Mass in

Kähler Geometry

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Stony Brook University

New Perspectives in Differential Geometry
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Joint work with

Joint work with

Hans-Joachim Hein
University of Maryland

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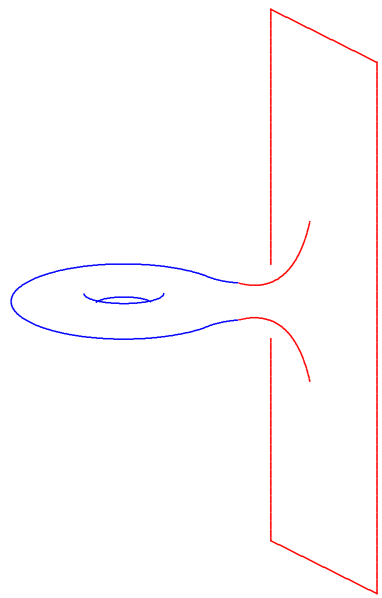
e-print: arXiv:1507.08885 [math.DG]

Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g)



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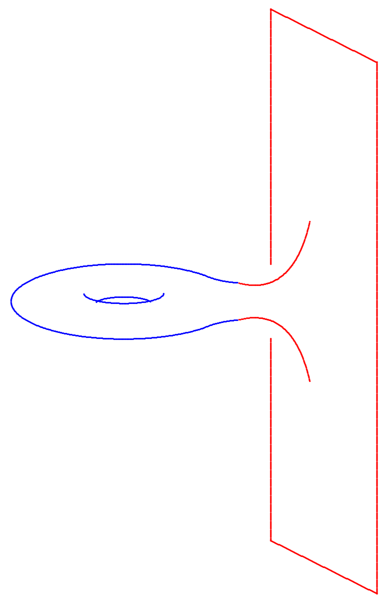

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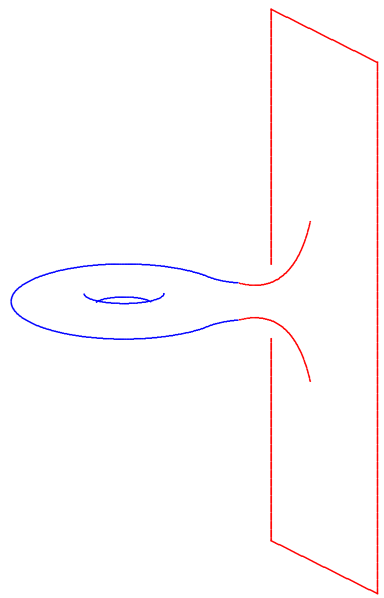

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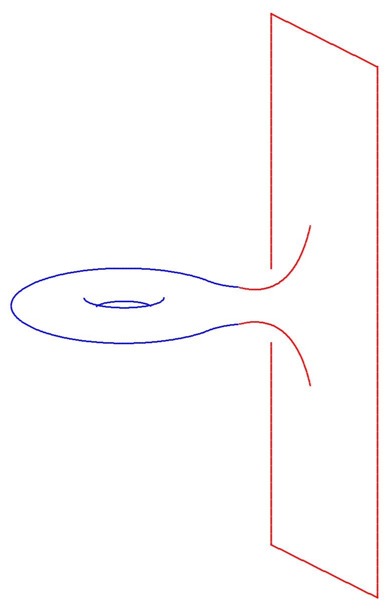

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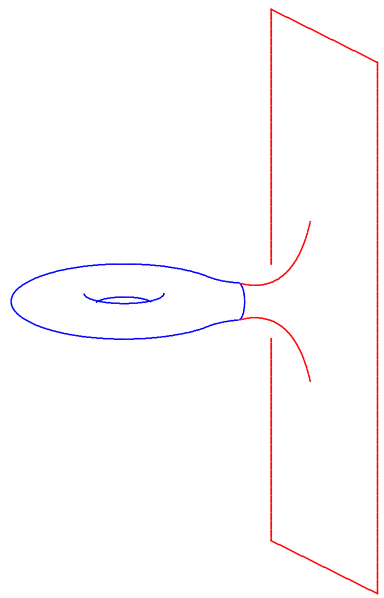

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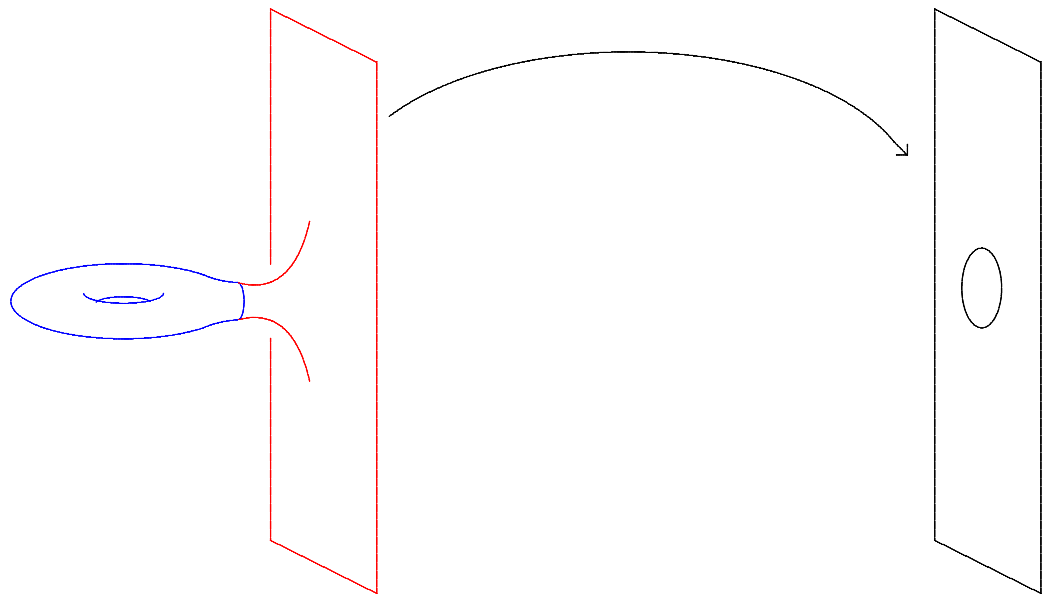

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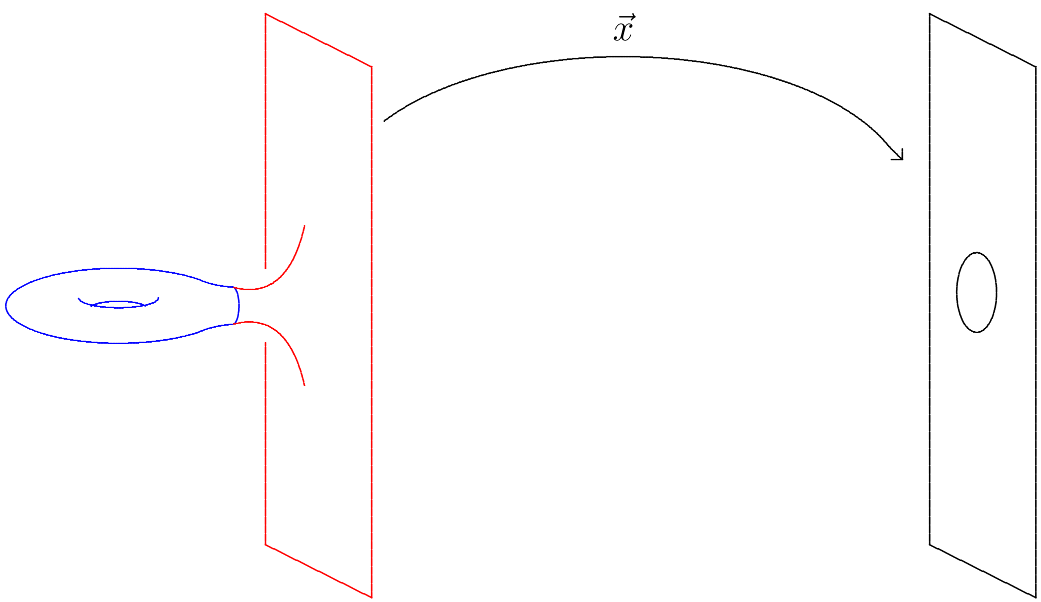

