Einstein Manifolds,

Self-Dual Weyl Curvature, &

Conformally Kähler Geometry

Claude LeBrun
Stony Brook University

Differential Geometry & Analysis Seminar
Princeton University, October 9, 2019

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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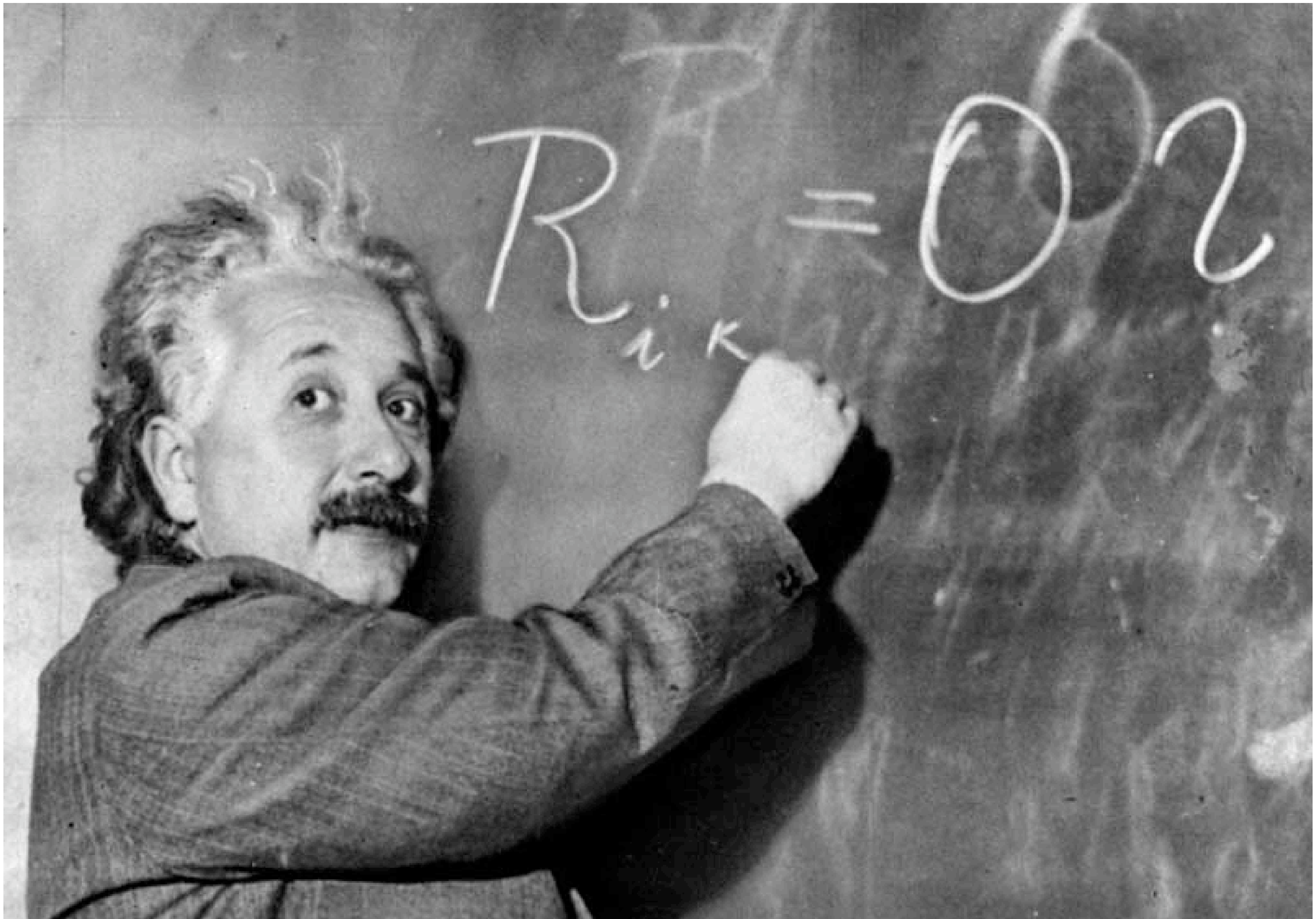

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$$s = r_j^j = \mathcal{R}^{ij}{}_{ij}.$$

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When $n = 4$, situation is more encouraging...

