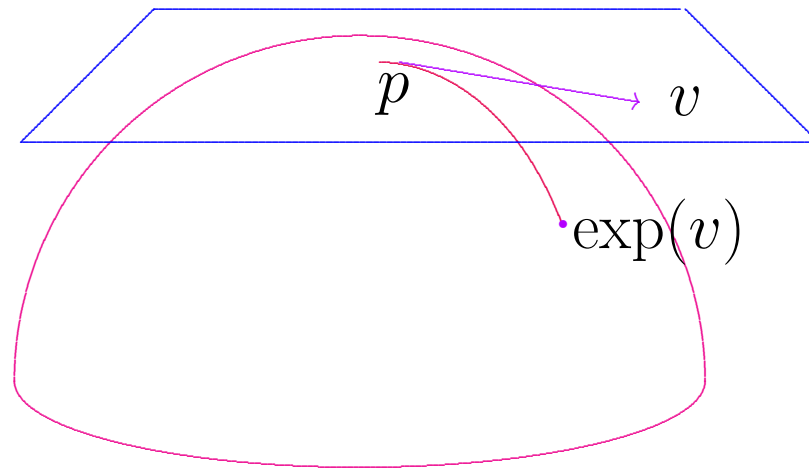
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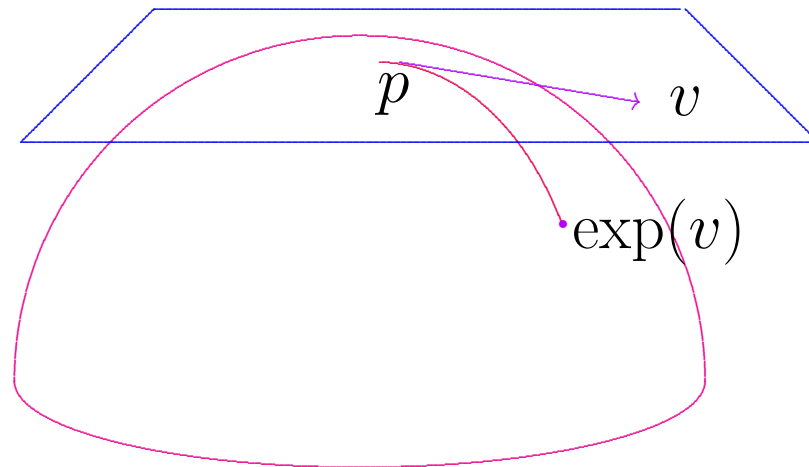
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Now choosing $T_p M \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal
basis gives us special coordinates on M .

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Components like \mathcal{R}_{1212} are “**sectional curvatures**” ...

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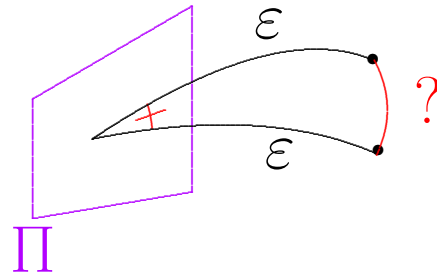
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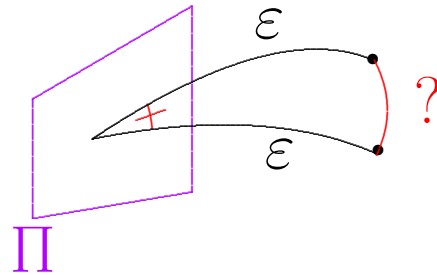


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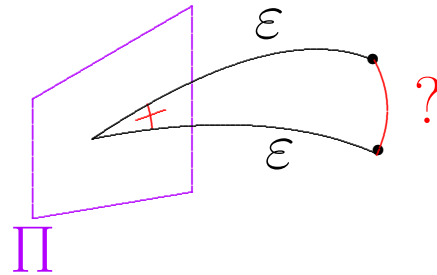
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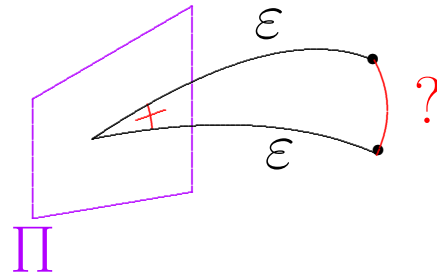
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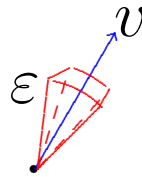
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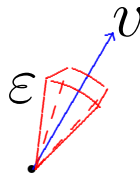


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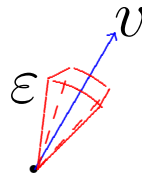


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The metric g is called *Ricci-flat* if it satisfies $r \equiv 0$.

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But this has a simple geometric interpretation...

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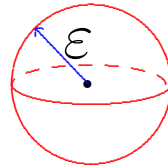
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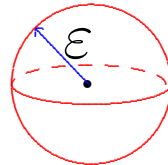


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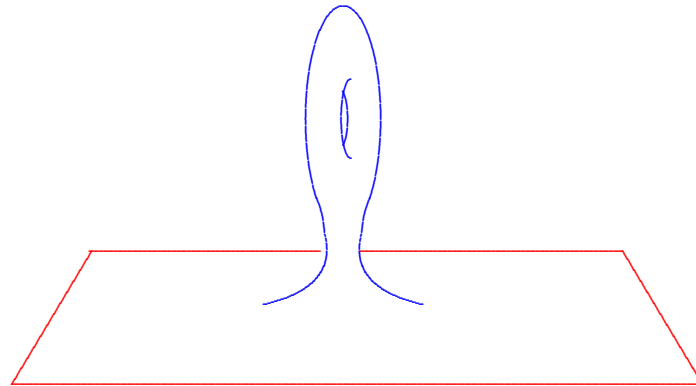


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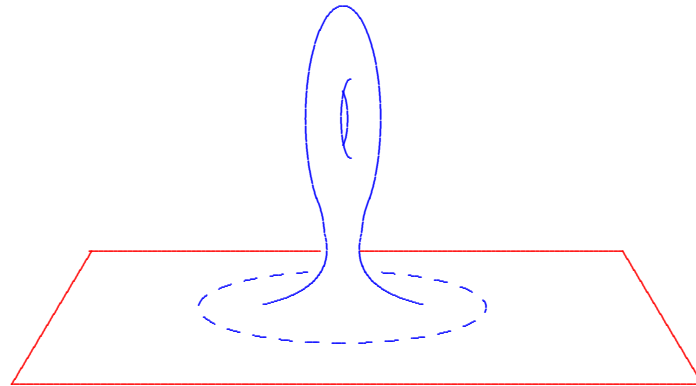
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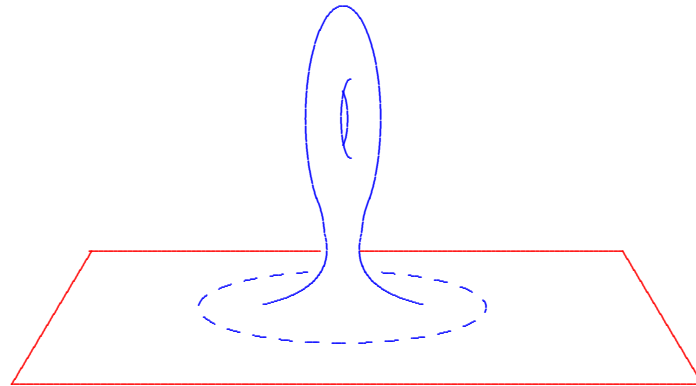
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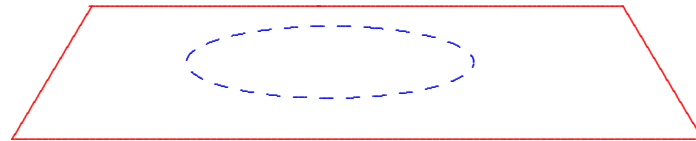


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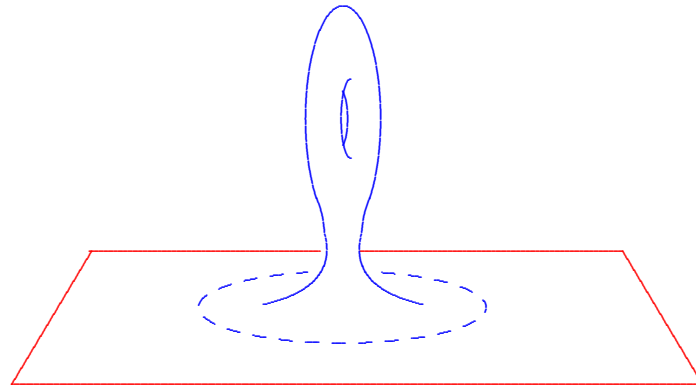


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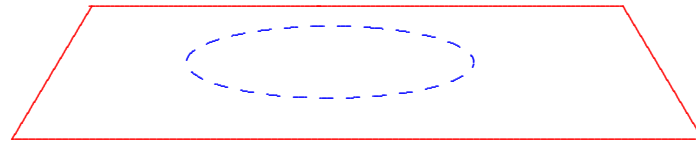


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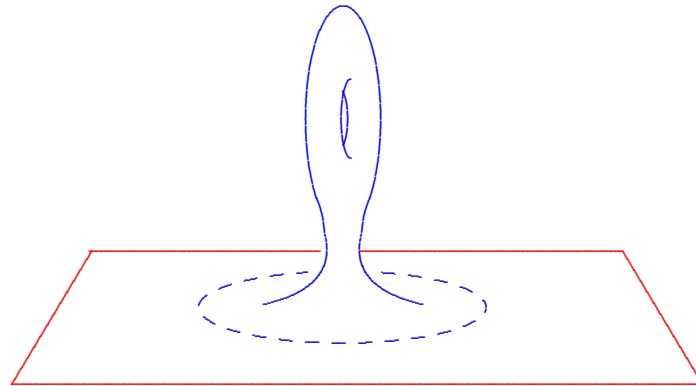


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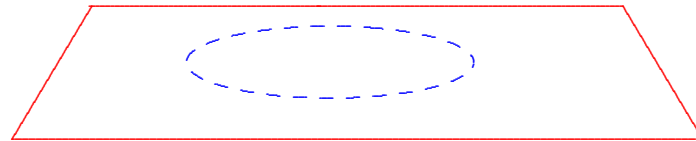


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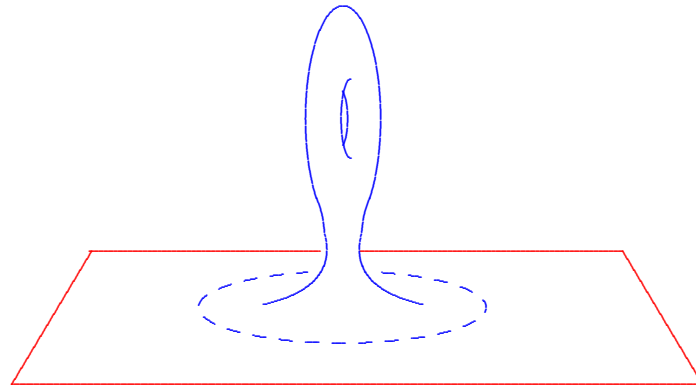


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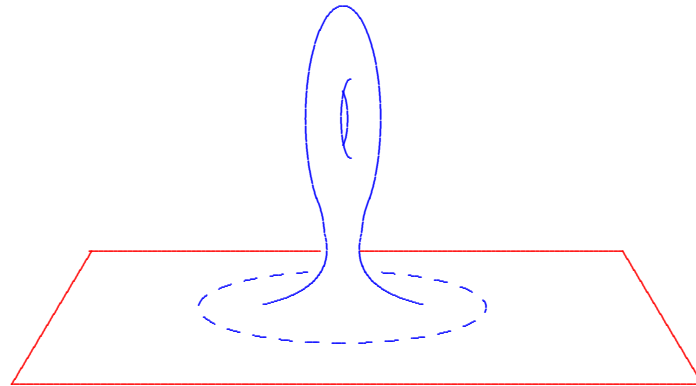


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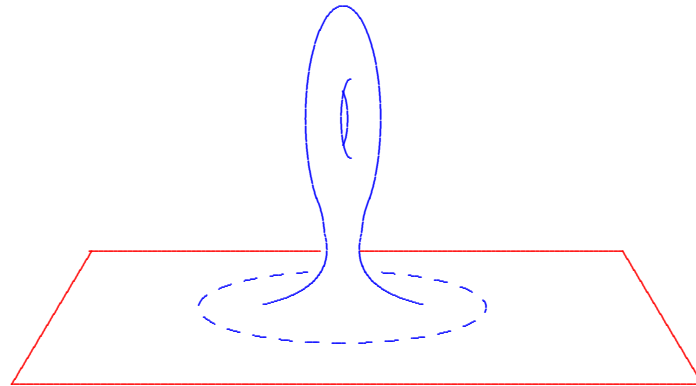
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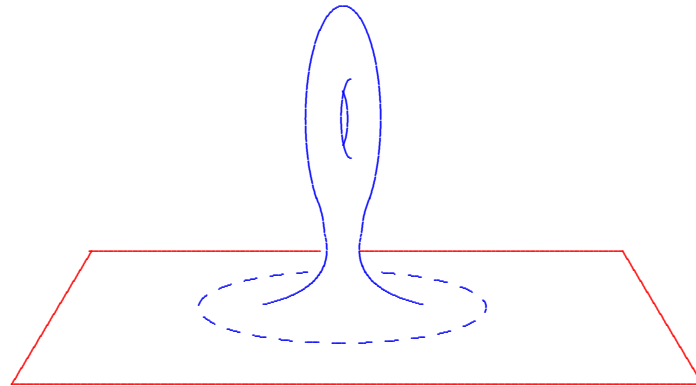
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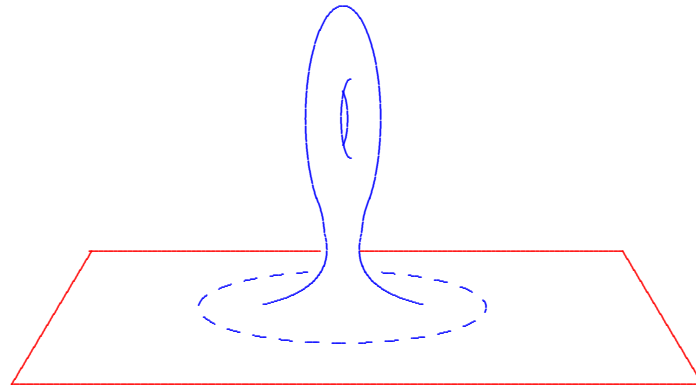
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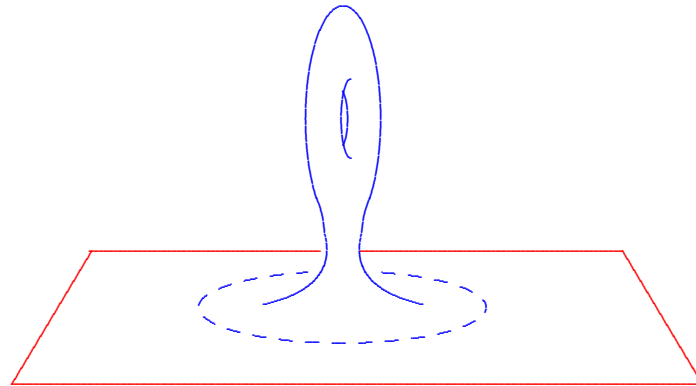
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Distance comparison...