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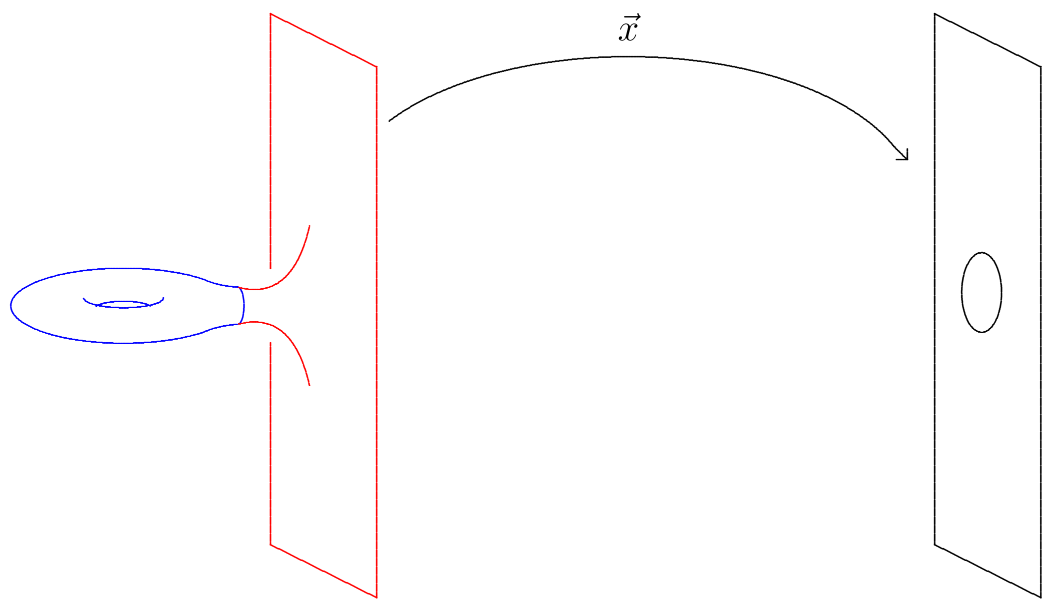

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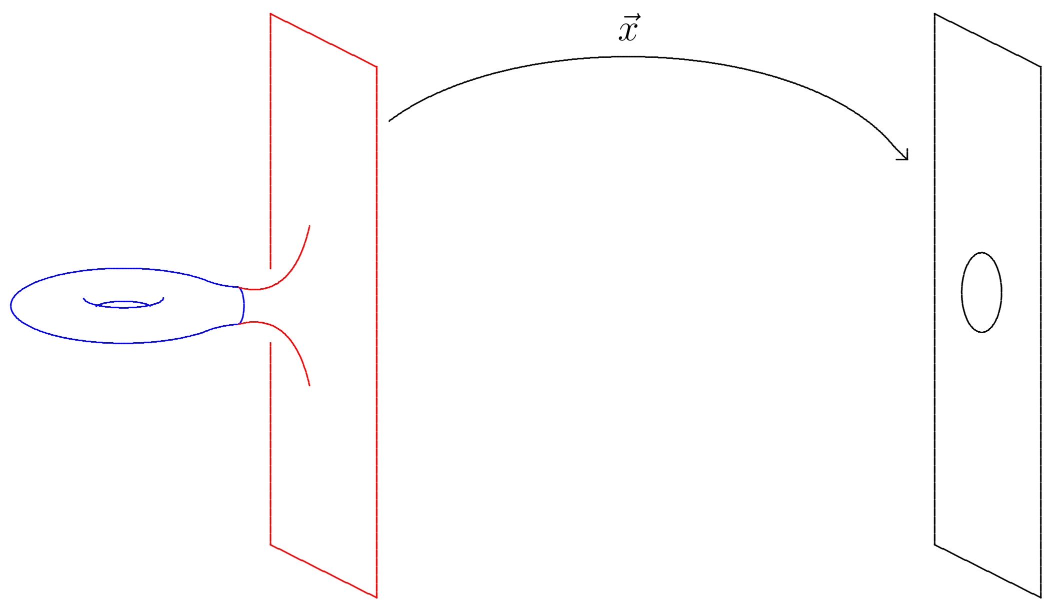

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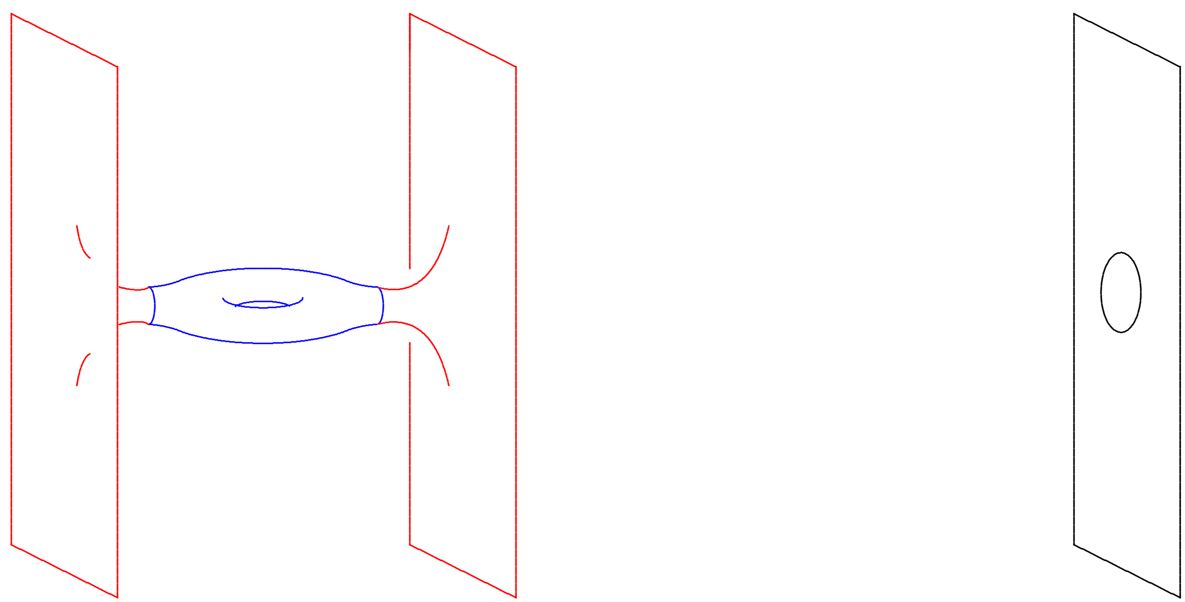