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i.e. a closed non-degenerate 2-form ω:

$$d\omega = 0, \qquad \omega \wedge \omega > 0.$$

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric h (unrelated to ω)? What if we also require $\lambda \geq 0$?*

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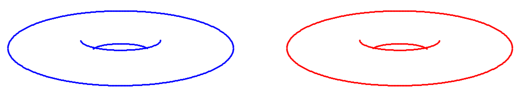

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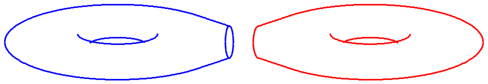

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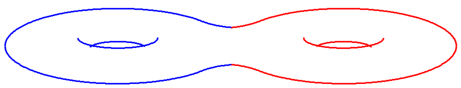

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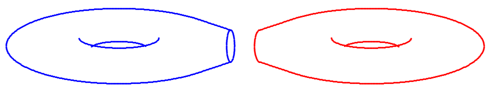

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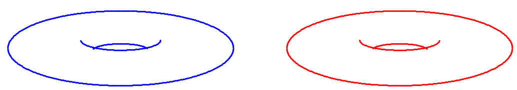

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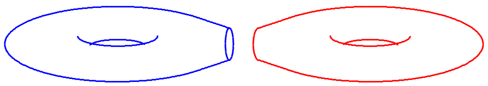

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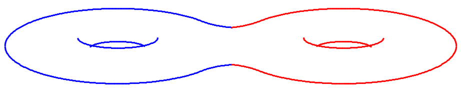

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Calabi/Yau: Admits Ricci-flat Kähler metrics.

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Know an Einstein metric on each manifold.

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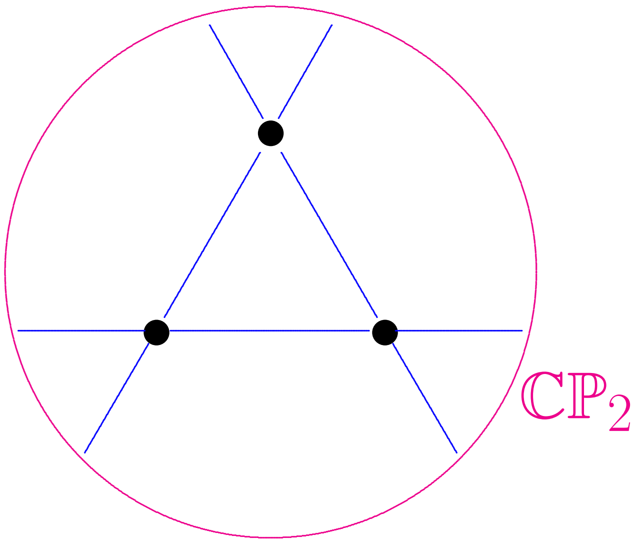

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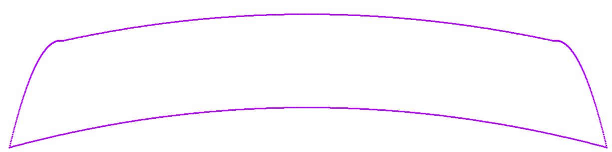

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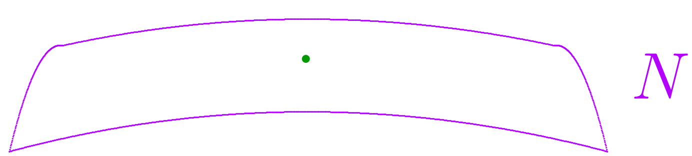

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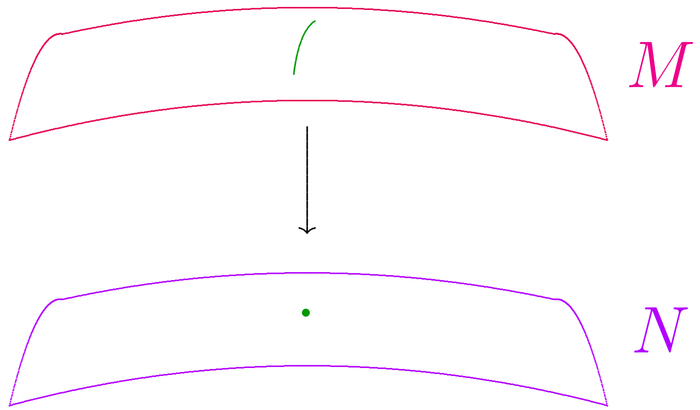