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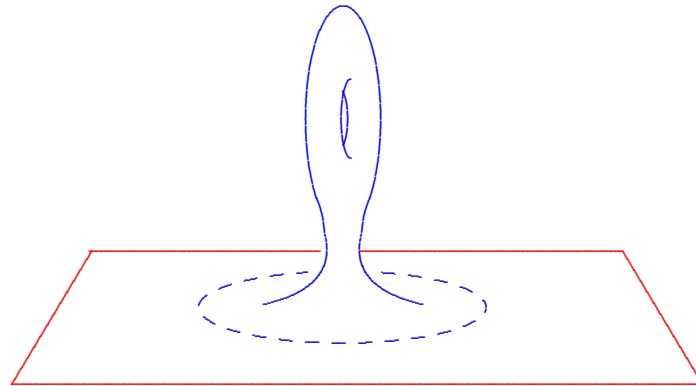
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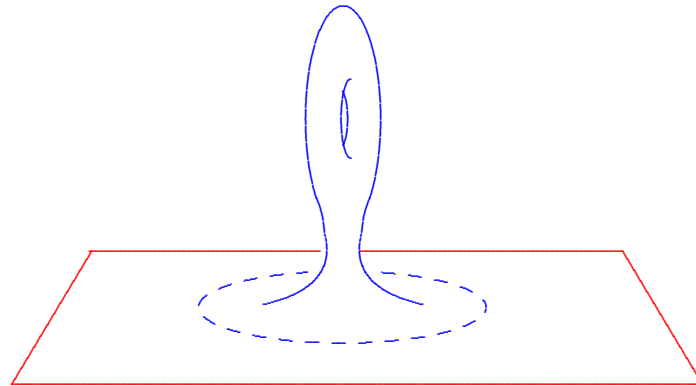
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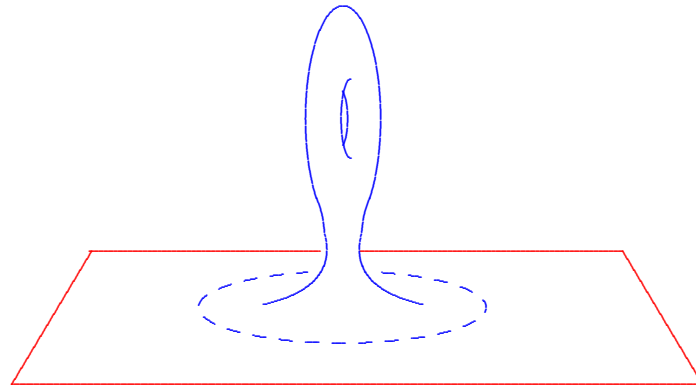
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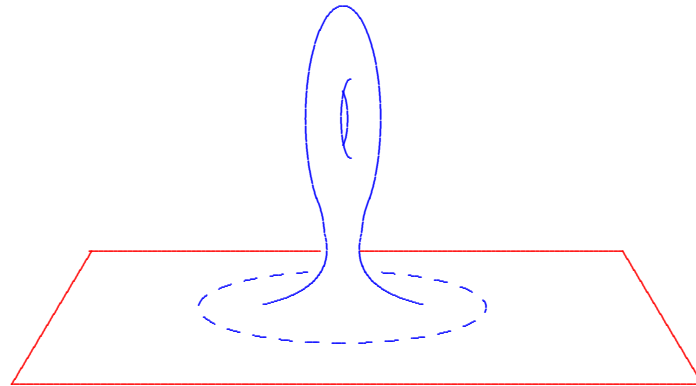
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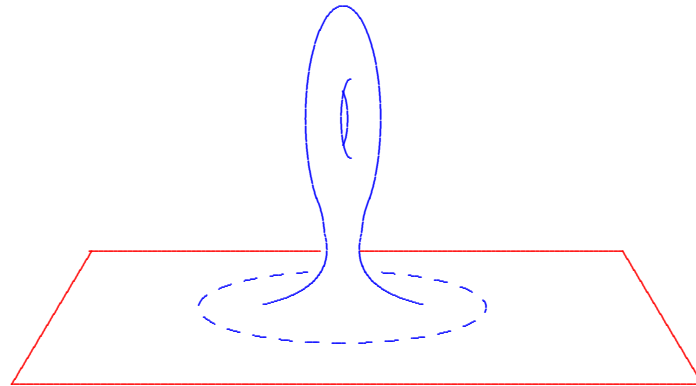
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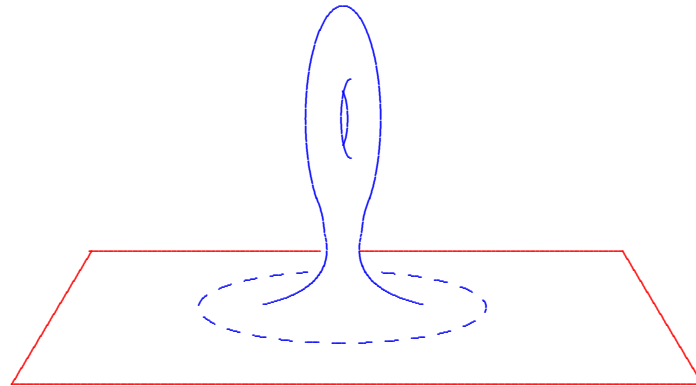
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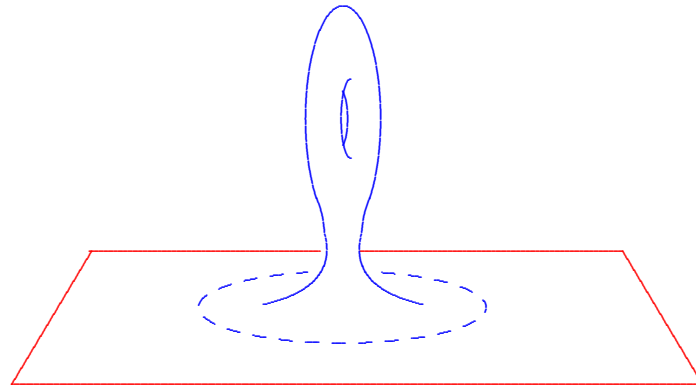
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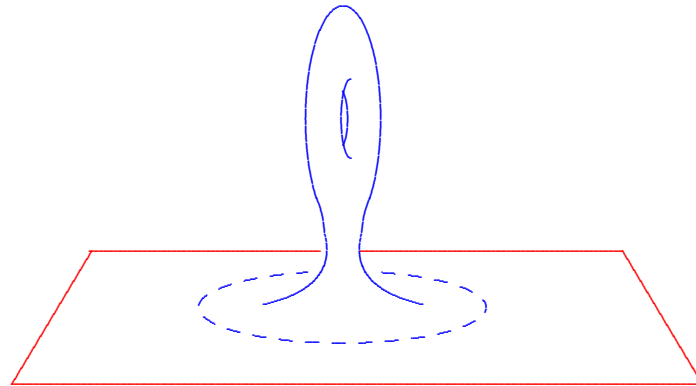
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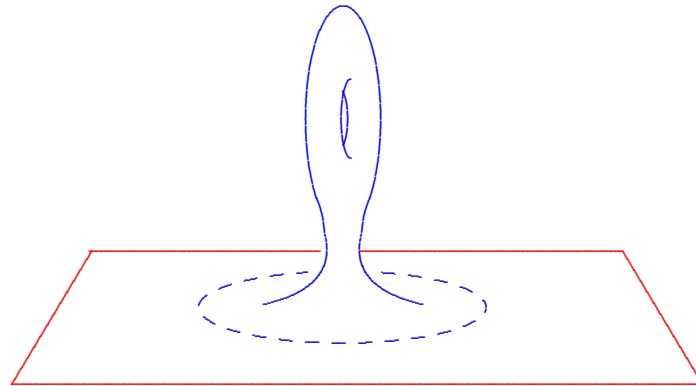
If M has scalar curvature ≥ 0 , is it flat? Yes!

This time, the inspiration comes from physics!

Simple, natural problem:

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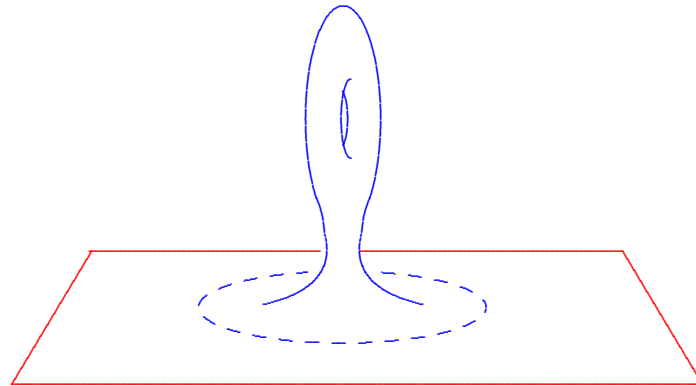
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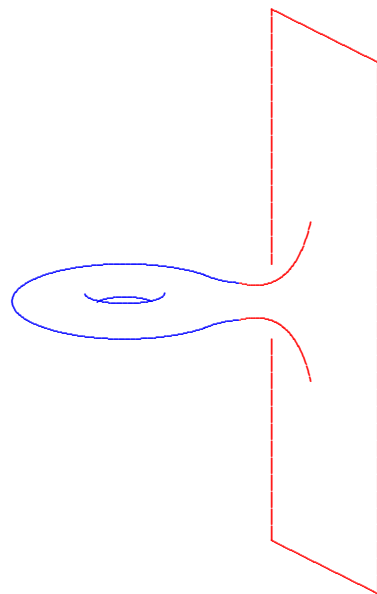
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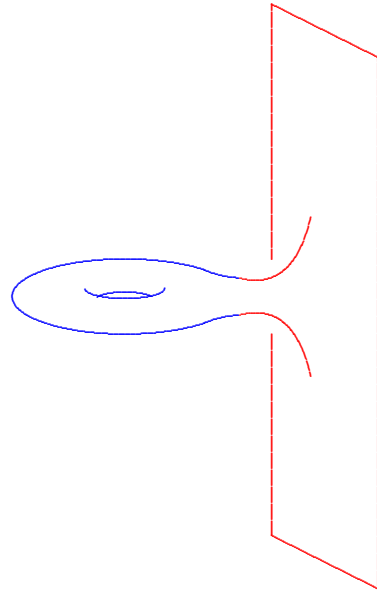
“Positive Mass Theorem”

Get result even with appropriate fall-off to Euclidean...

Definition. A complete, non-compact Riemannian n -manifold (M^n, g)

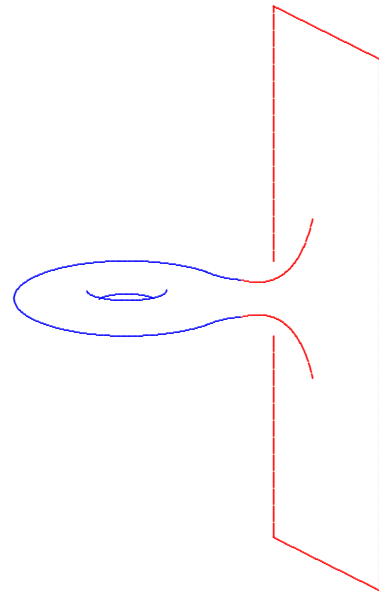


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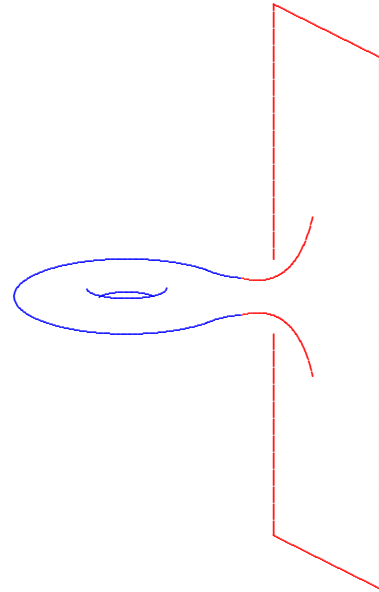
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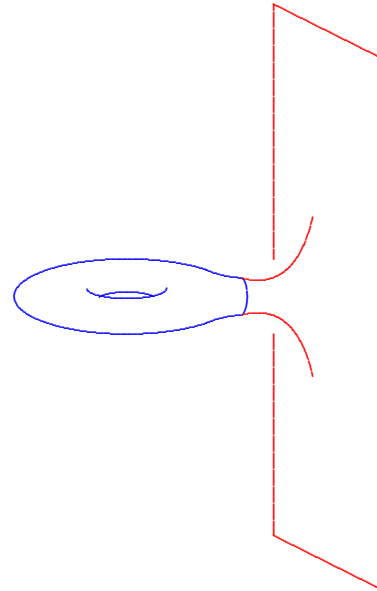
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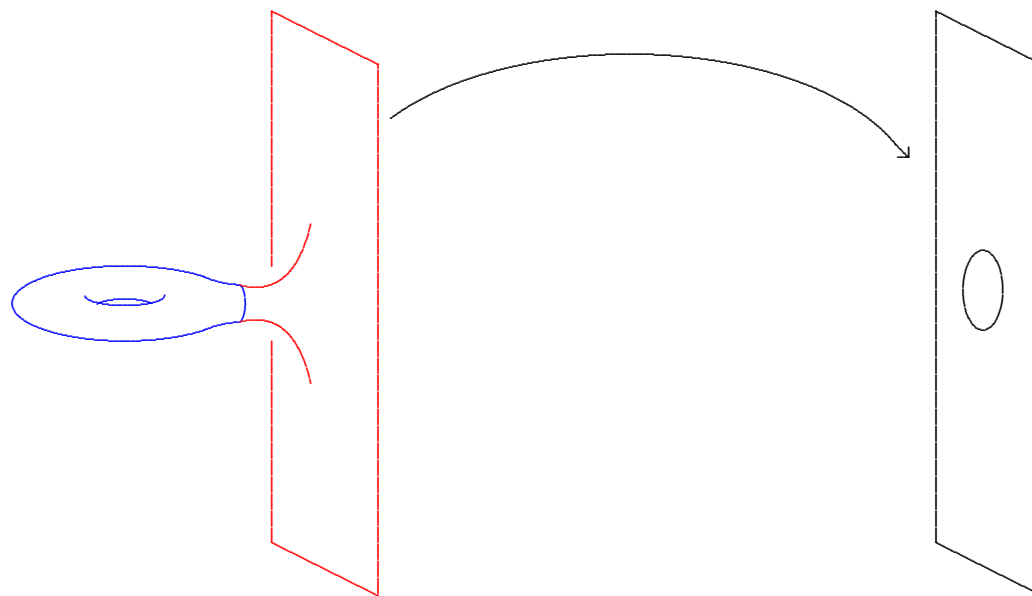


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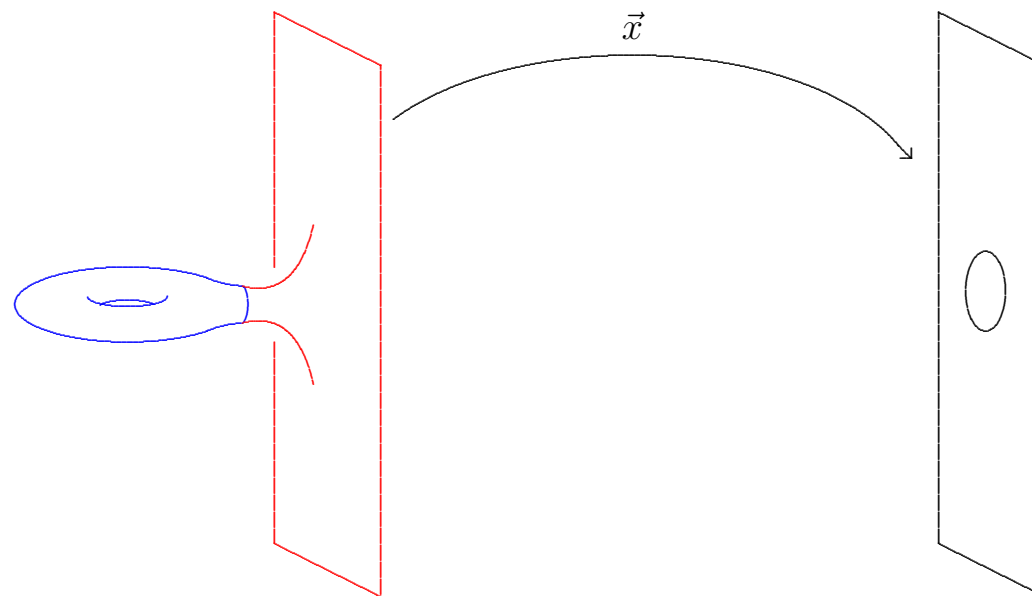
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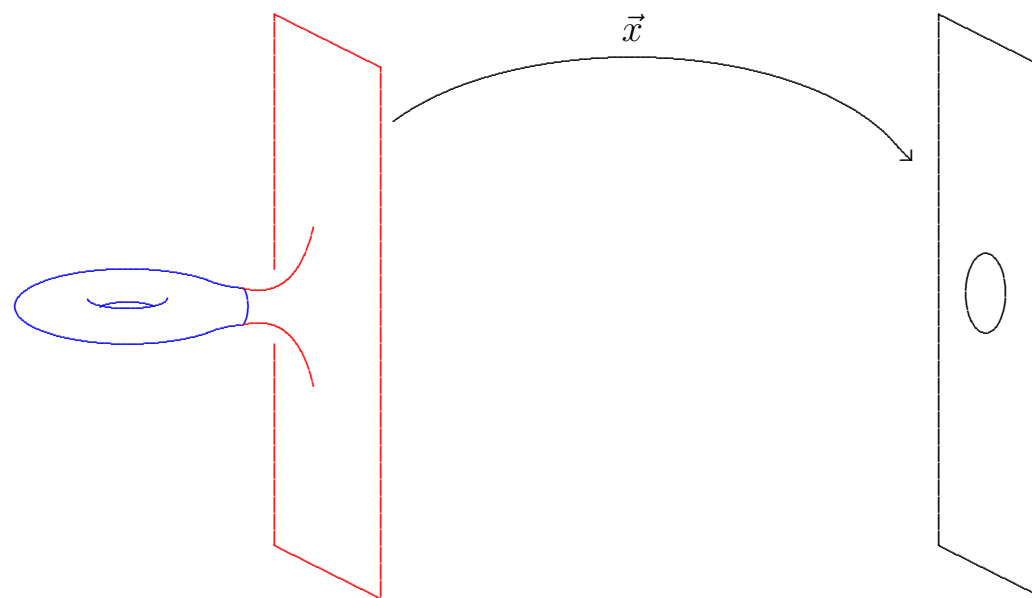


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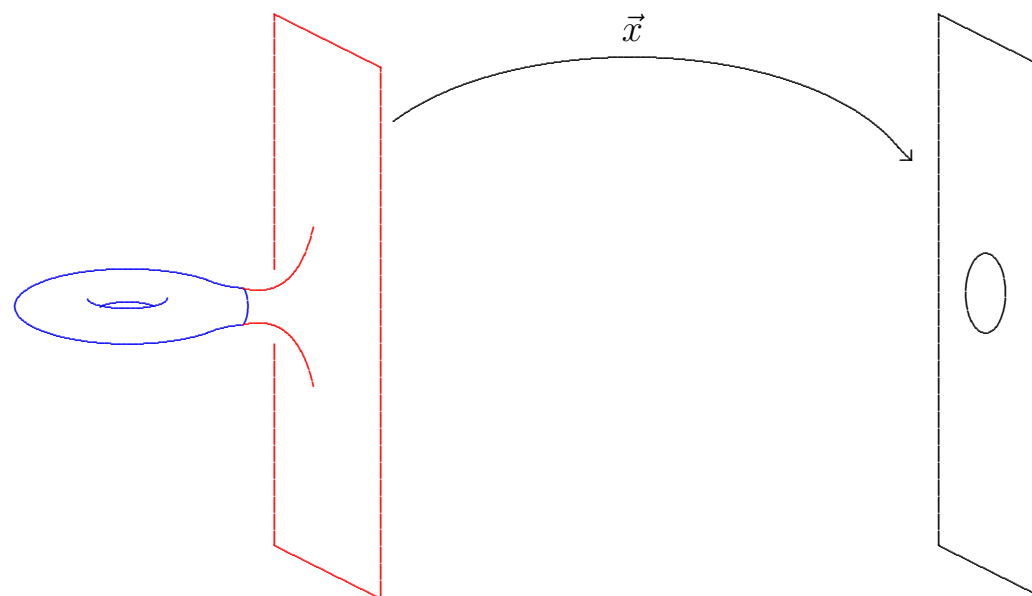
$$g_{jk} = \delta_{jk} + \text{terms that fall-off at infinity}$$