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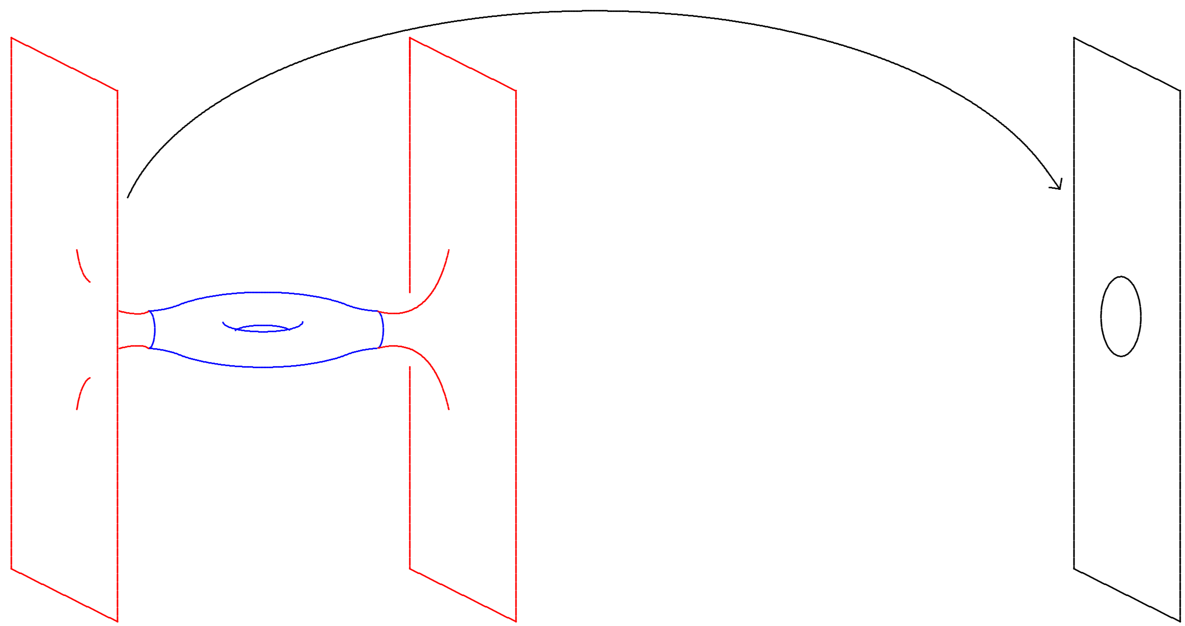

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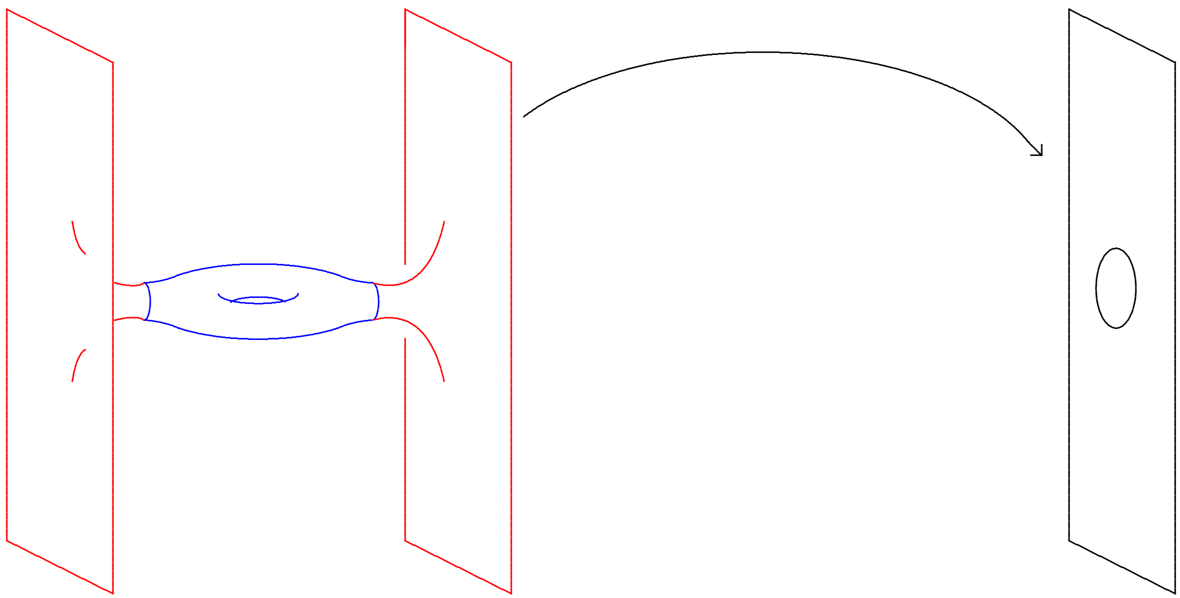

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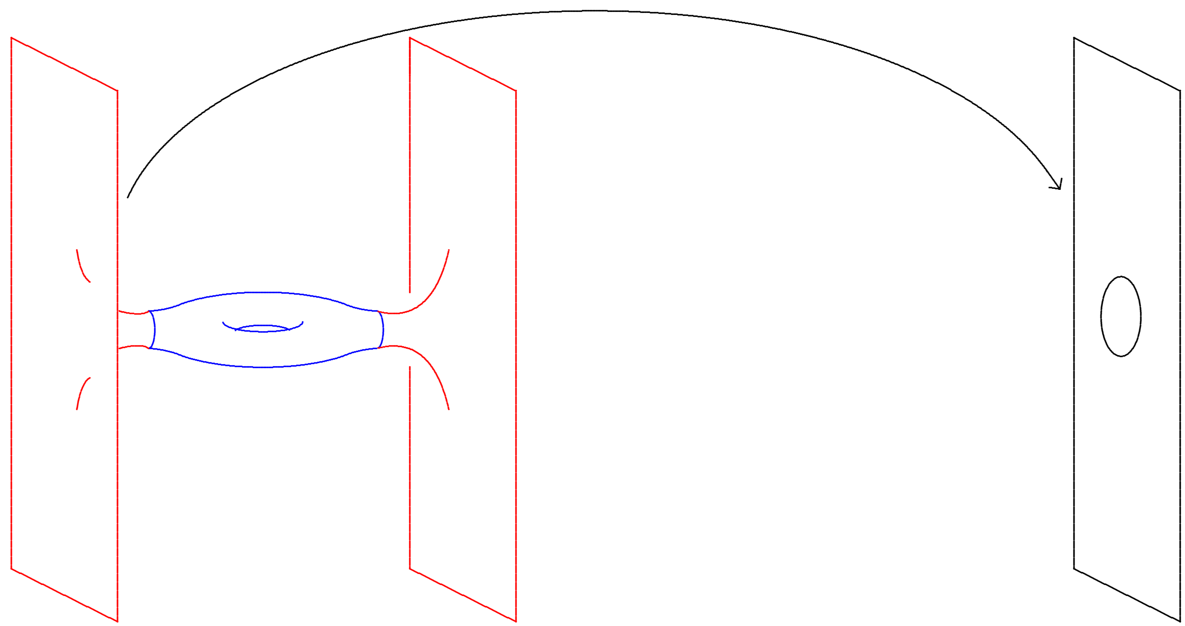