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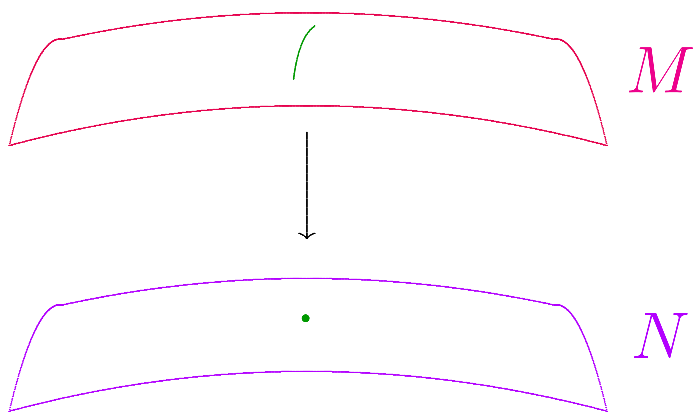
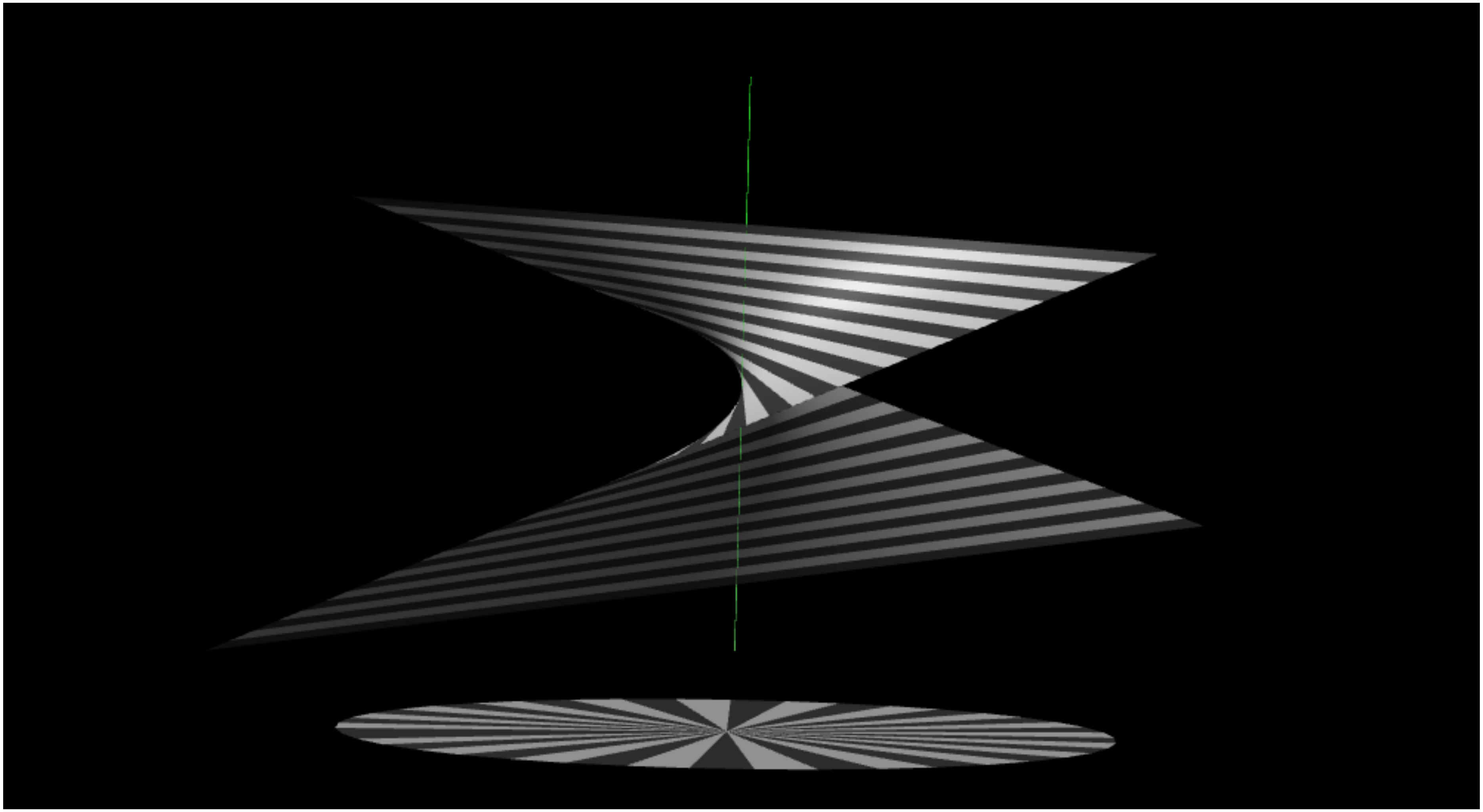
$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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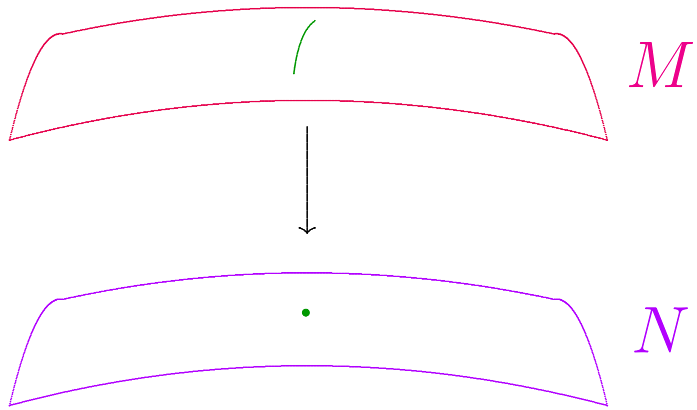