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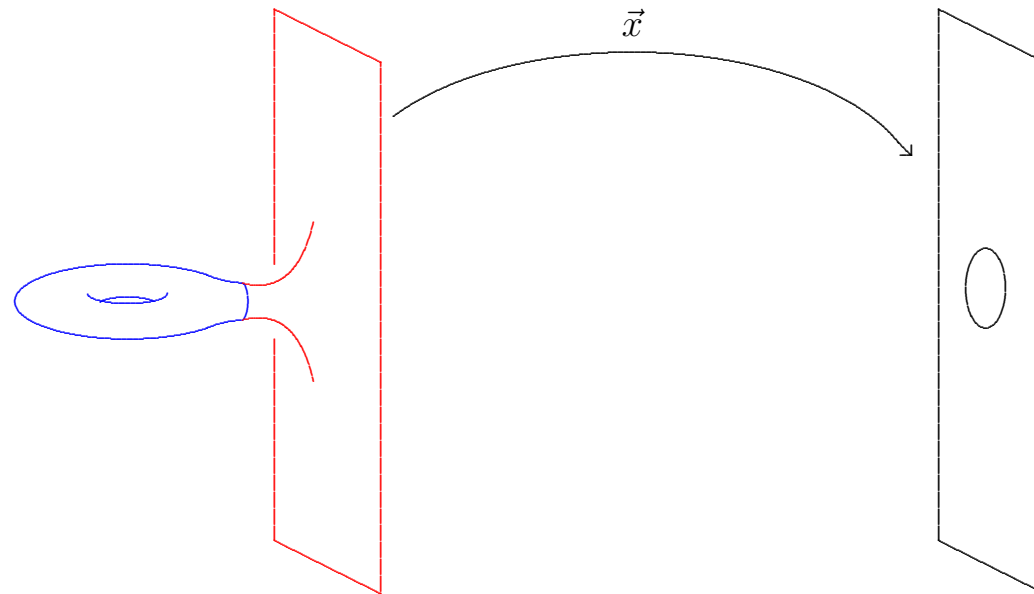
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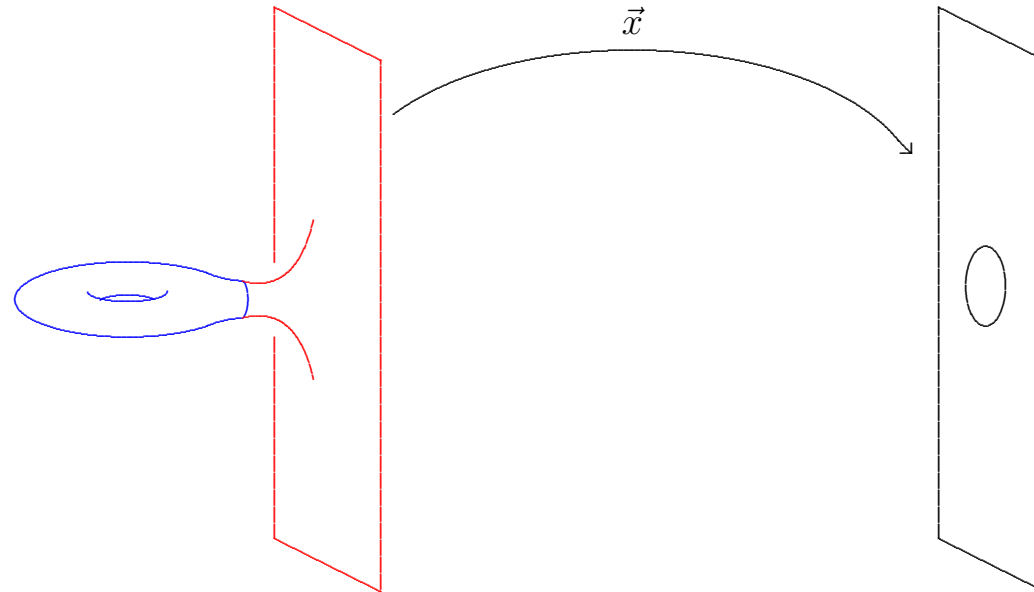
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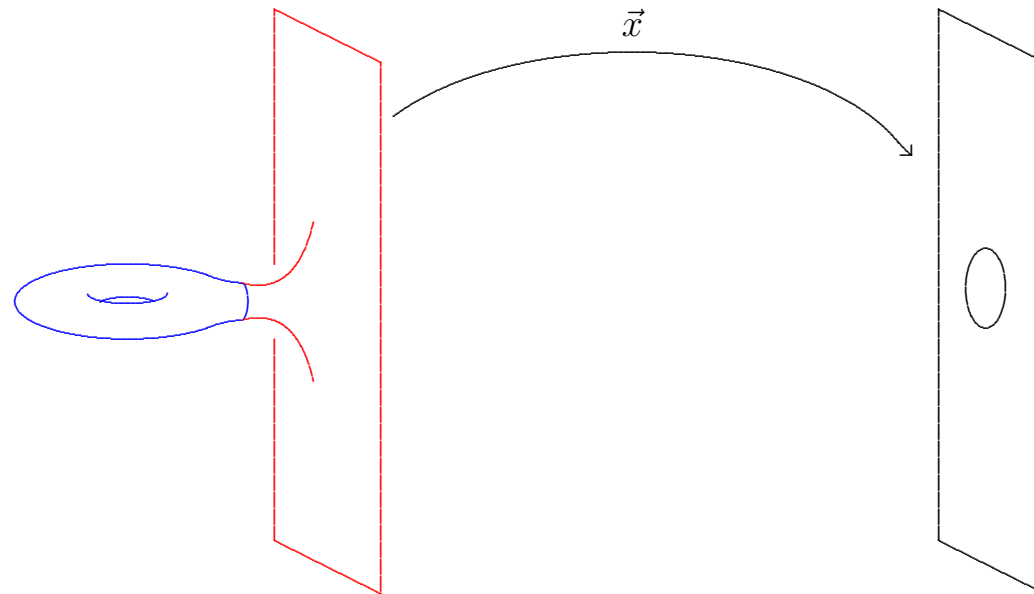
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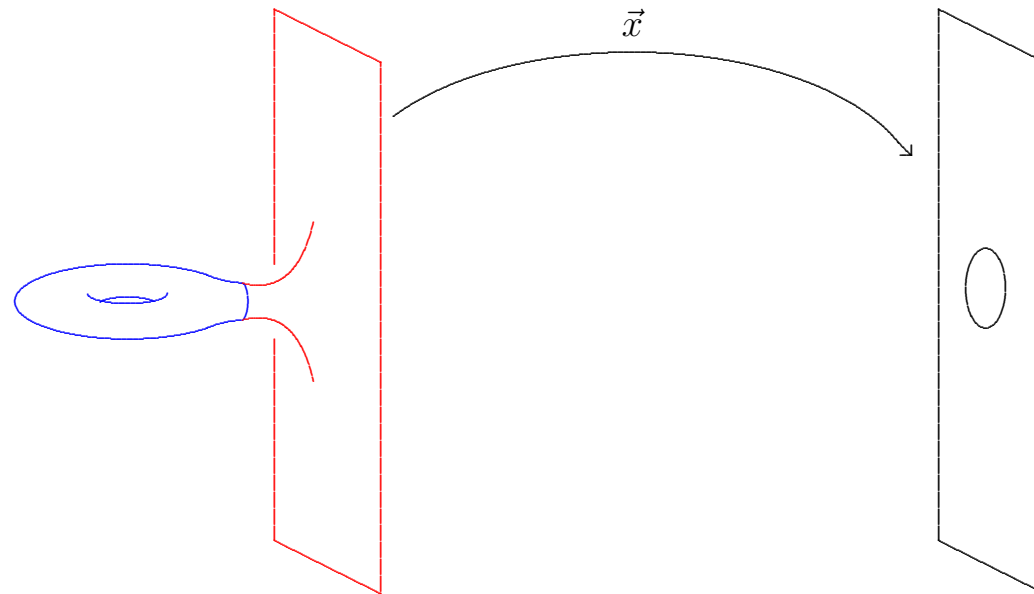
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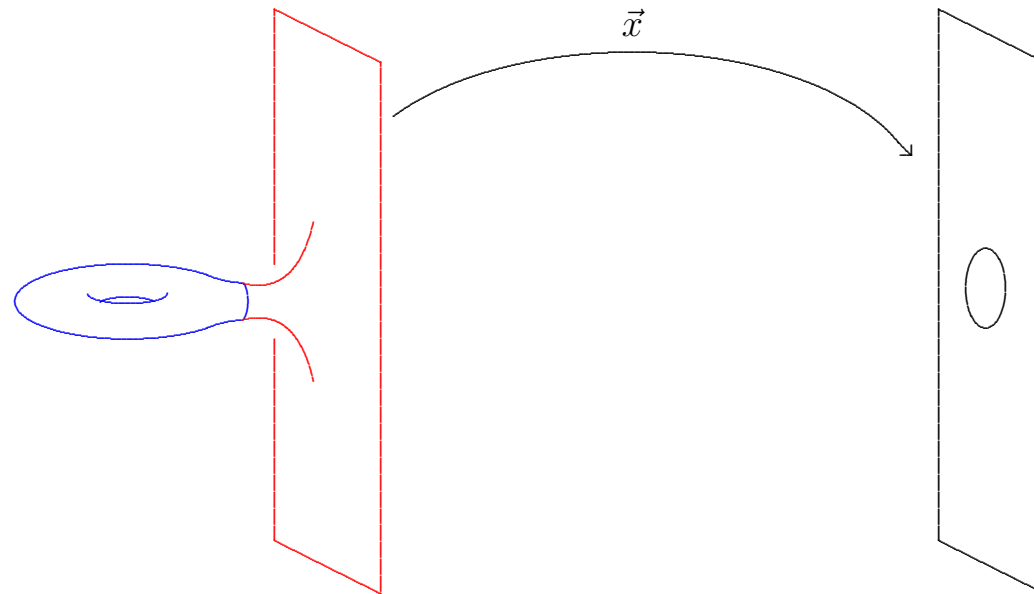
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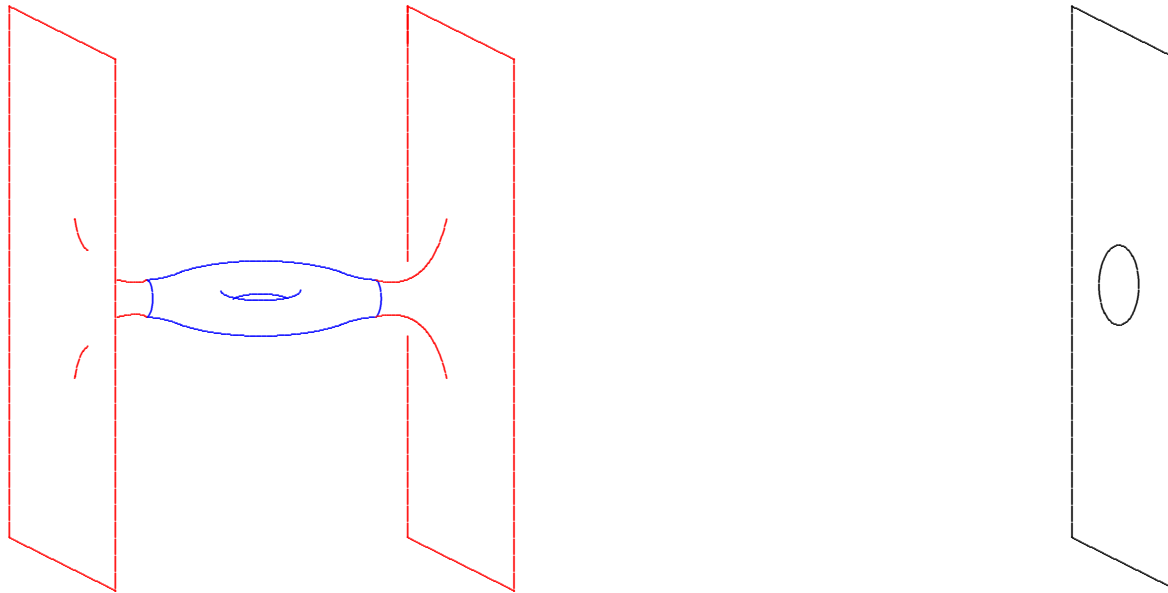
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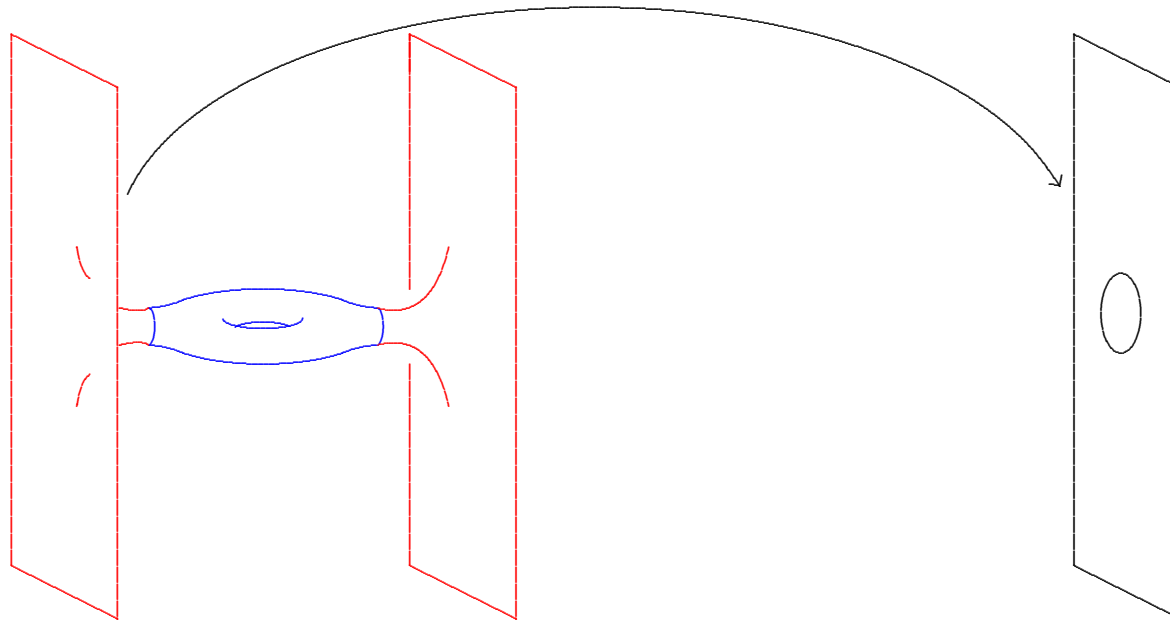
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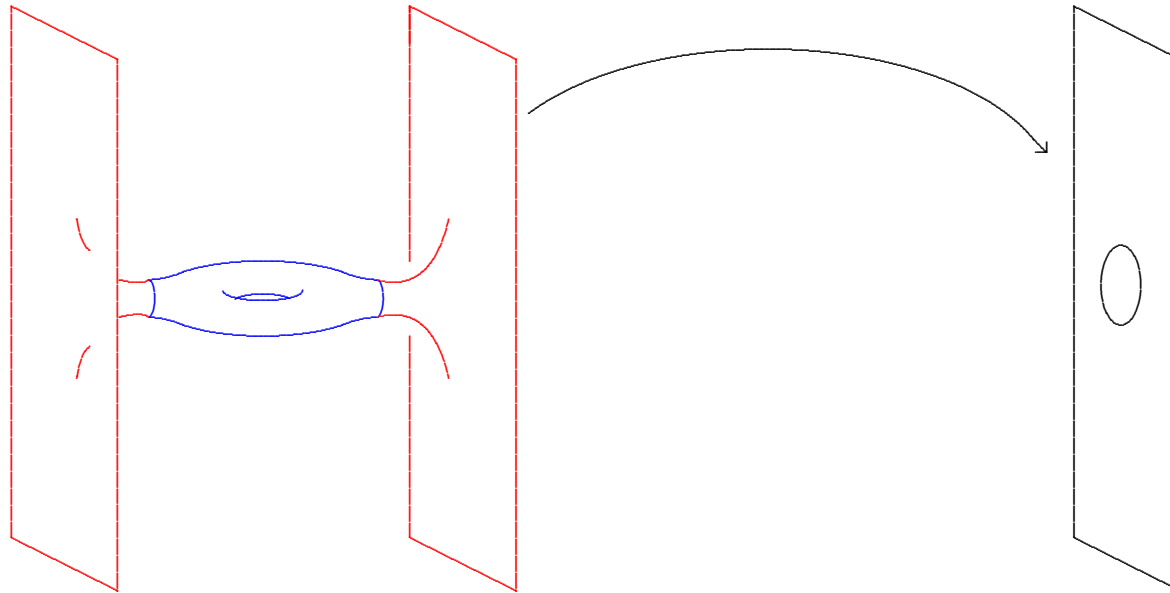
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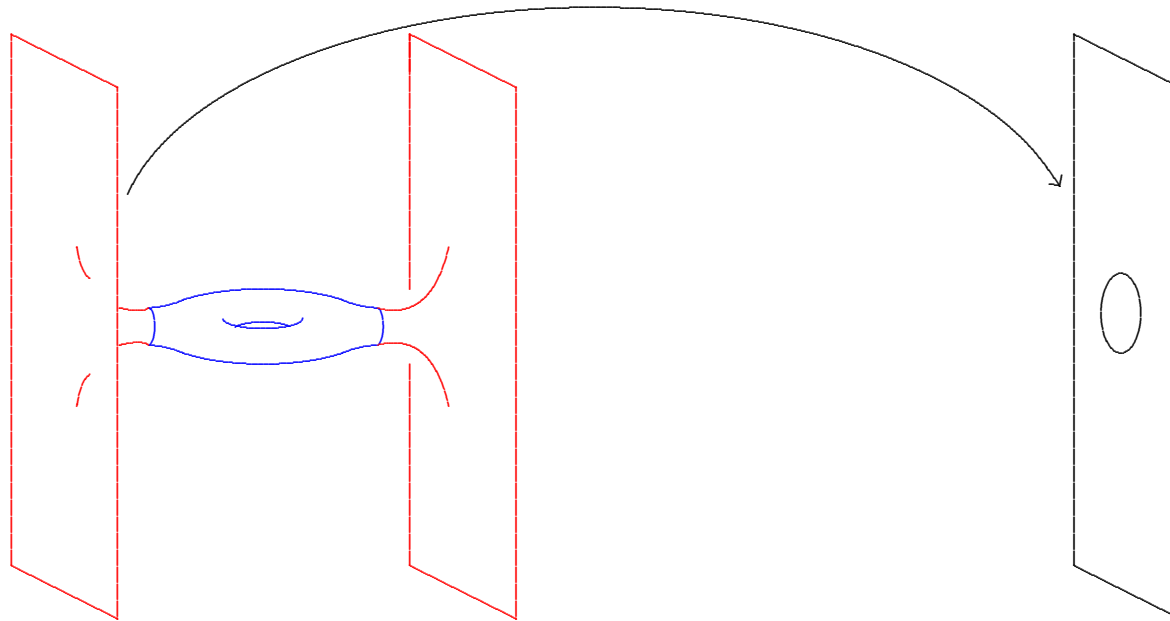
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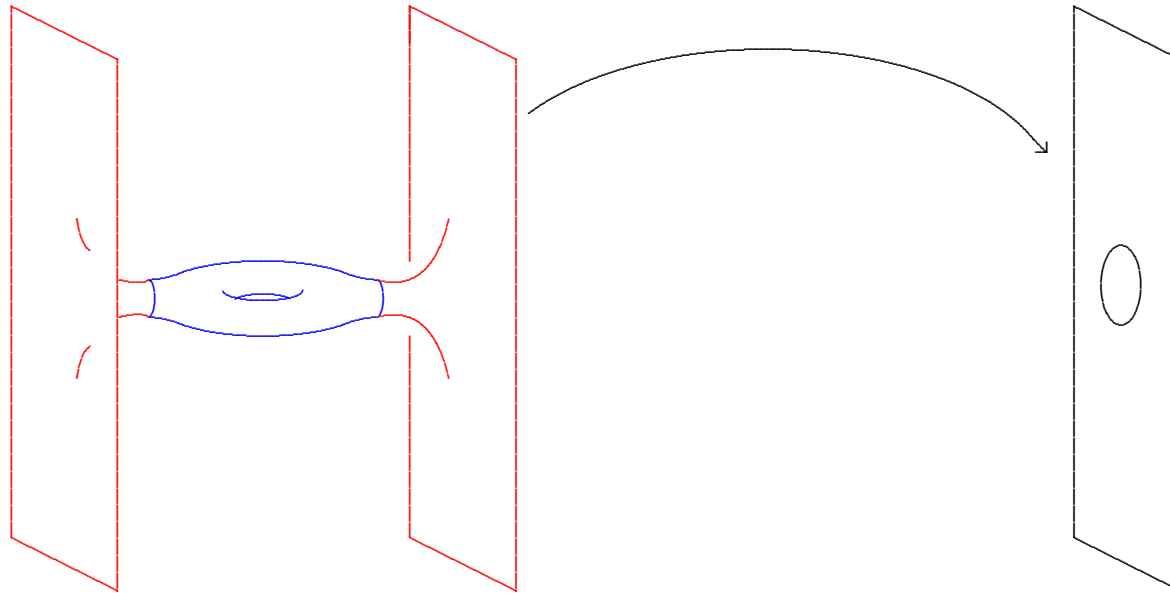
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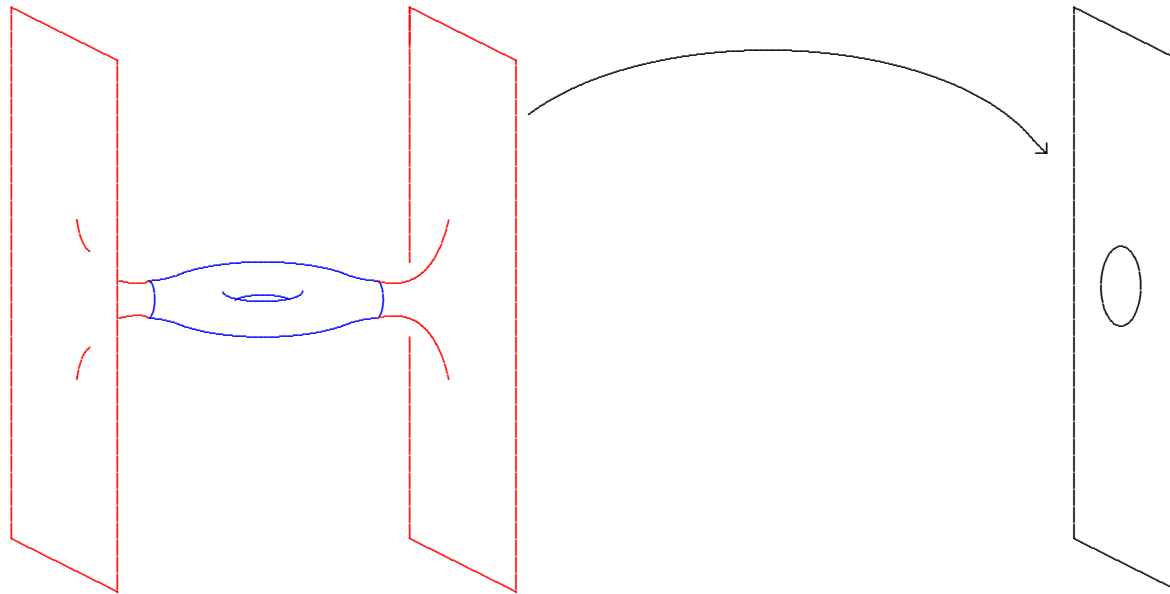
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Seems to depend on choice of coordinates!