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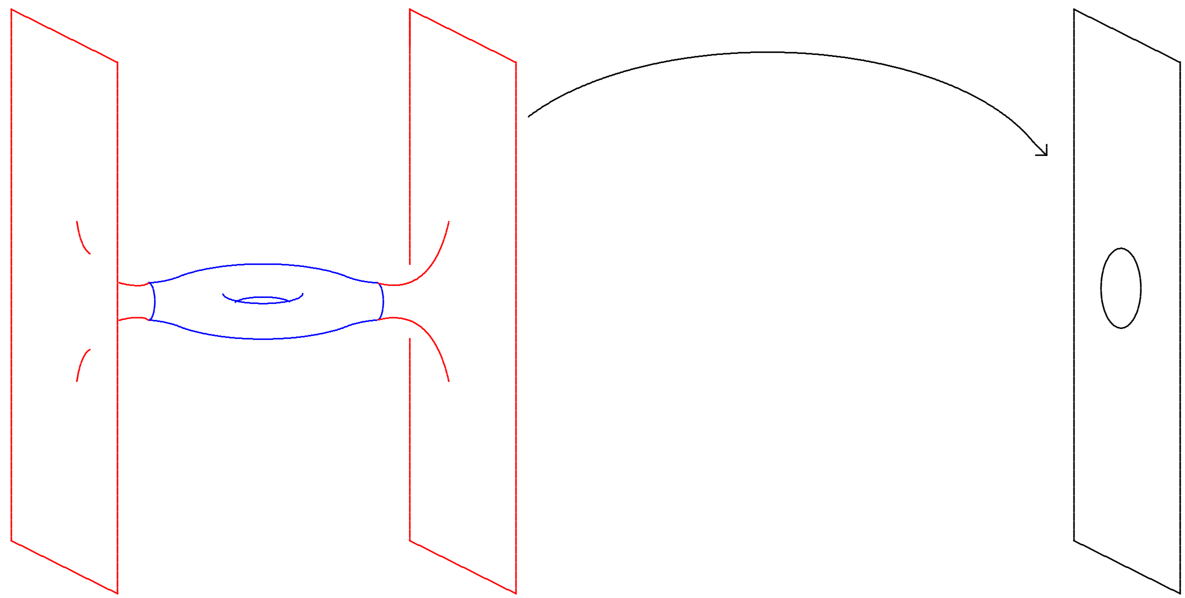

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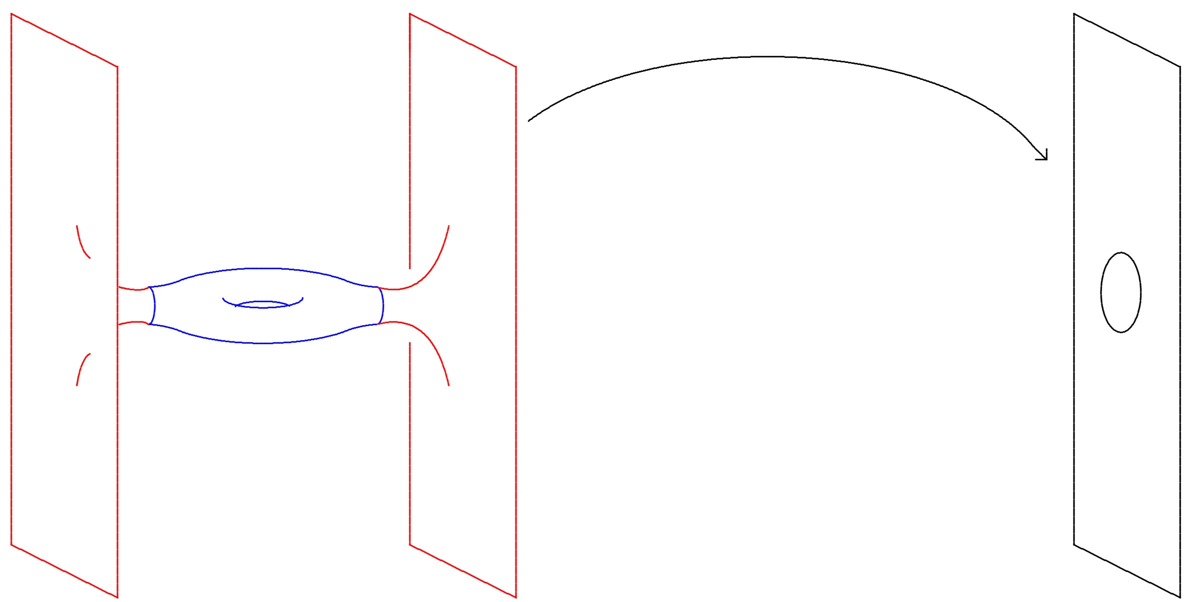

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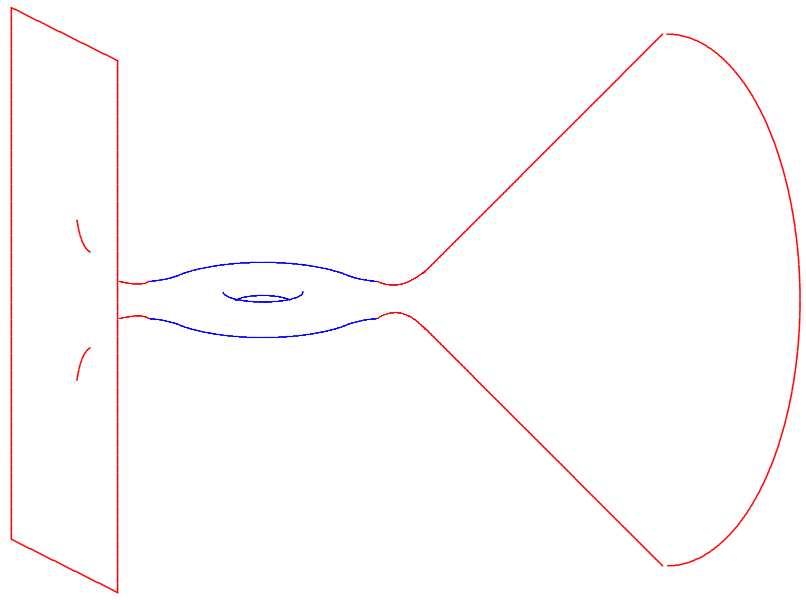

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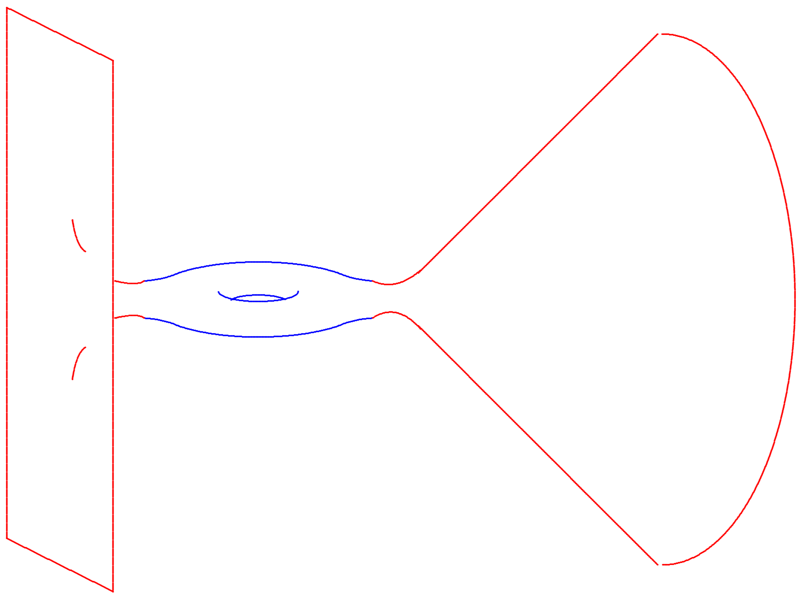

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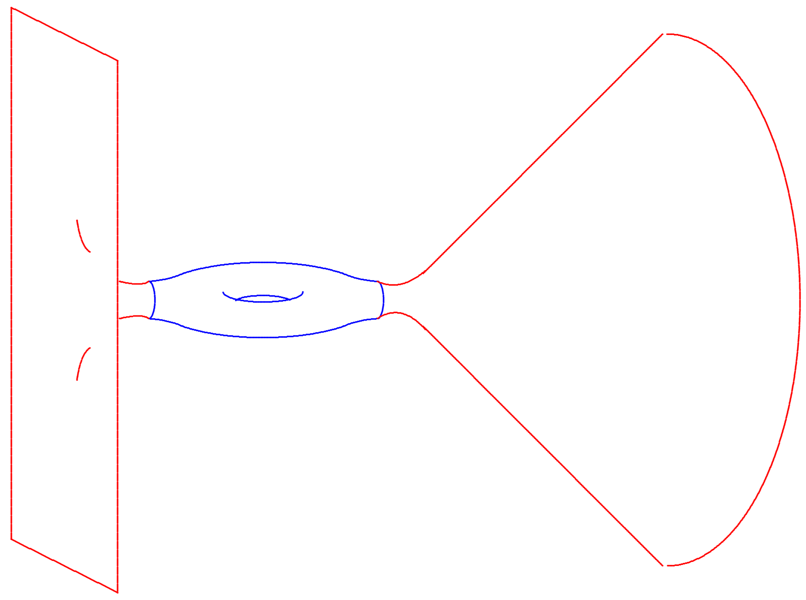