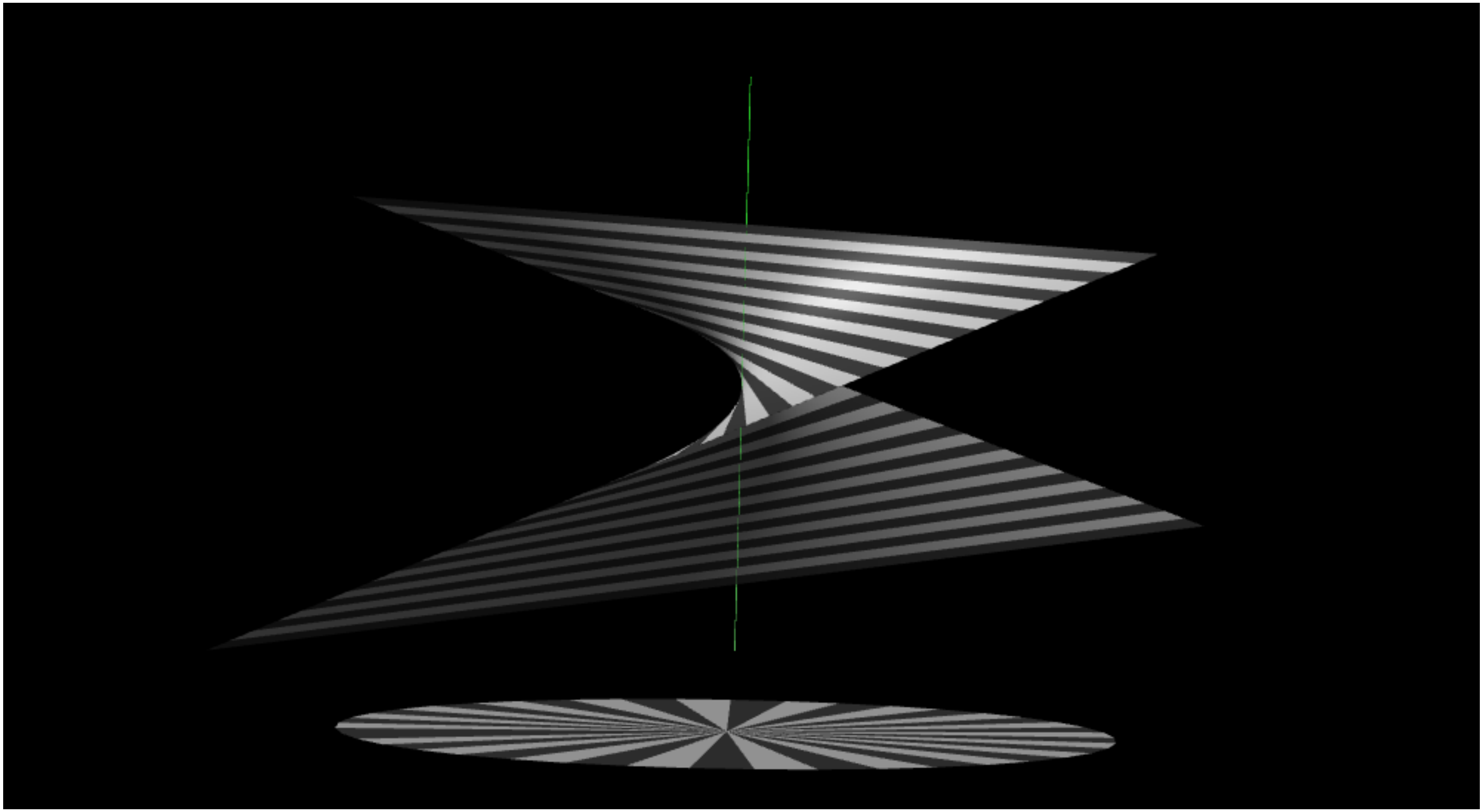

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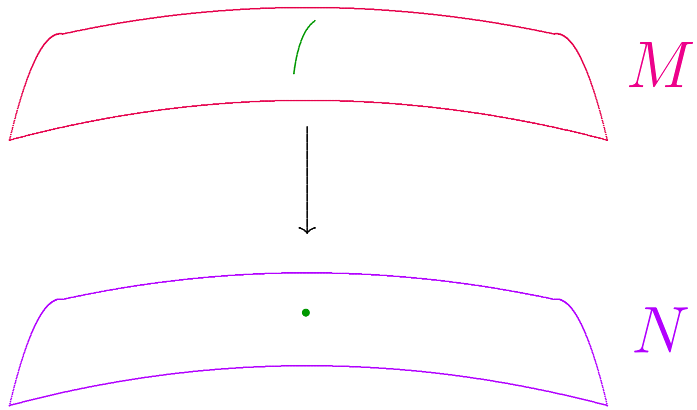