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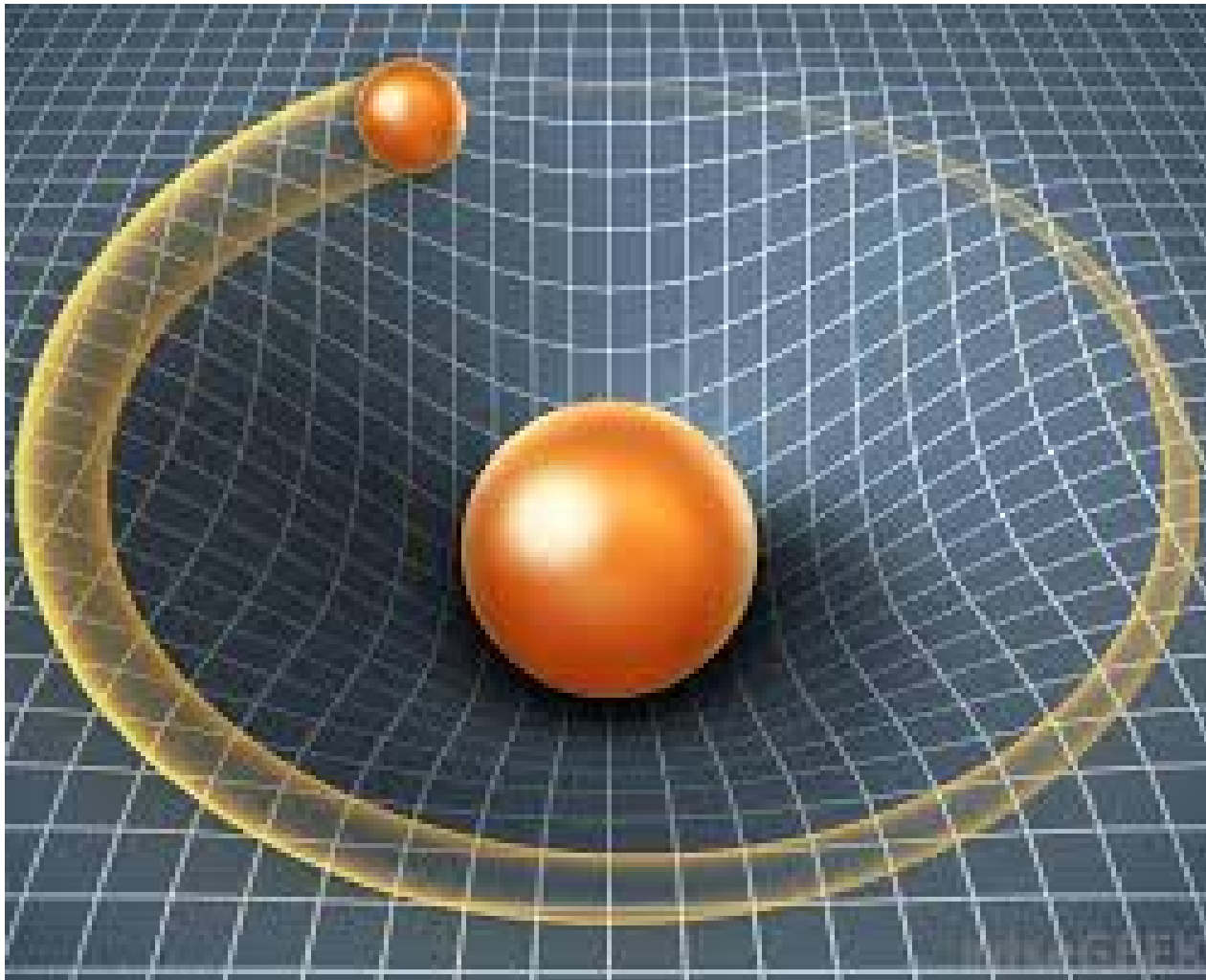
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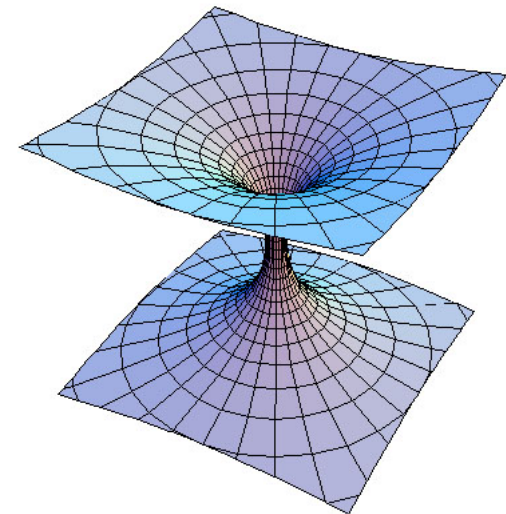
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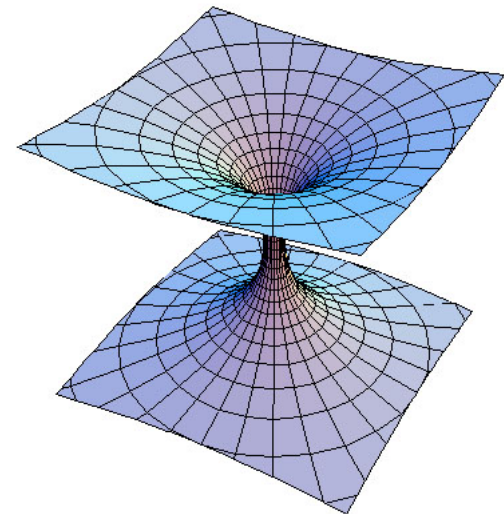
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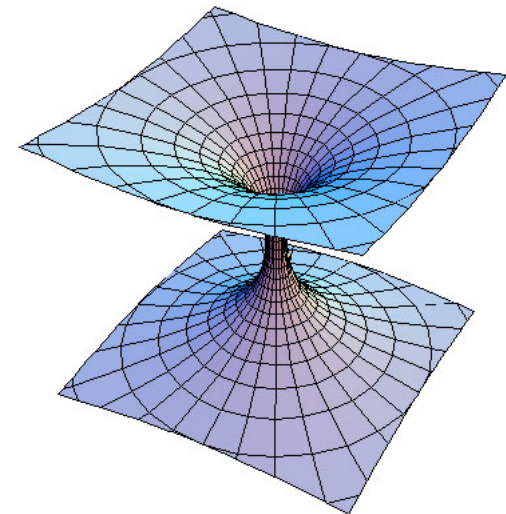
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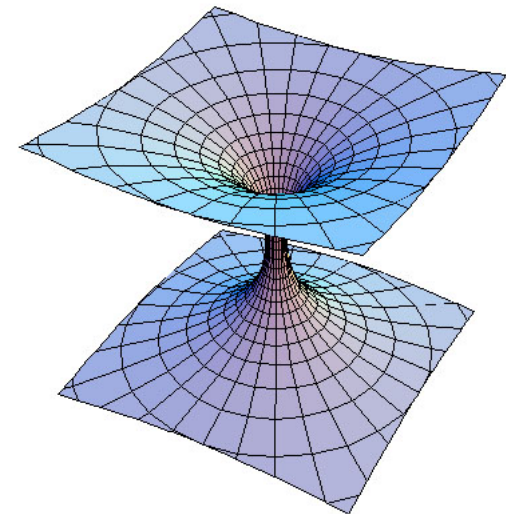
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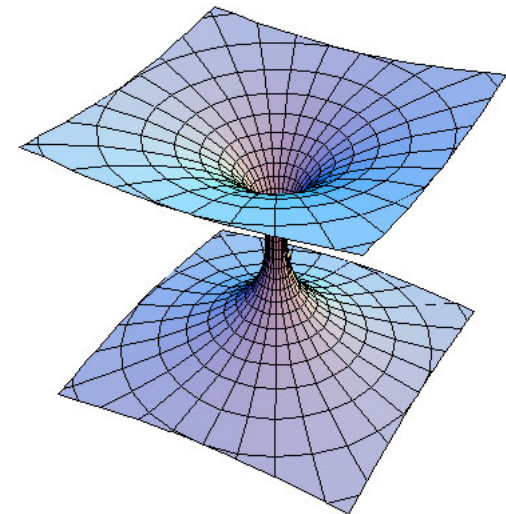
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Motivation:

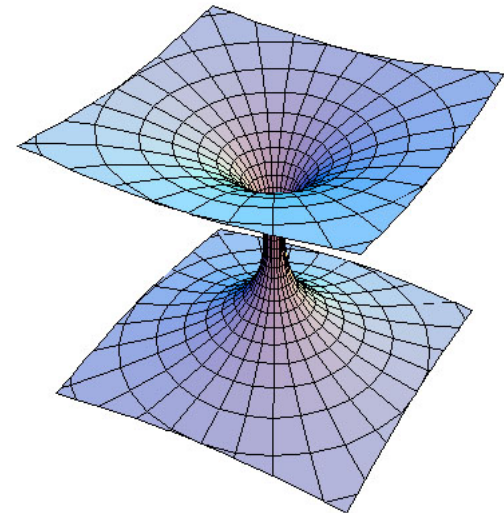
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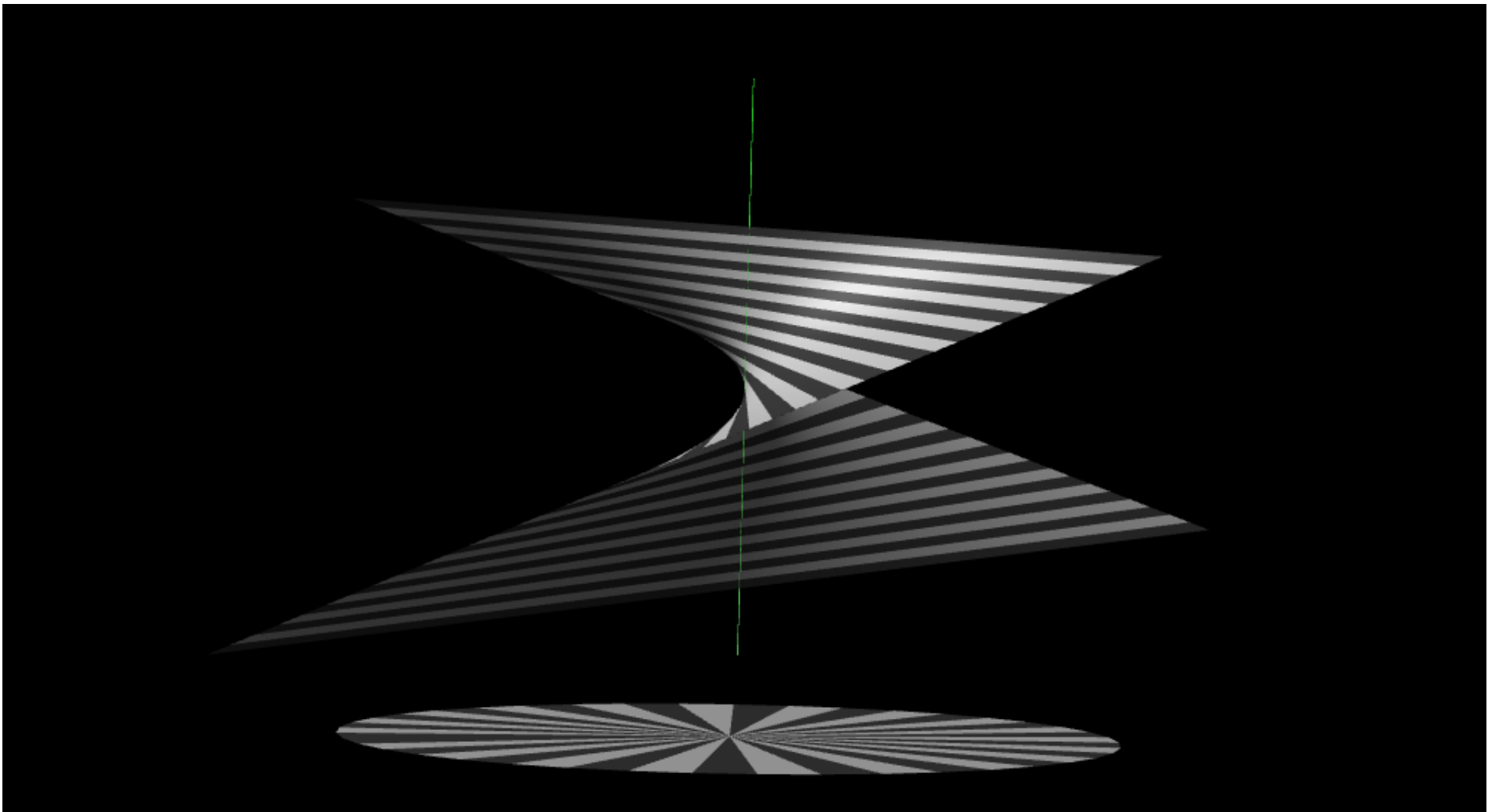
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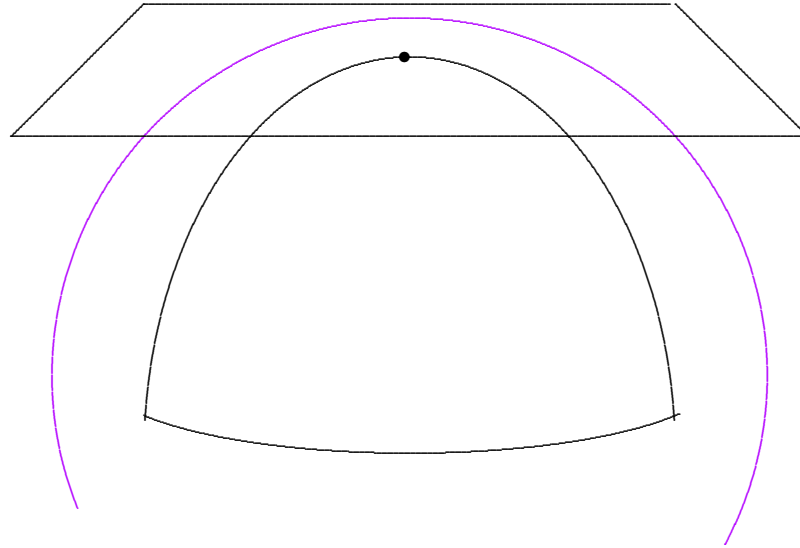
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holonomy