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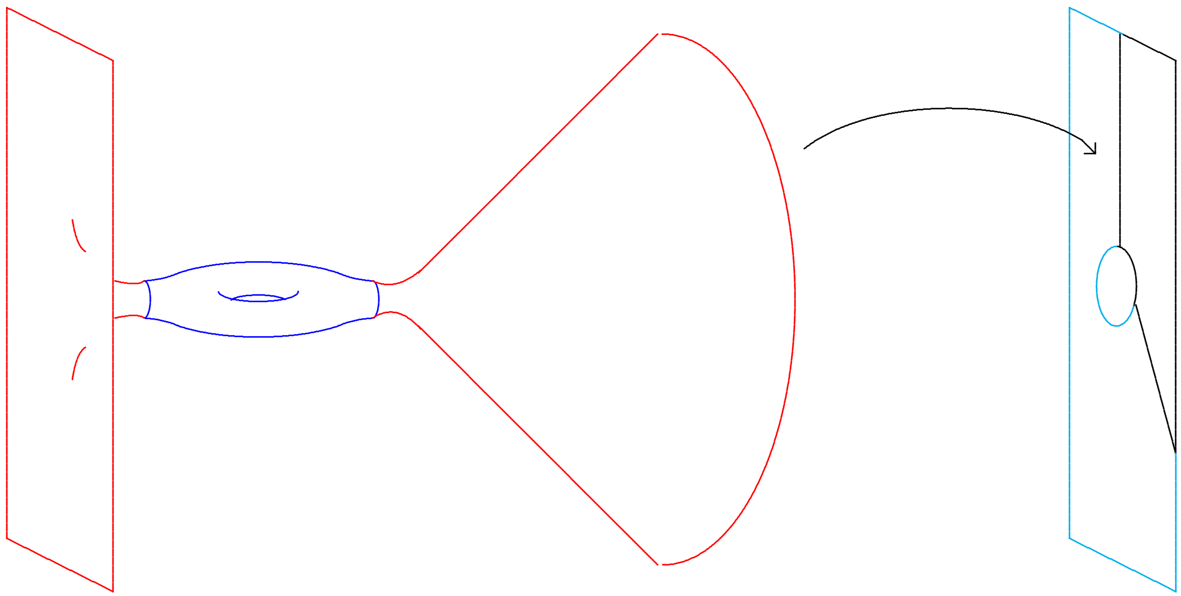

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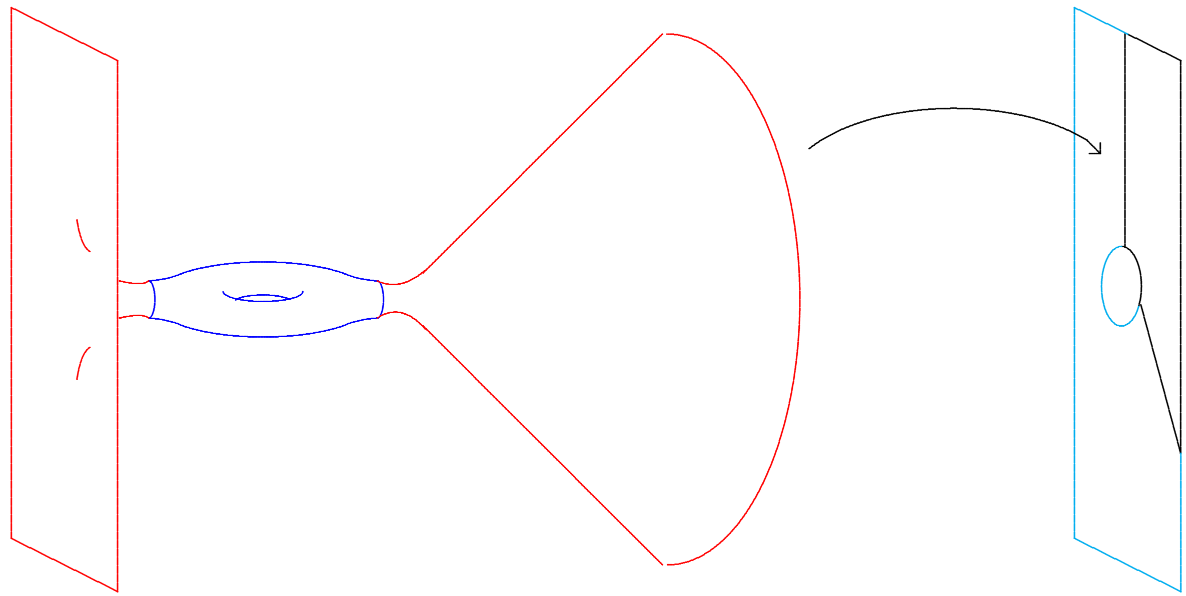

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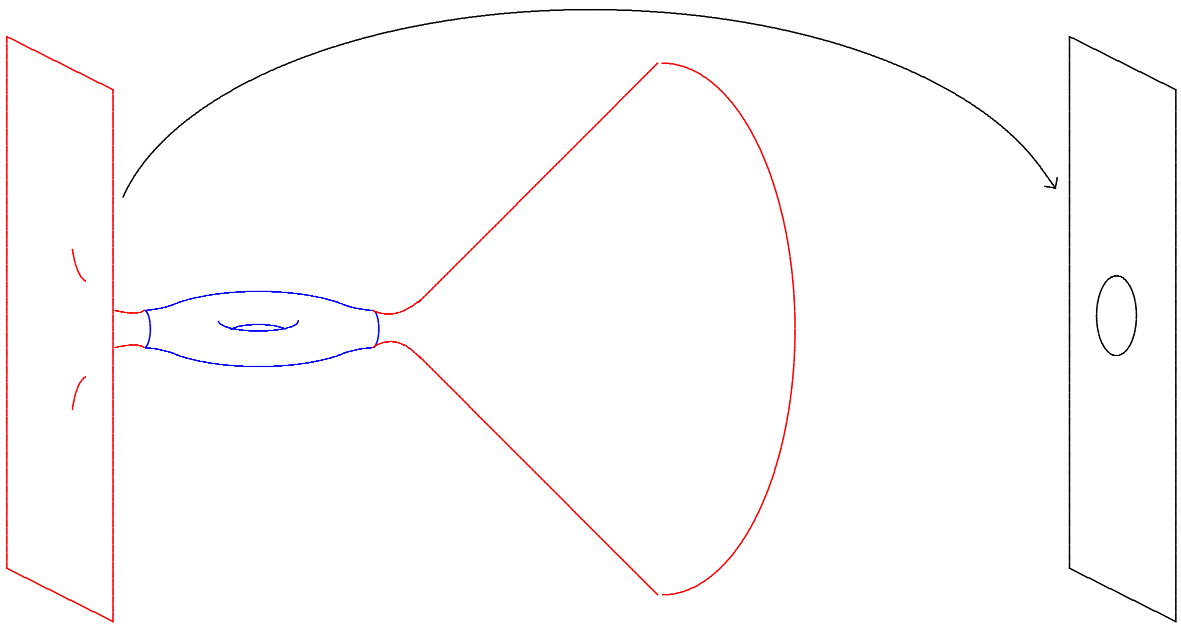