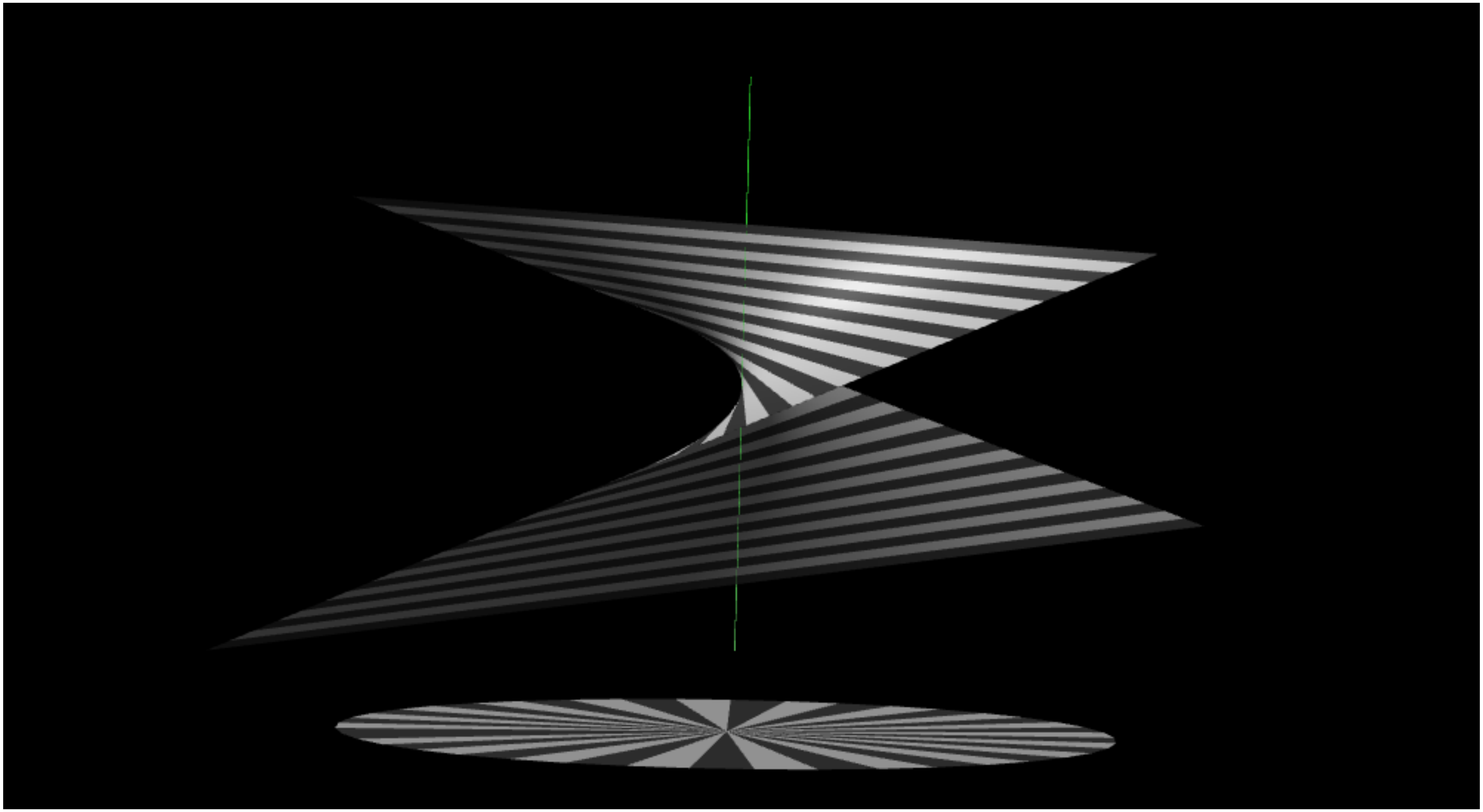

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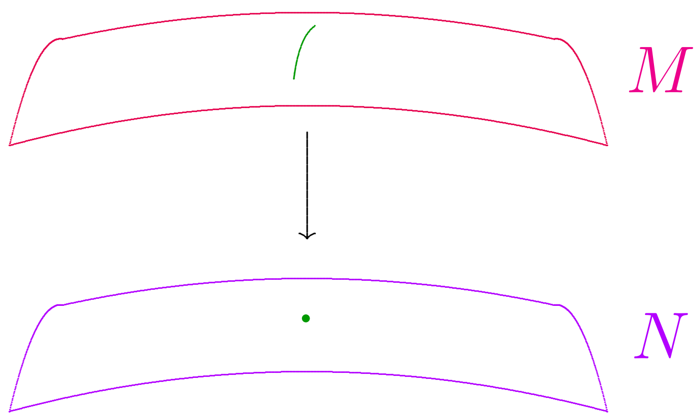