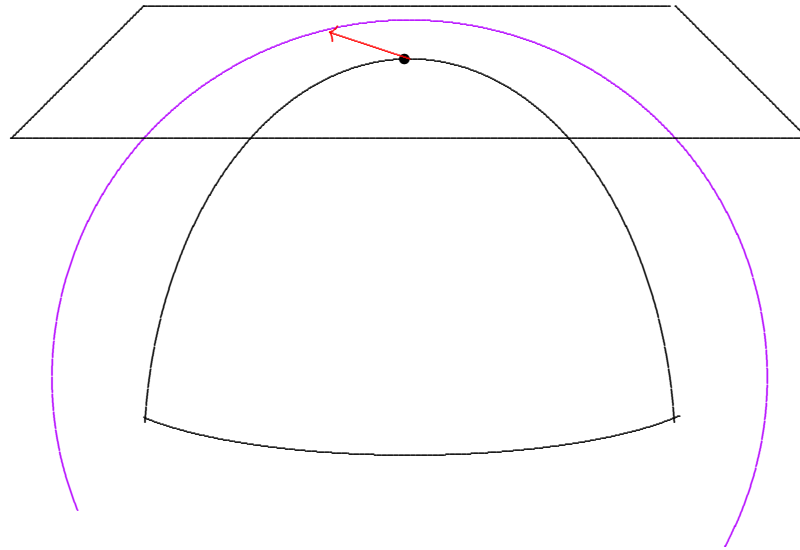
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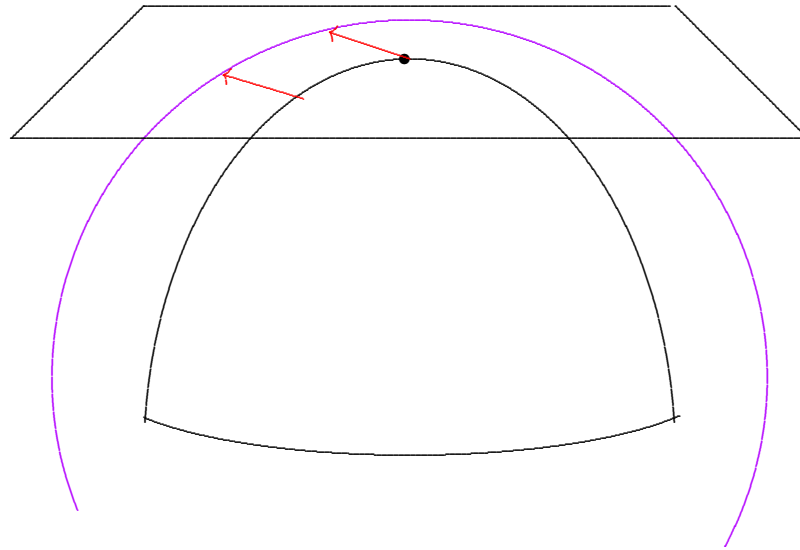
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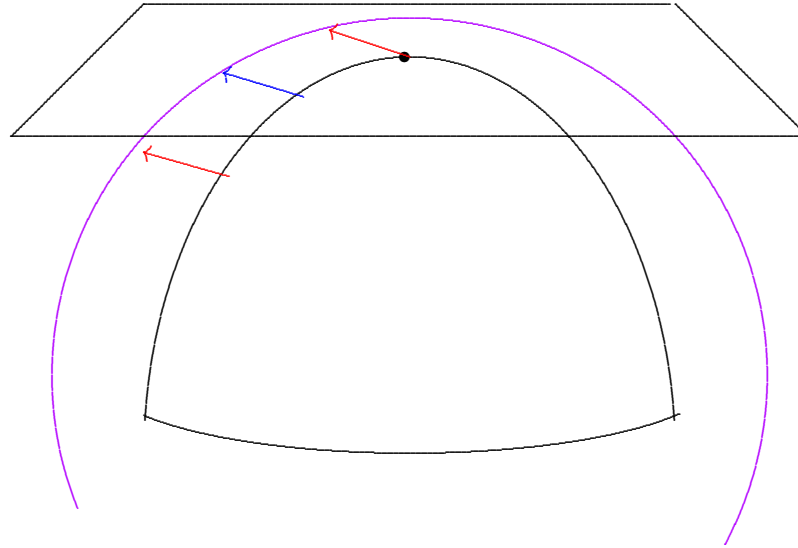
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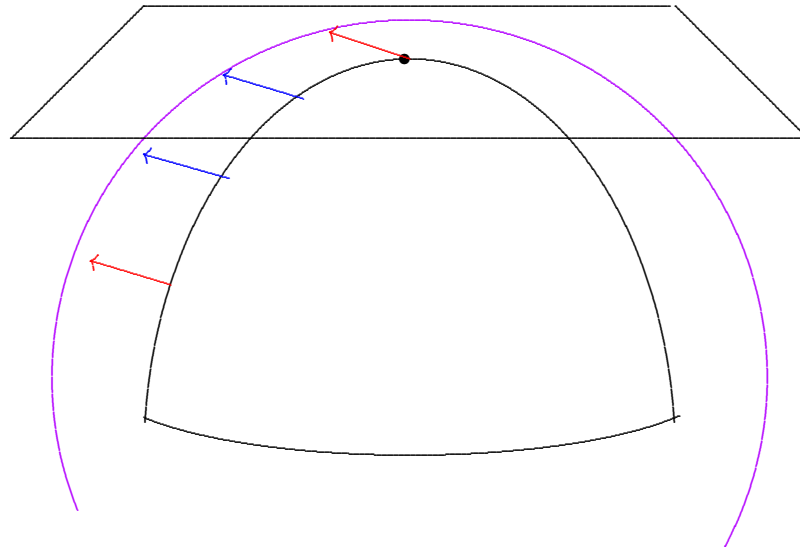
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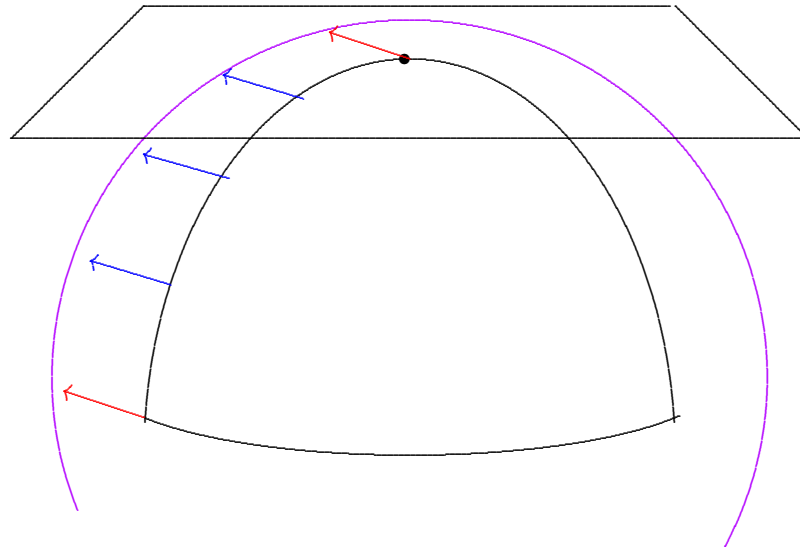
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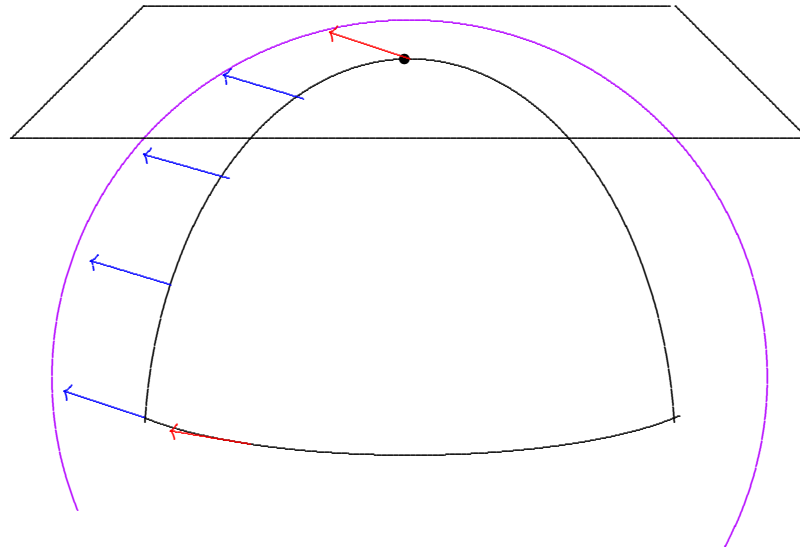
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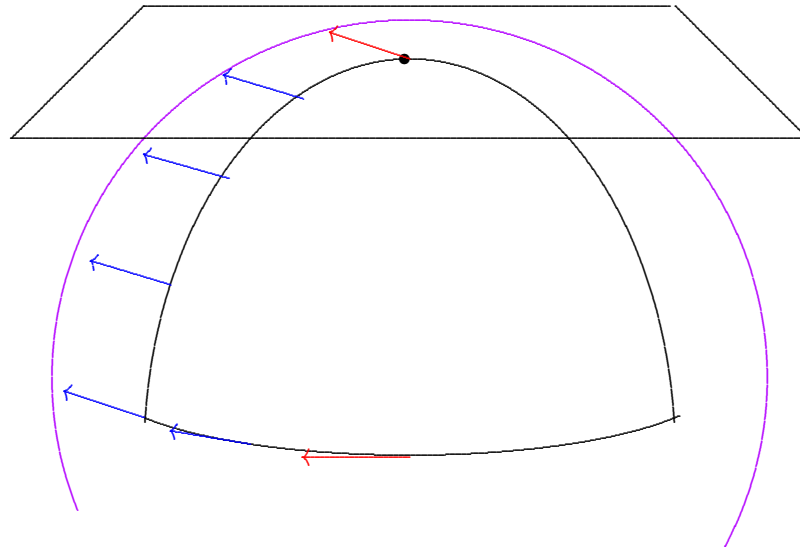
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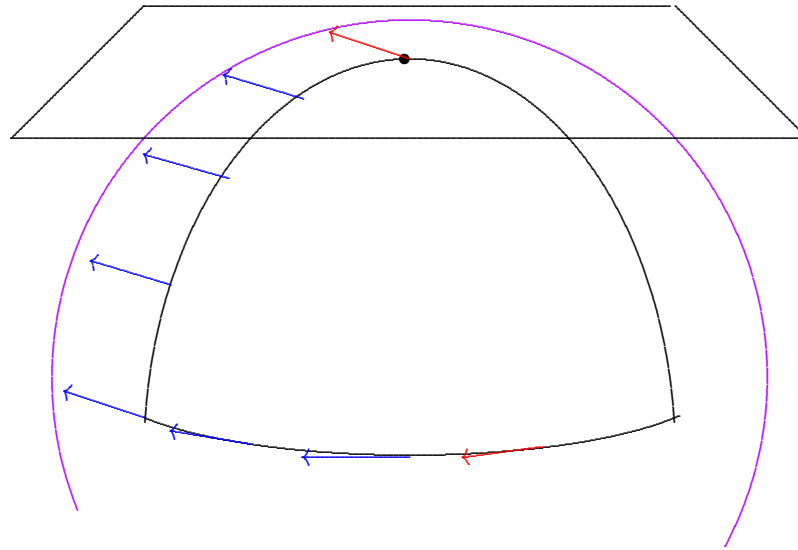
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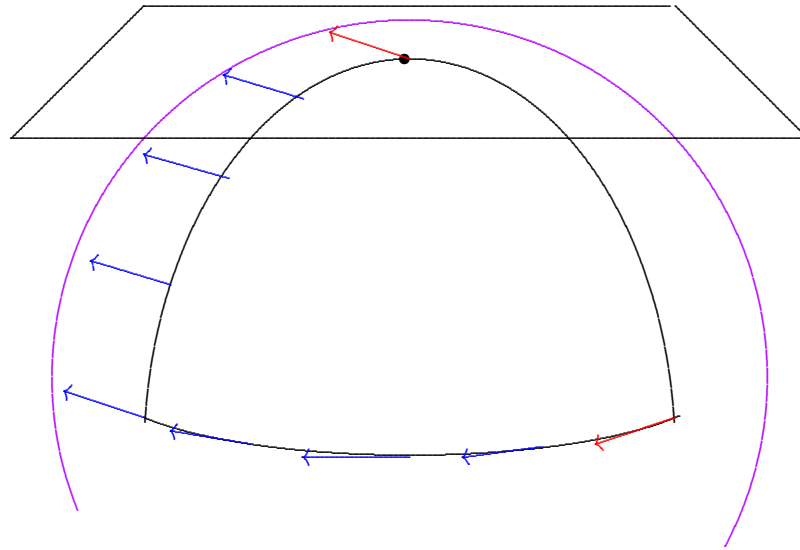
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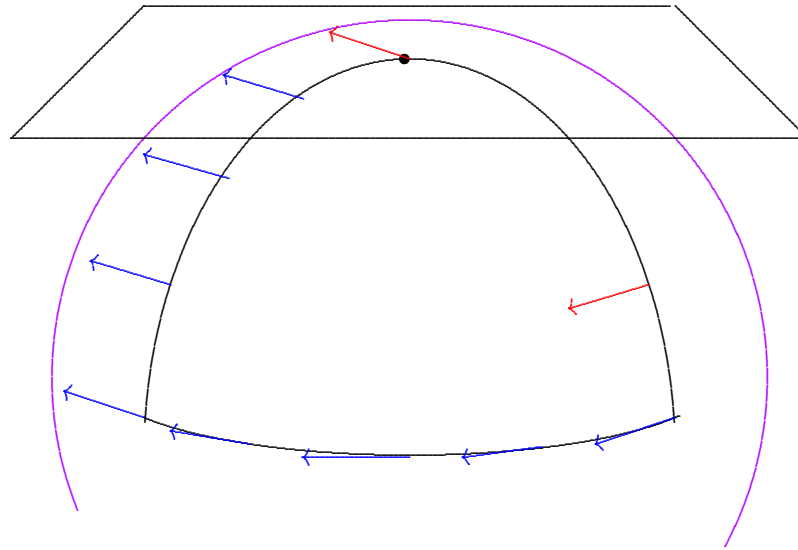
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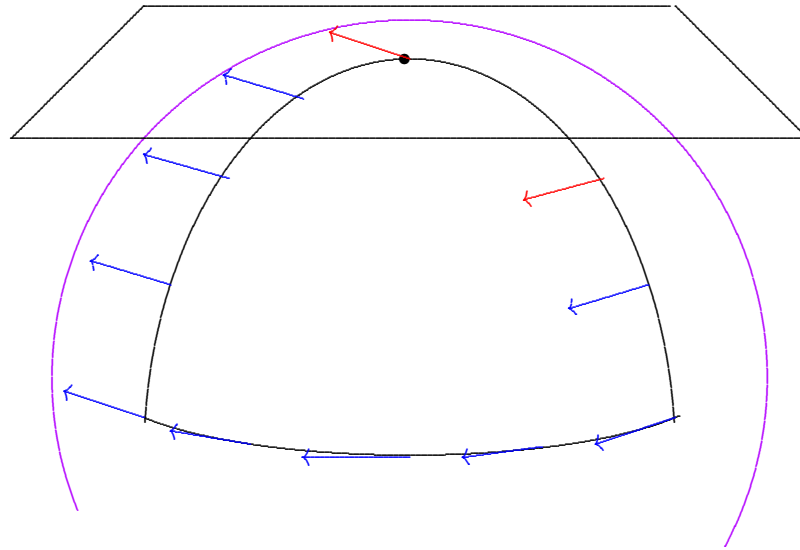
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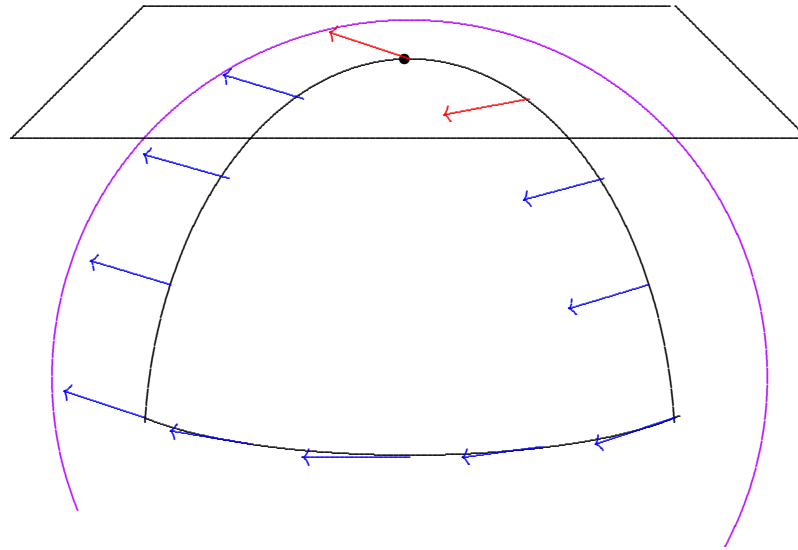
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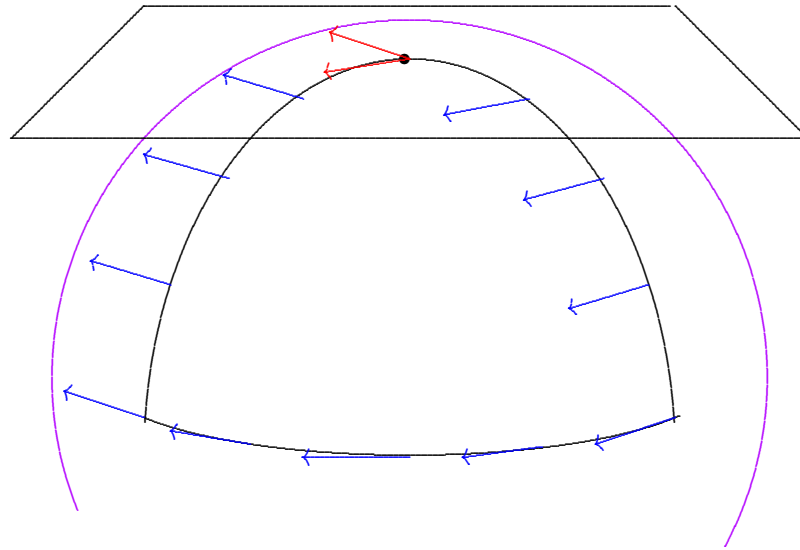
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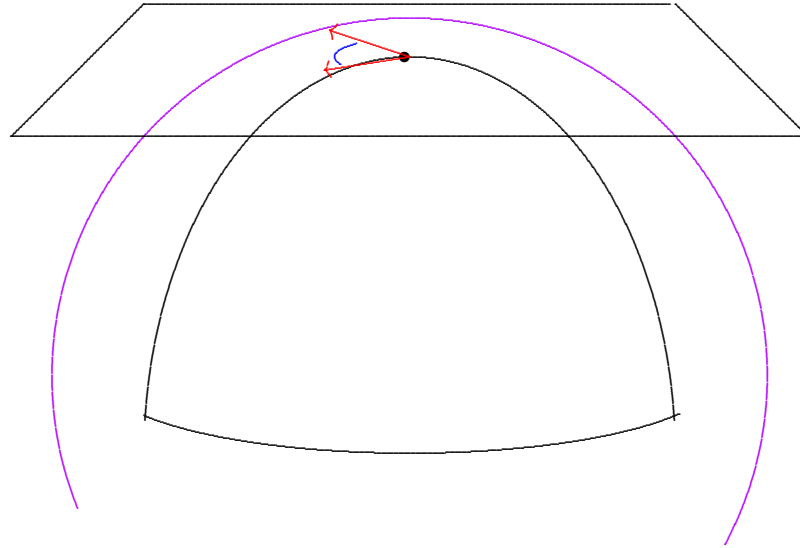
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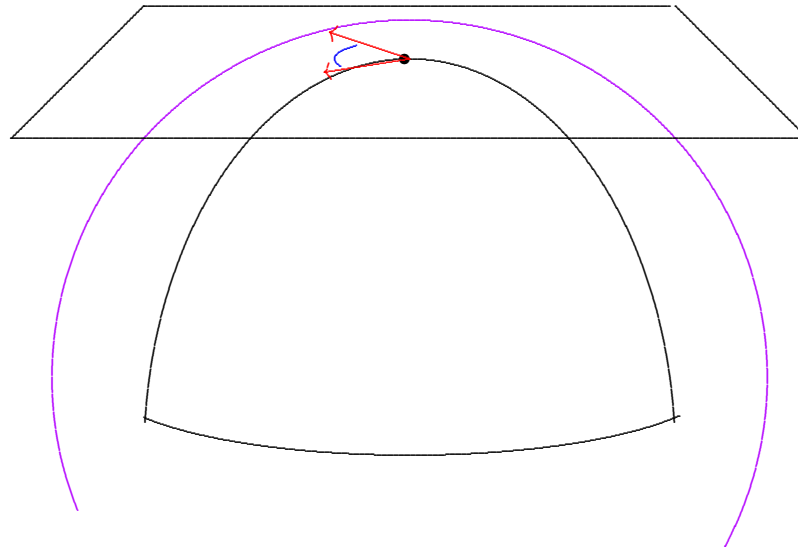
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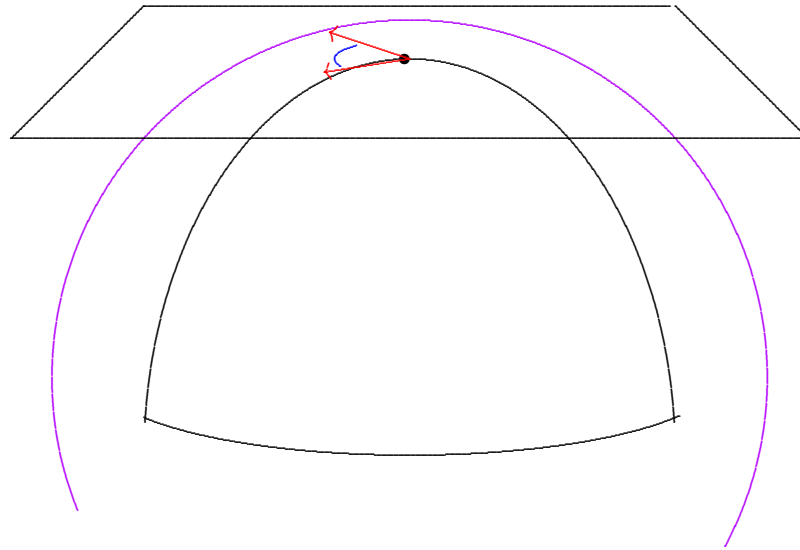
holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