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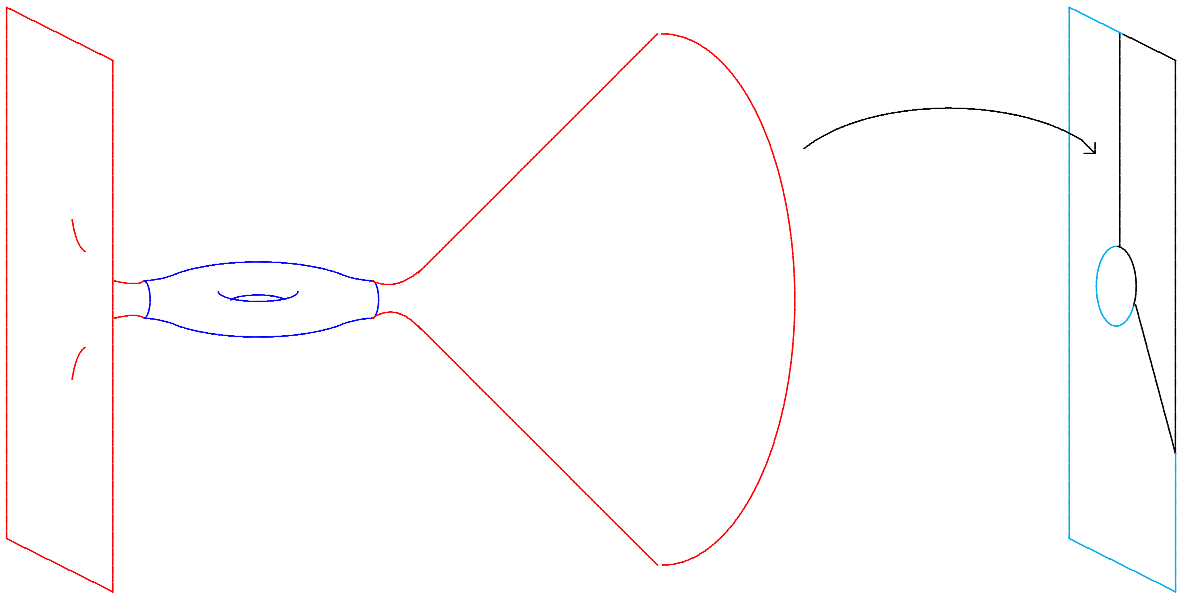

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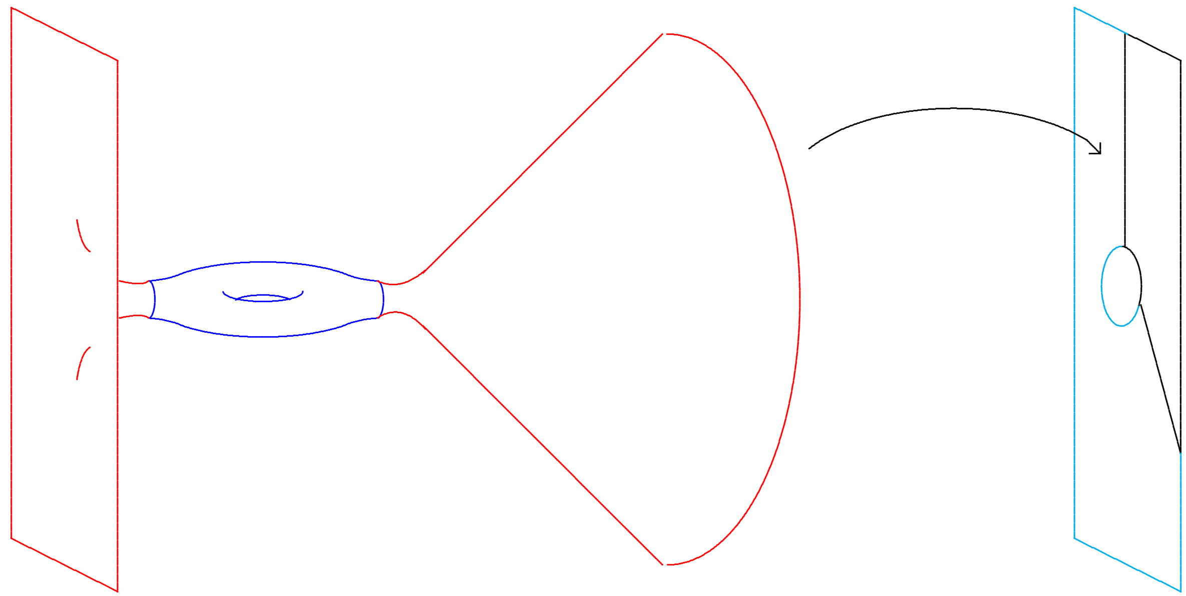

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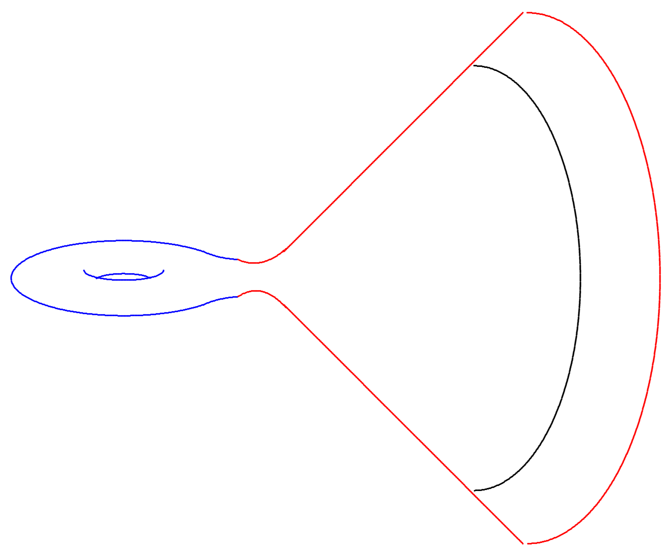

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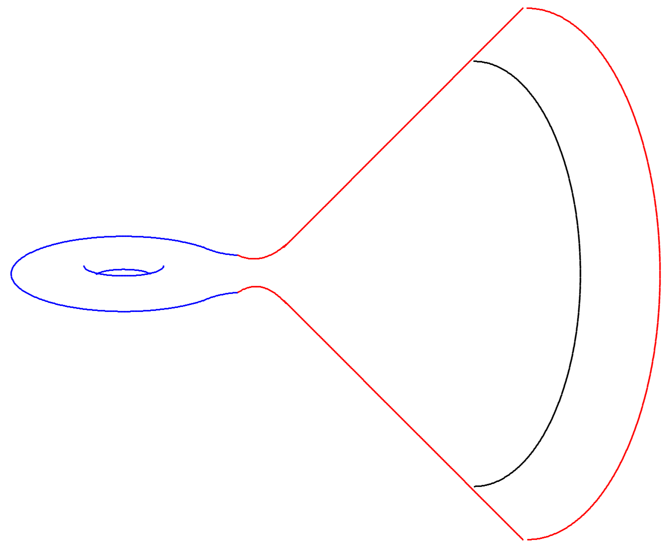