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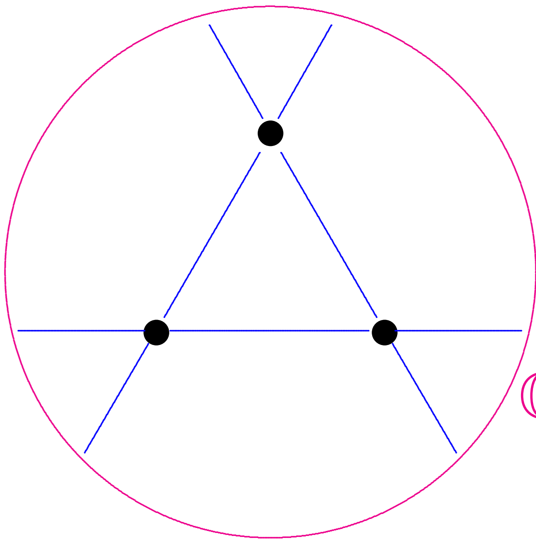

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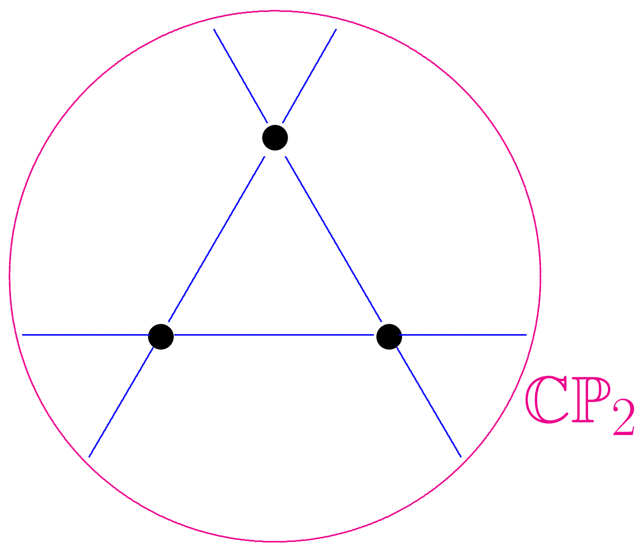

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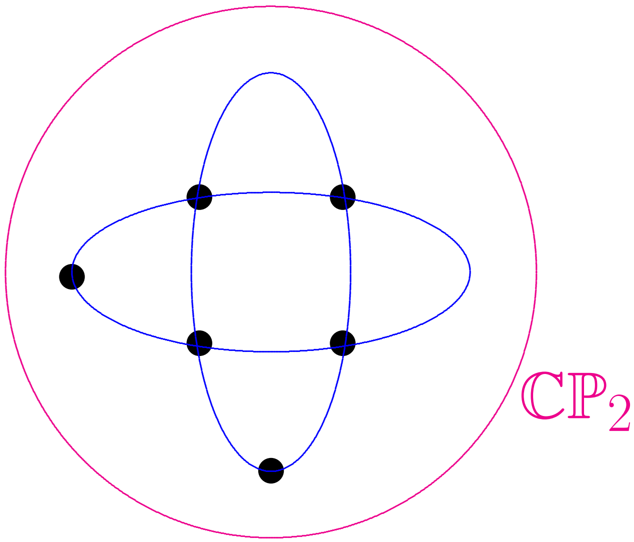

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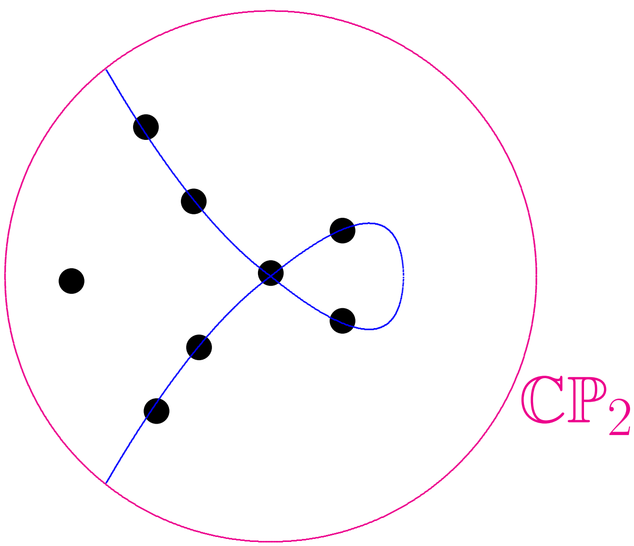

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Uniqueness: Bando-Mabuchi '87, L '12.