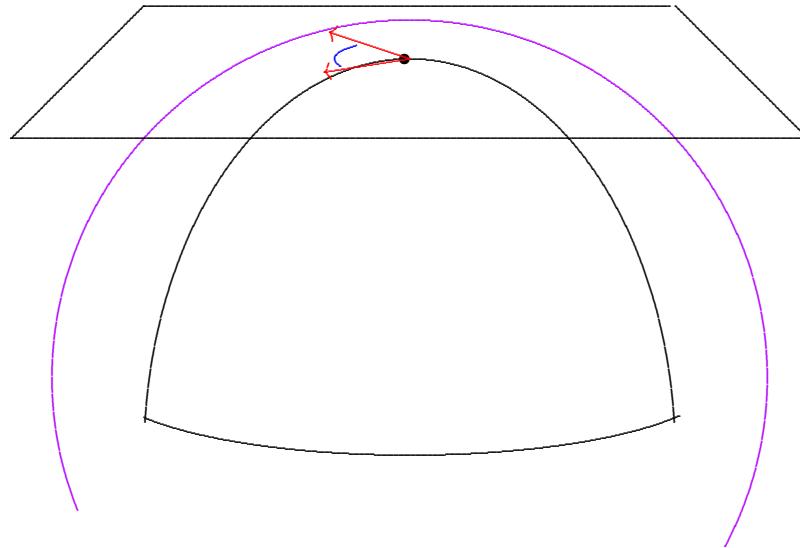
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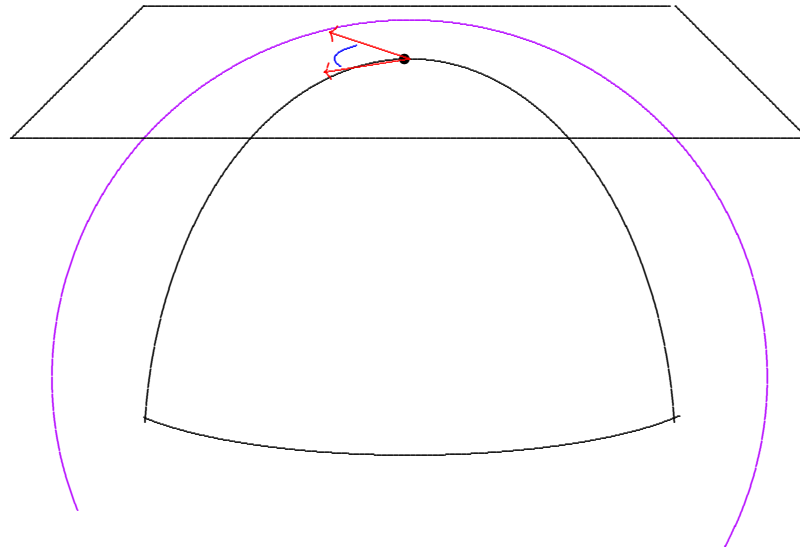
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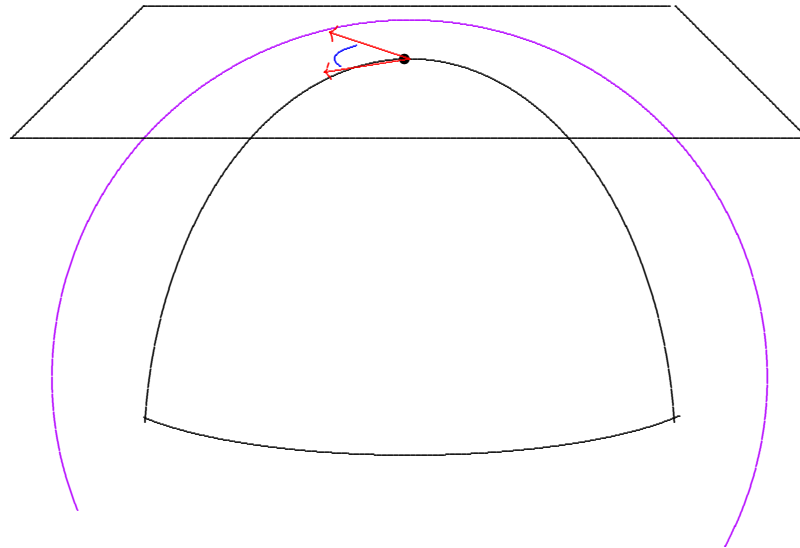
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$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

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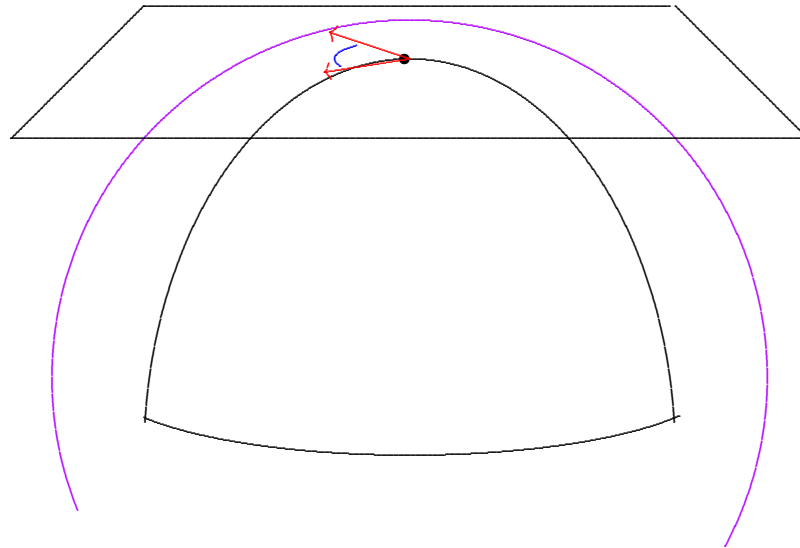
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Makes tangent space a complex vector space!

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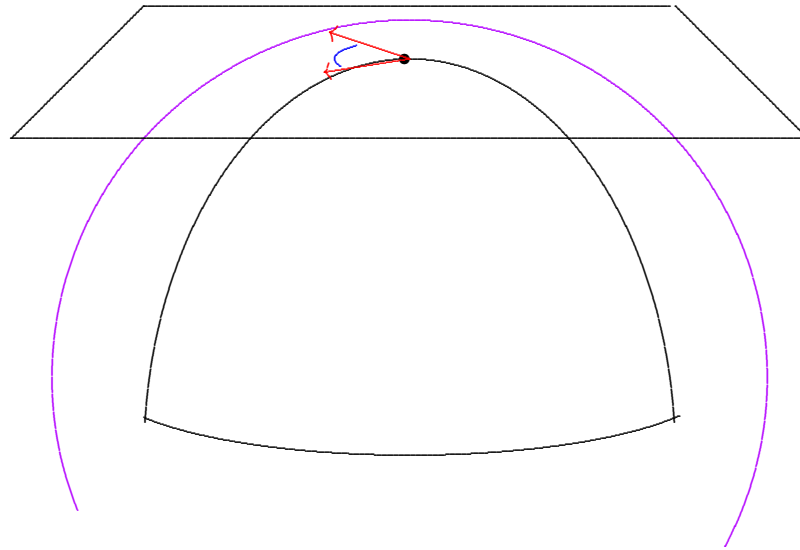
Makes tangent space a complex vector space!

$$J : TM \rightarrow TM, \quad J^2 = -\text{identity}$$

“almost-complex structure”

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Makes tangent space a complex vector space!

Invariant under parallel transport!

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Kähler magic:

$$r = - \sum_{j,k=1}^m \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det[g_{p\bar{q}}] \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

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If we define the Ricci form by

$$\rho = r(J\cdot, \cdot)$$

then $i\rho$ is curvature of canonical line bundle $\Lambda^{m,0}$.