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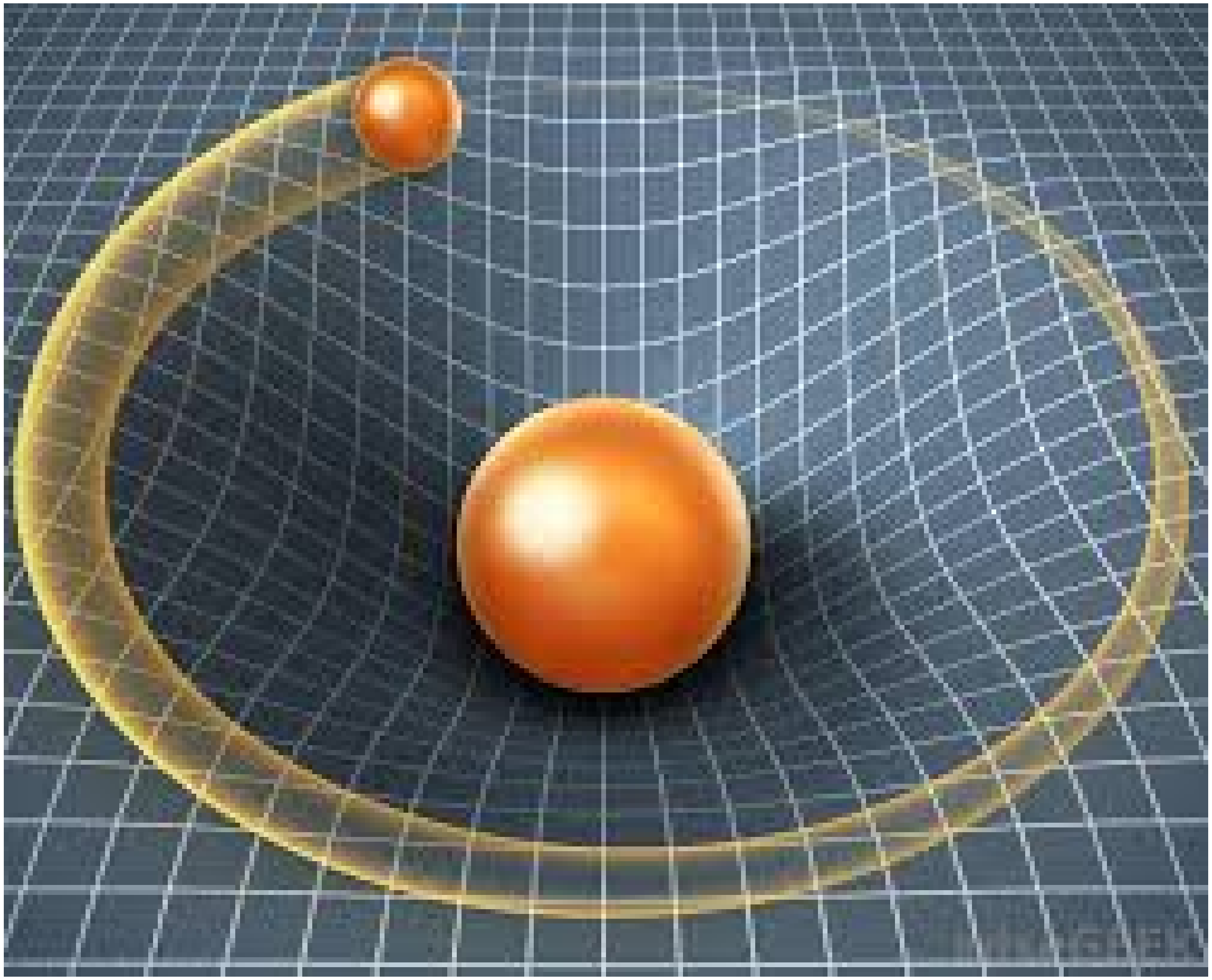

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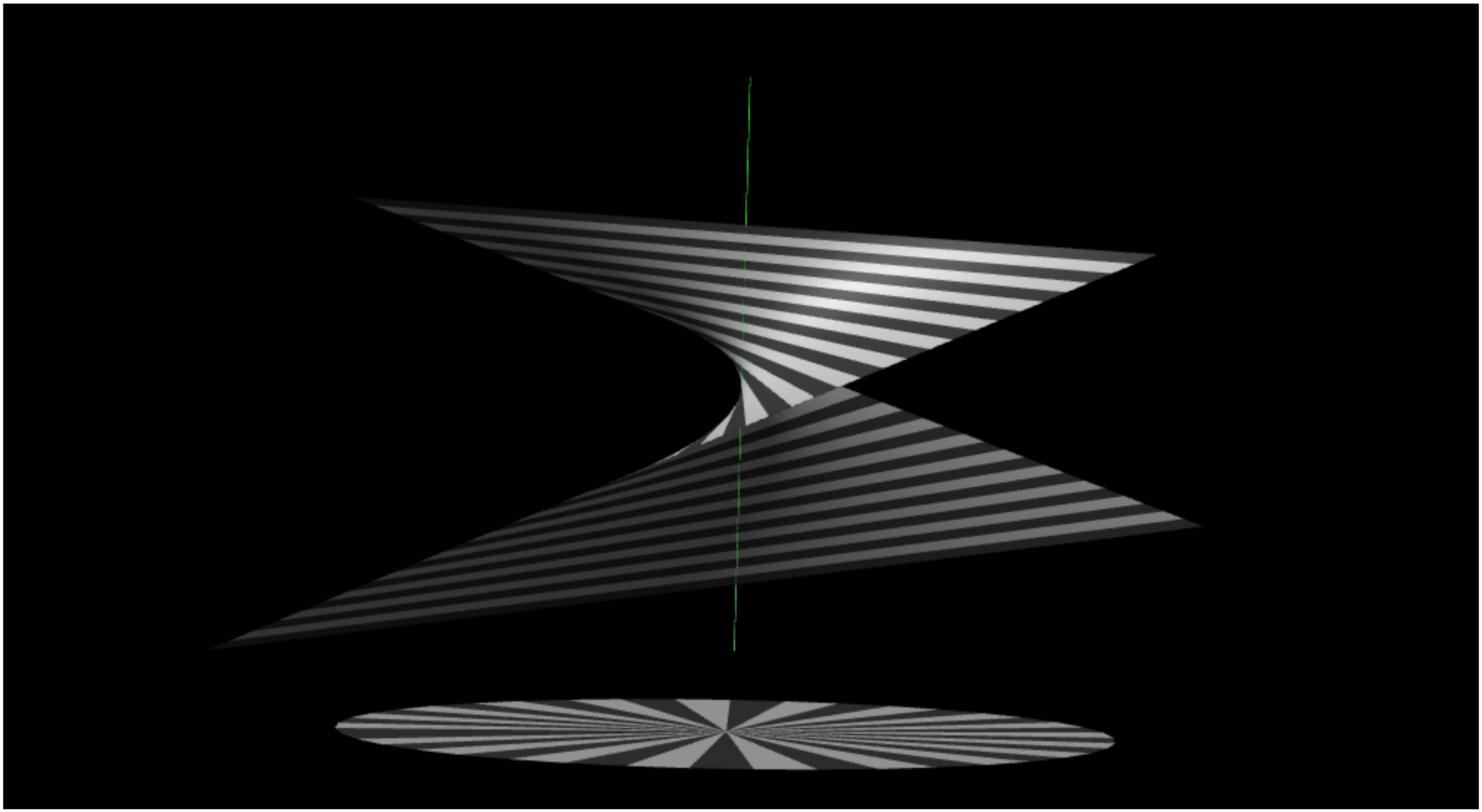

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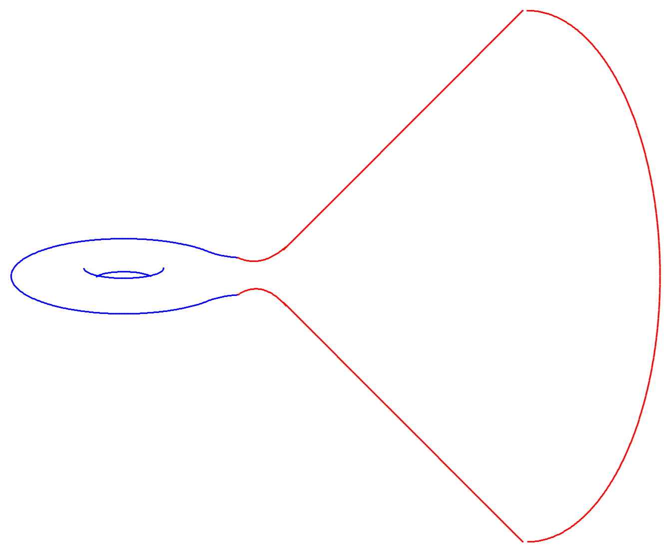