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This all depends on . . .

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More generally, their dimensions

$$b_\pm(M) = \dim \mathcal{H}^\pm_h$$

are completely metric-independent, and
are oriented homotopy invariants of M.

One Riemannian characterization:

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Up to sign, $\forall\ h$, $\exists!$ self-dual harmonic 2-form ω:

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method also proves more general results.

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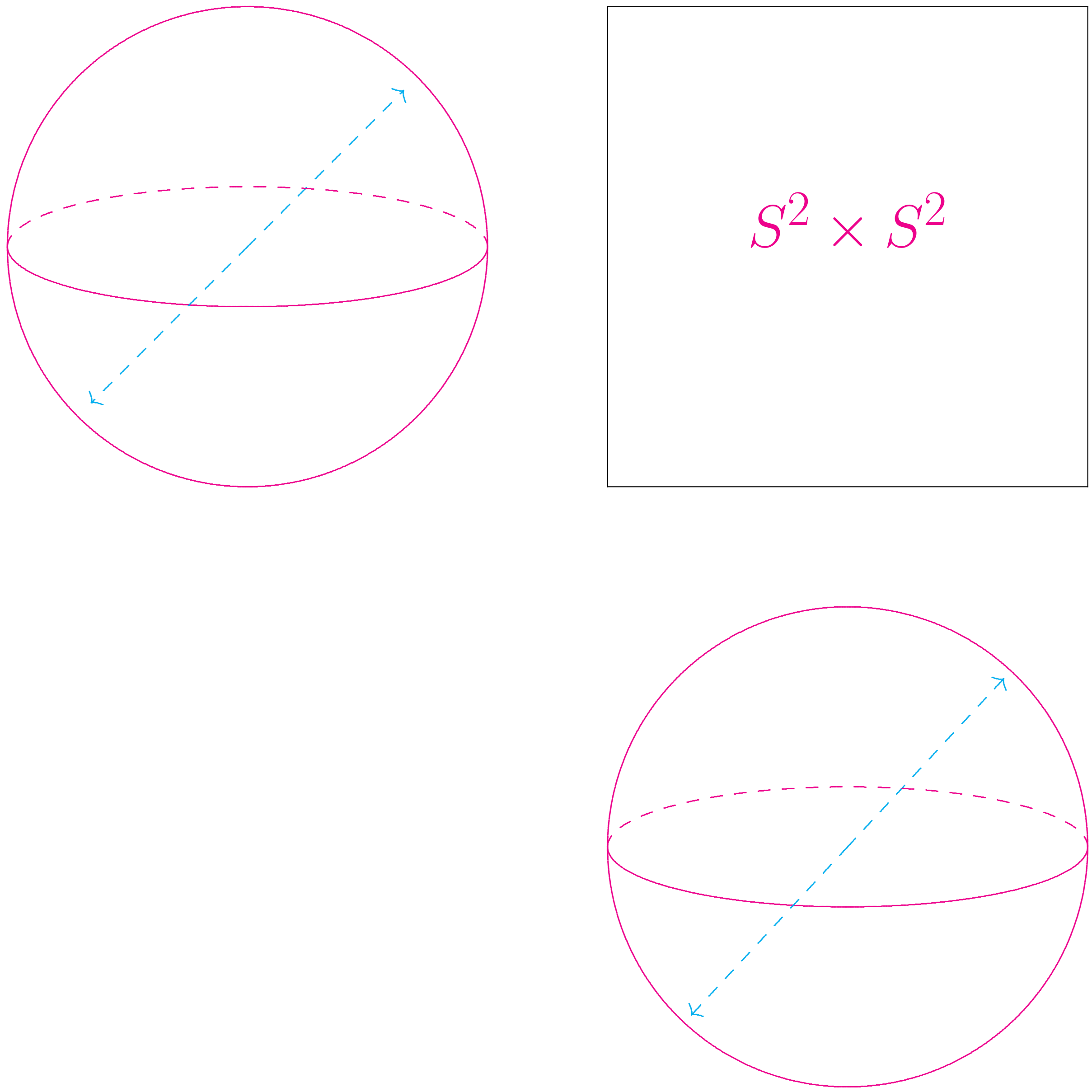
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However, this example is as bad as it gets. . .

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But actually more widely applicable!