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$$[\omega] \in H^2(M)$$

“Kähler class”

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ω non-degenerate closed 2-form: symplectic form

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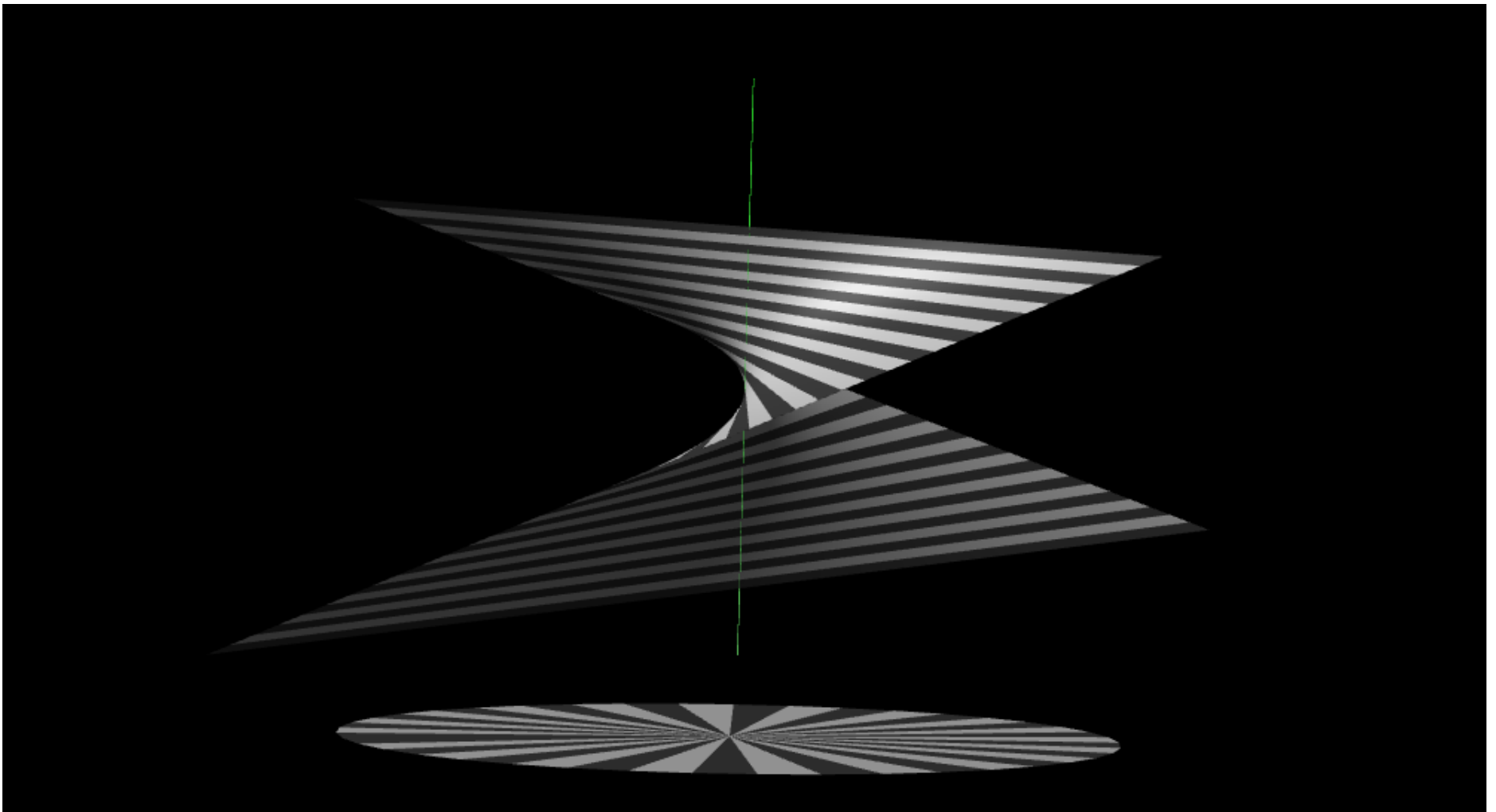
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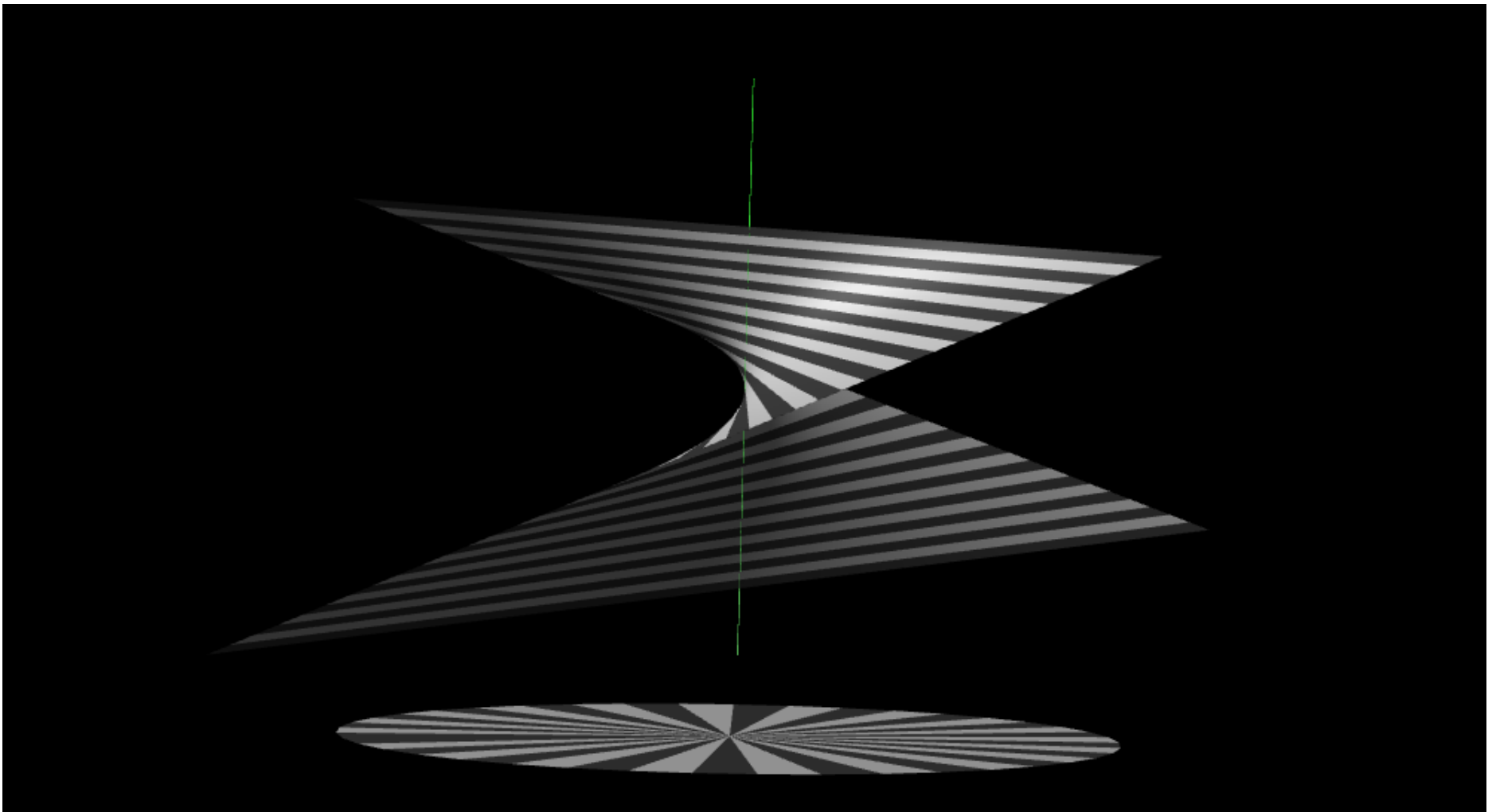
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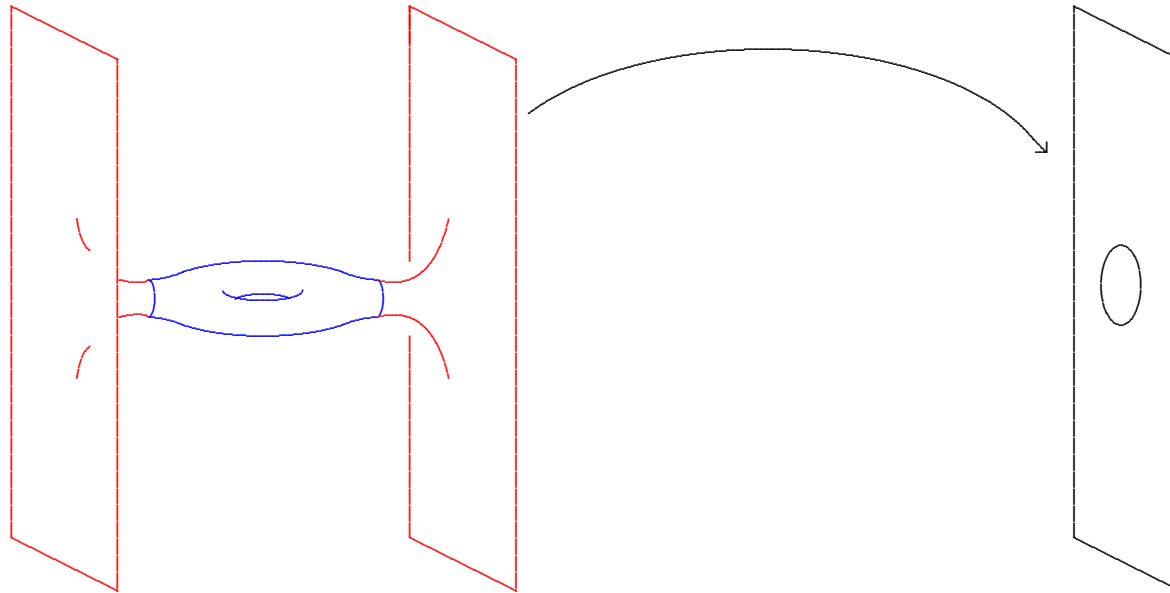
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also has mass m . Again measures “size of throat.”

Definition. A complete, non-compact Riemannian n -manifold (M^n, g) is called **asymptotically Euclidean (AE)** if there is a compact set $K \subset M$ such that each “end” is diffeomorphic to $\mathbb{R}^n - D^n$ in such a manner that

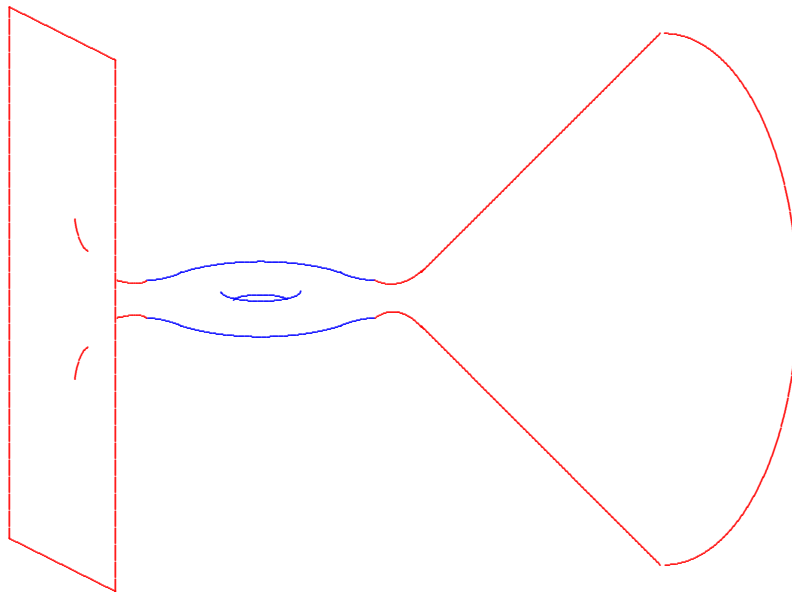


$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

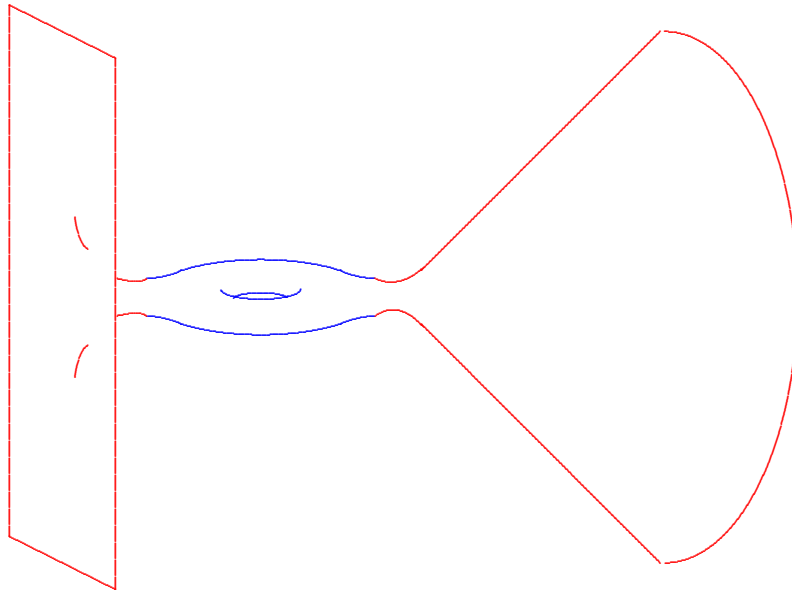
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Interesting generalization...

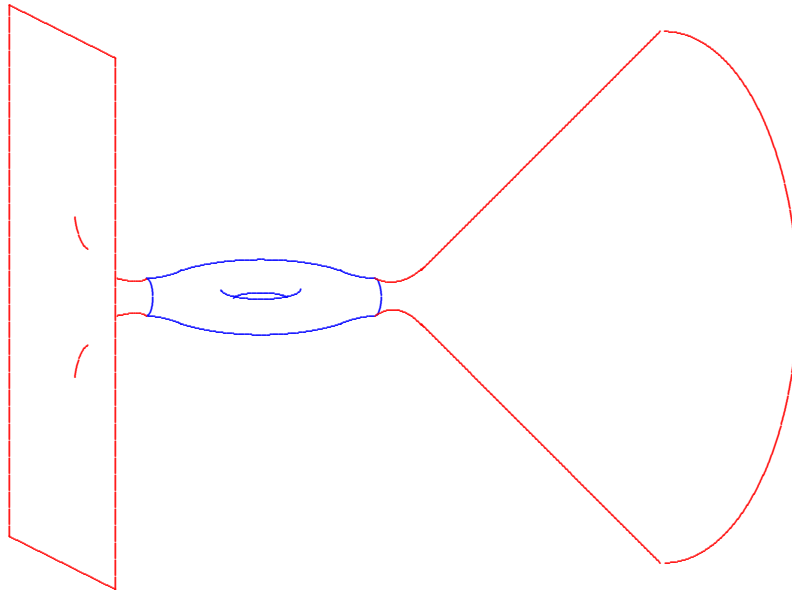
Definition. Complete, non-compact n -manifold (M^n, g) is asymptotically locally Euclidean



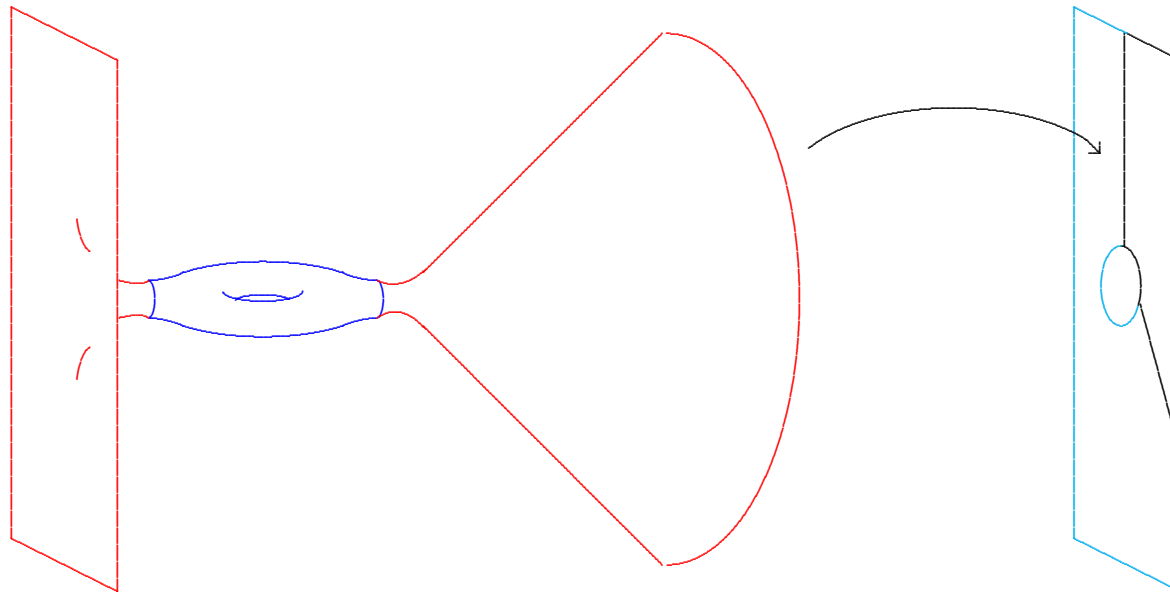
Definition. Complete, non-compact n -manifold (M^n, g) is asymptotically locally Euclidean (ALE)