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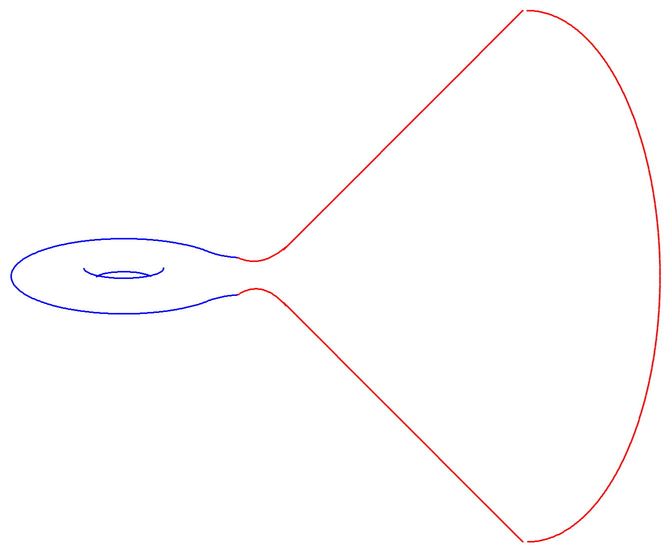

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Theorem A.

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

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Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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So **Theorem A** is an immediate consequence!

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$$g^{jk}\left(g_{j\ell,k} - g_{jk,\ell}\right)\nu^\ell \alpha_E = -\star d \log\left(\sqrt{\det g}\right) + O(\varrho^{-3-\varepsilon}).$$

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However, since $s = 0$,

$$d(\theta \wedge \omega) = \rho \wedge \omega = \frac{s}{4}\omega^2 = 0.$$

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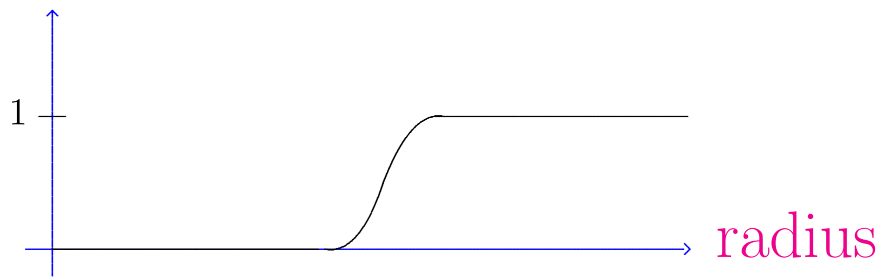

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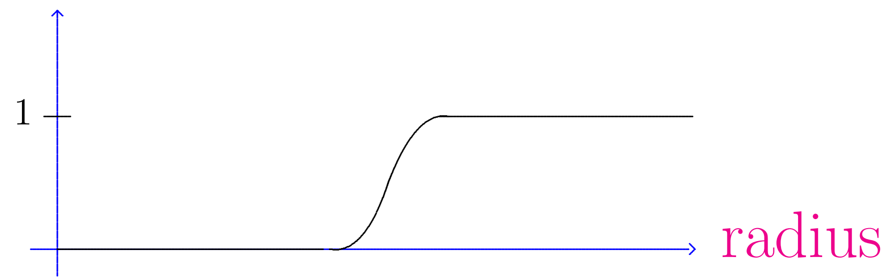

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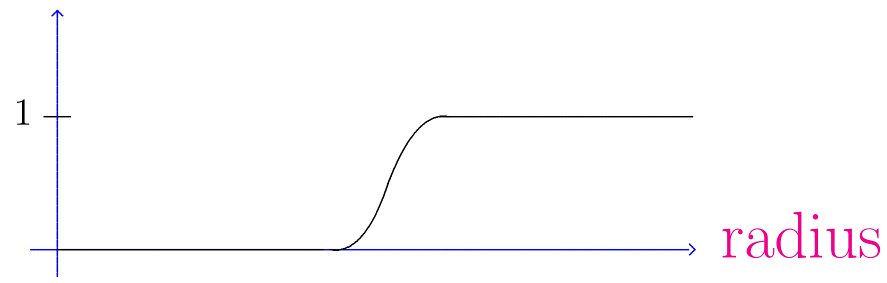

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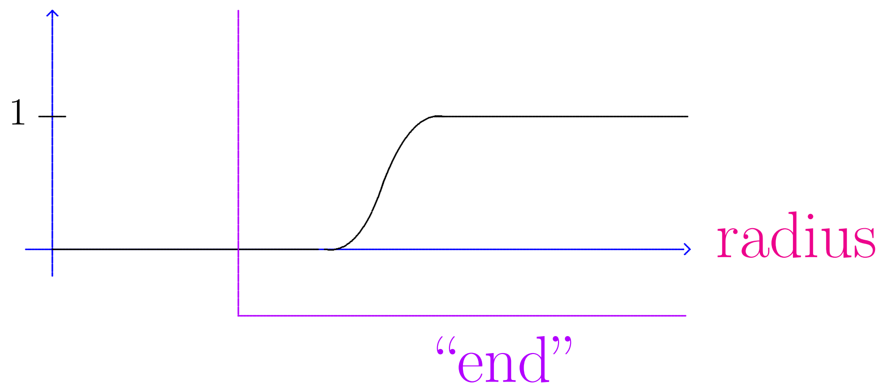

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Compactly supported, because $d\theta = \rho$ near infinity.

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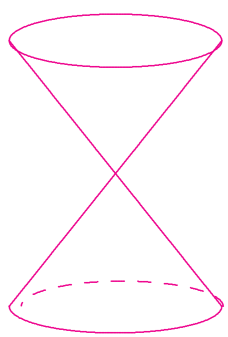

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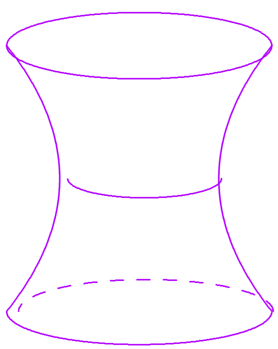

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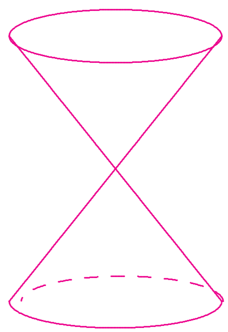

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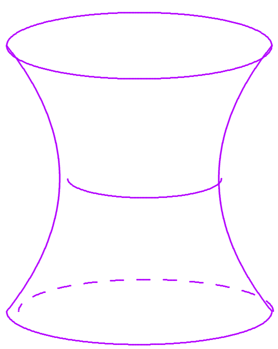

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$$J = J_0 + O(\varrho^{-3}), \qquad \nabla J = O(\varrho^{-4})$$

in suitable asymptotic coordinates adapted to g.

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To understand J at infinity:

Let $\widetilde{M}_\infty$ be universal over of end M_∞.

Cap off $\widetilde{M}_\infty$ by adding $\mathbb{CP}_{m-1}$ at infinity.

Added hypersurface $\mathbb{CP}_{m-1}$ has normal bundle $\mathcal{O}(1)$.

Belongs to m-dimensional family of hypersurfaces.

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This has some interesting consequences. . .

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Proof actually shows something stronger!

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so the mass formula implies the claim.

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Tanti auguri! And Happy Birthday!