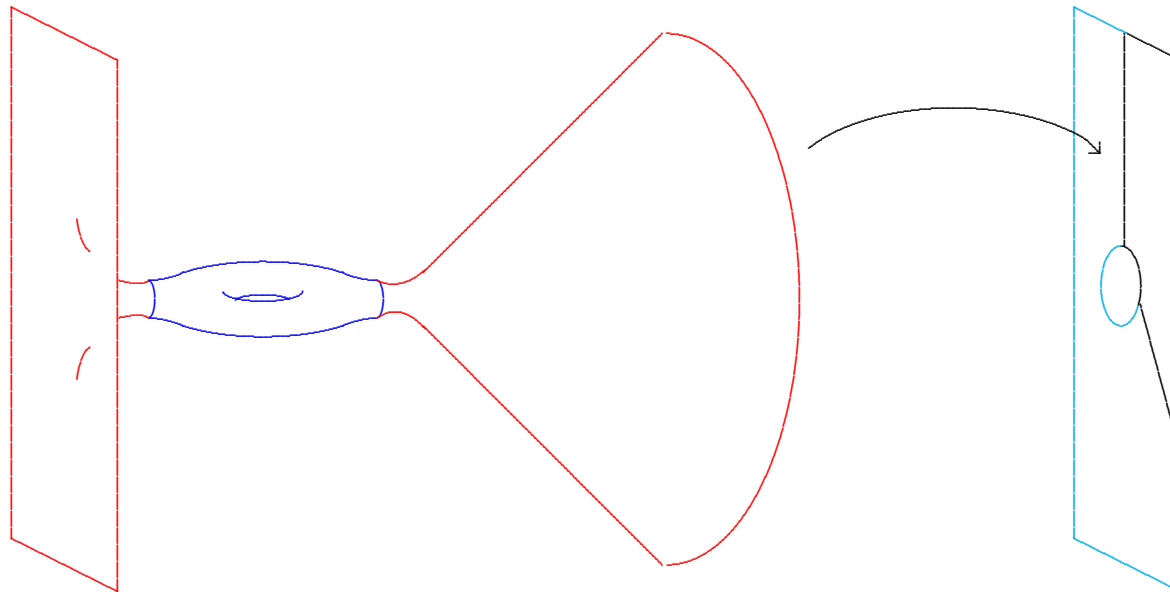
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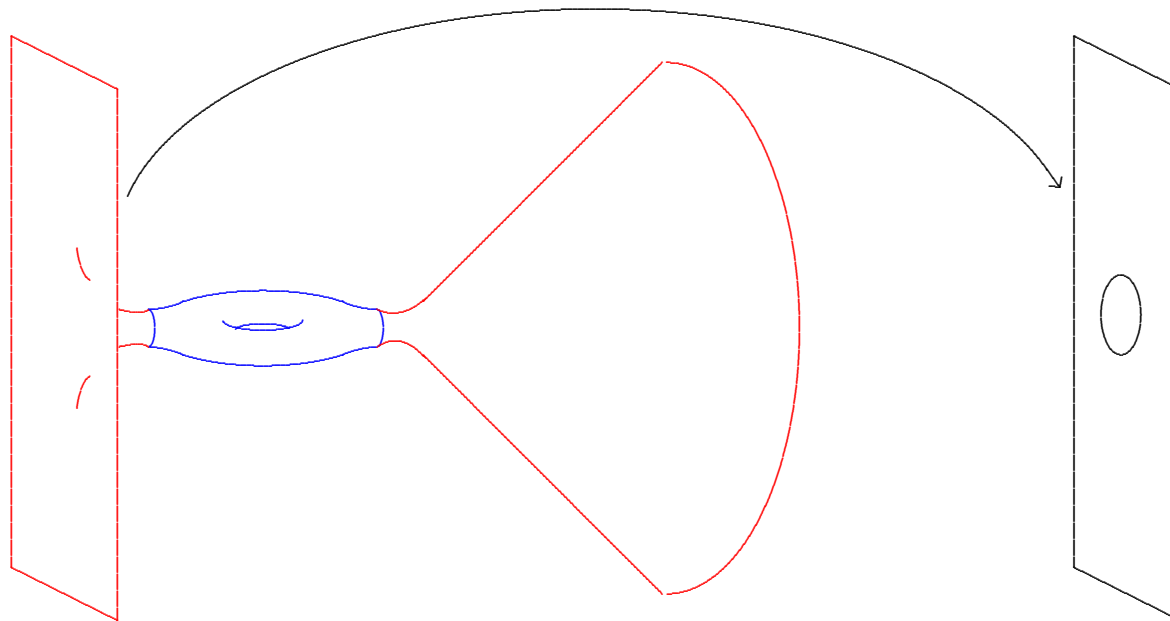
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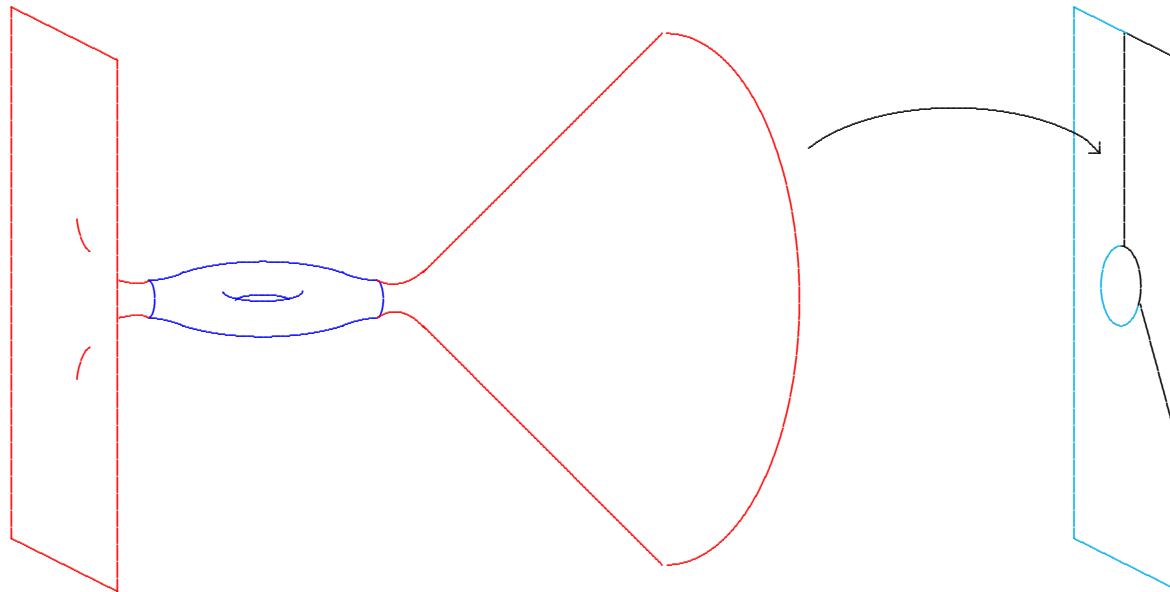
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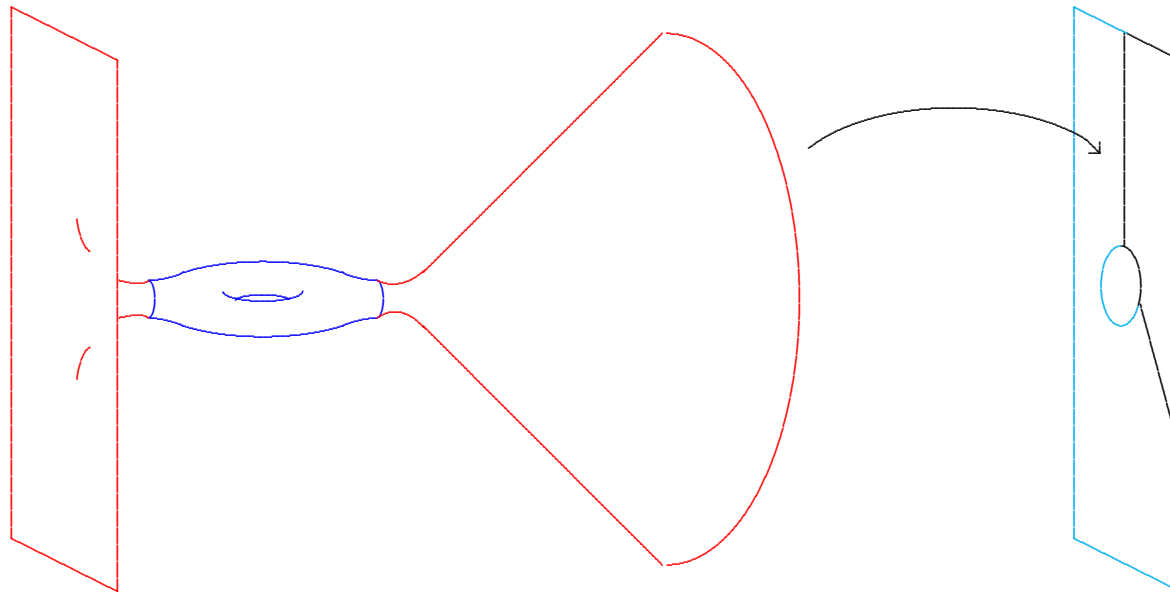
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Why consider *ALE* spaces?

Key examples:

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Gravitational Instantons:

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ALE Ricci-flat Kähler manifolds.

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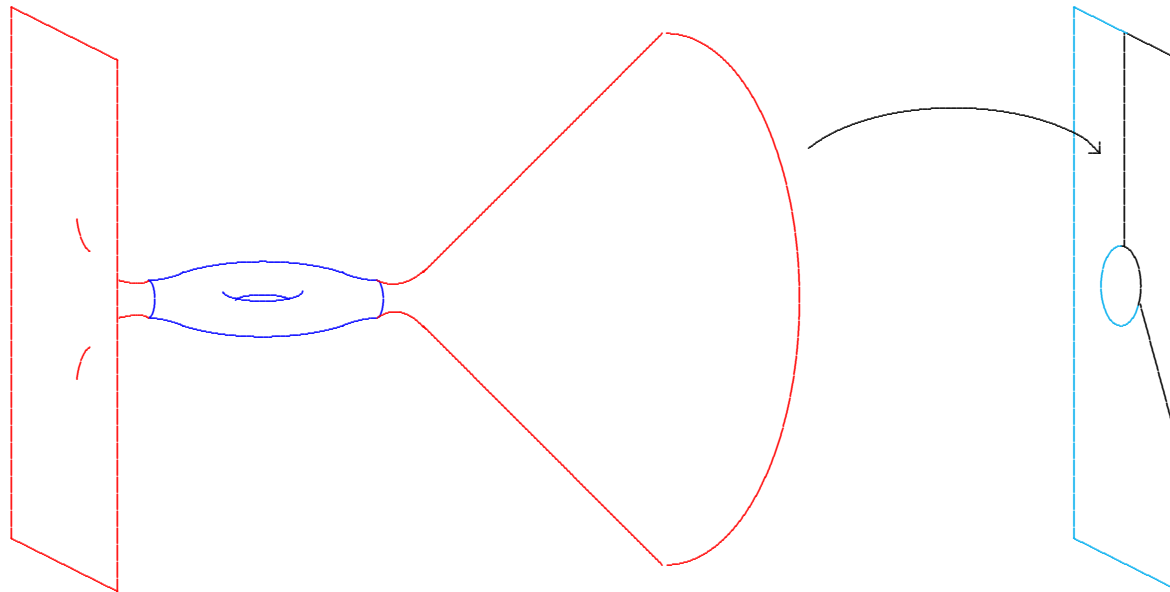
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Mass still meaningful in this context...

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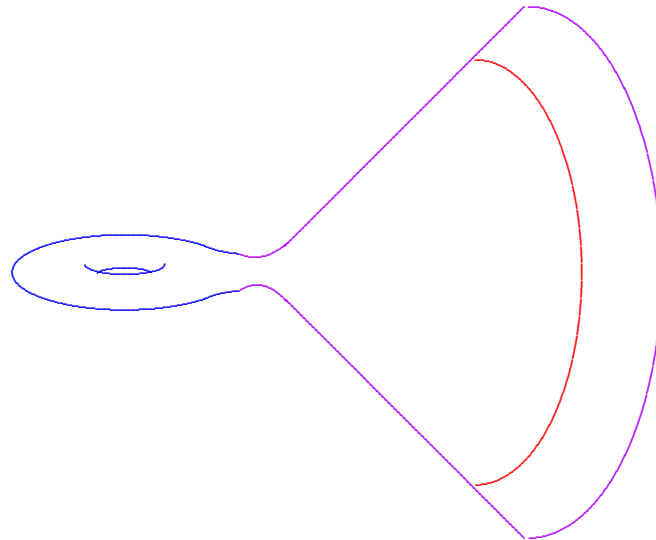
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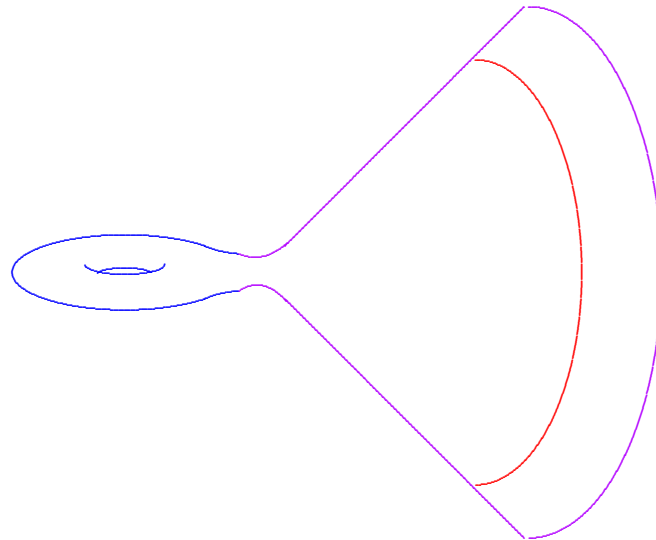


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Bartnik/Chruściel (1986): With weak fall-off conditions, the mass is well-defined & coordinate independent.

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Lemma.

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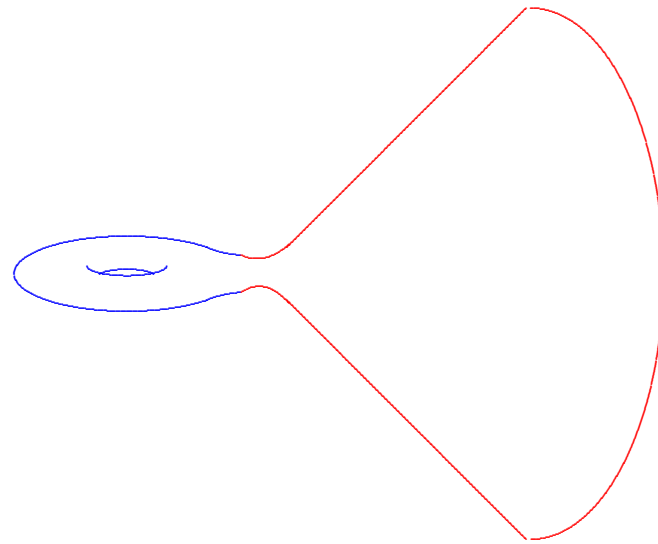
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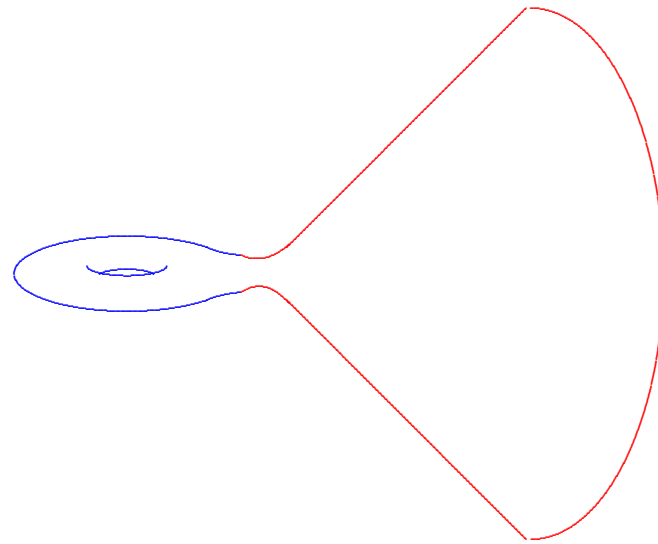
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Does not depend on the choice of an end!

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Theorem A.

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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Non-minimal resolutions typically admit families of such metrics for which the mass can be continuously deformed from negative to positive.

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So the mass is a “boundary correction” to the topological formula for the total scalar curvature.

Theorem C. Any *ALE Kähler manifold* (M, g, J) of complex dimension m has mass given by

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So **Theorem A** is an immediate consequence!

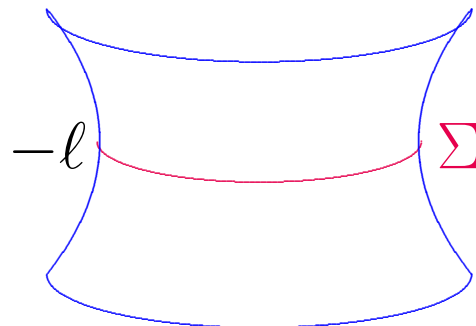
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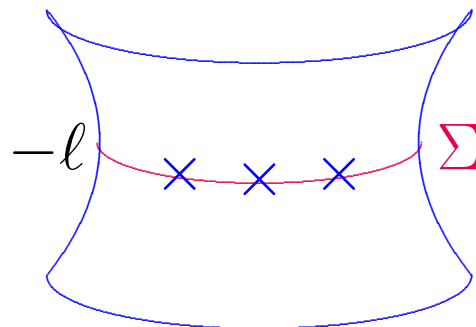
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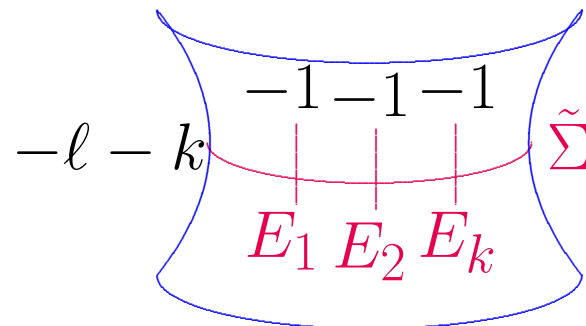
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Proof actually shows something stronger!

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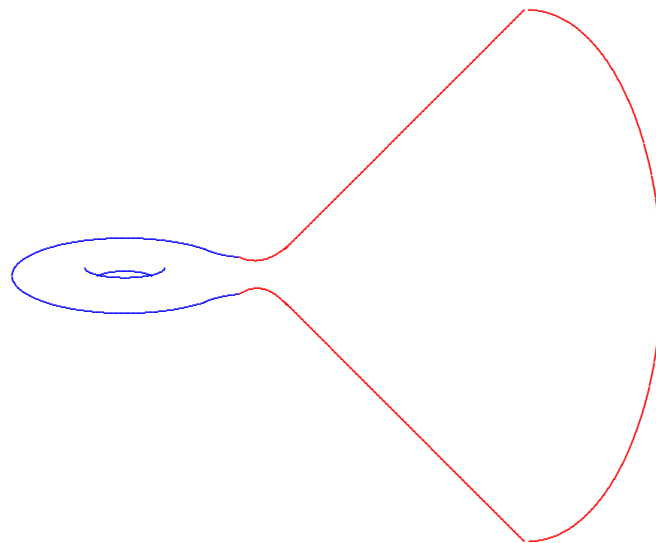
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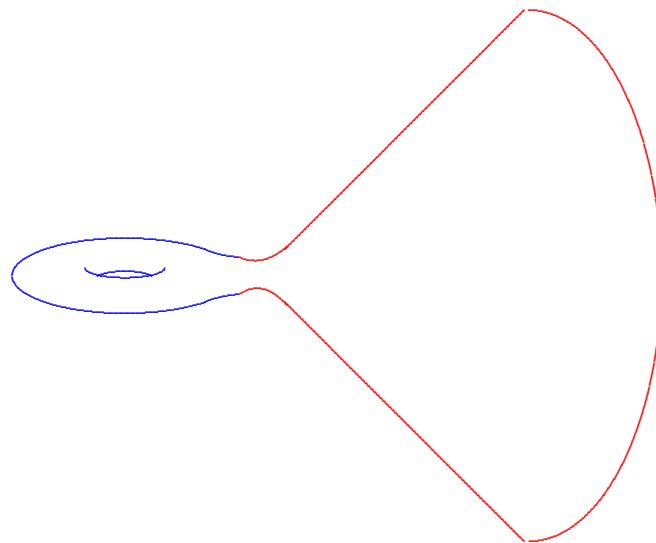
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