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Corollary. *A smooth compact oriented* M^4 *with* $b_+(M) \neq 0$ *admits metrics with* $\delta W^+ := 0$ *and* $\det(W^+) > 0$ *if and only if it is diffeomorphic to a rational or ruled complex surface.*

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$$M \overset{diff}{\approx} \begin{cases} (\Sigma^2 \times S^2) \# k\overline{\mathbb{CP}}_2, & k \geq 0 \\ \Sigma^2 \varkappa S^2, \text{ or} & \\ \mathbb{CP}_2. & \end{cases}$$

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Corollary. *A smooth compact oriented M^4 with $b_+(M) \neq 0$ admits metrics with $\delta W^+ := 0$ and $\det(W^+) > 0$ if and only if it is diffeomorphic to a rational or ruled complex surface. When such metrics exist, their moduli space is always infinite dimensional.*

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Key to all this:

Correctly understanding equation $\delta W^+ = 0$.

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$$0 = \nabla^*\nabla W^+ + \frac{s}{2}W^+ - 6W^+ \circ W^+ + 2W^+|^2 I$$

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Derdziński: $\langle W^+, \underline{\quad} \rangle$ and integrate:

$$\int_M \det(W^+) d\mu = \frac{1}{36}\int_M \left[2|\nabla W^+|^2 + s|W^+|^2\right] d\mu$$

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provided $(M^4, h) \neq$ standard S^4 or $\overline{\mathbb{CP}}_2$!

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Wu's criterion:

Instead demand $\det(W^+) > 0$ everywhere.

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But we will follow a different path.

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with $\omega \otimes \omega$, and integrate by parts.

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We'll choose self-dual 2-form ω adapted to problem,

take L^2 inner product of the Weitzenböck formula

$$0 = \nabla^*\nabla(fW^+) + \frac{s}{2}fW^+ - 6fW^+ \circ W^+ + 2f|W^+|^2 I$$

with $\omega \otimes \omega$, and integrate by parts. This yields:

$$0 = \int_M \left[\langle W^+, \nabla^*\nabla(\omega\otimes\omega)\rangle + \frac{s}{2}W^+(\omega,\omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2\right] f\, d\mu$$

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holds whenever $h = f^2 g$ satisfies $\delta W^+ = 0$.

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This identity has many applications.

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Example. If $\exists$ harmonic ω with $W^+(\omega, \omega) > 0$,

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Example. If $\exists$ harmonic ω with $W^+(\omega, \omega) > 0$, then $\omega \neq 0$ everywhere.

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Example. If $\exists$ harmonic ω with $W^+(\omega, \omega) > 0$, then $\omega \neq 0$ everywhere. Choose $g = f^{-2} h$ so that

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This g is almost-Kähler.

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This g is almost-Kähler. Above identity becomes

$$0=\int_M \left(8|W^+|^2-sW^+(\omega,\omega)+4|W^+(\omega)^\perp|^2\right) f\, d\mu,$$

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and this eventually turns out to imply

$$0 \geq \int_M W^+(\omega, \omega) |\nabla \omega|^2 f \, d\mu,$$

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thus showing that g must actually be Kähler.

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So $\alpha = \alpha_h : M \to \mathbb{R}^+$ a smooth function. Set

$$f = {\alpha_h}^{-1/3}, \qquad g = f^{-2}h = {\alpha_h}^{2/3}h.$$

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Now choose $\omega \in \Gamma\Lambda^+$ so that

$$W_g^+(\omega) = \alpha\,\omega, \quad |\omega|_g \equiv \sqrt{2},$$

after at worst passing to double cover $\hat{M} \to M$.

$$0 = \int_{\hat{M}} \Big[\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle + \frac{s}{2} W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 2|W^+|^2 |\omega|^2 \Big] f \, d\mu$$

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because

$$W_g{}^+(\omega) = \alpha\omega$$

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$$0 \geq \int_M \Big[-2W^+(\nabla_e \omega, \nabla^e \omega) + 2\alpha\langle \omega, \nabla^*\nabla\omega\rangle$$
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because

$$|W_g{}^+|^2 \geq \frac{3}{2}\alpha^2$$

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$$|\omega|_g^2 = 2 \quad \Longrightarrow \quad (\nabla_e \omega) \perp \omega$$

$$\det(W^+) > 0 \quad \Longrightarrow \quad -W^+(\nabla_e\omega, \nabla^e\omega) \geq 0$$

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But

$$\alpha f \equiv 1$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle + \frac{s}{2} \alpha |\omega|^2 - 3|\omega|^2 \alpha \right] d\mu$$

$$0 \geq \int_M \left[2\langle \omega, \nabla^* \nabla \omega \rangle - 3W^+(\omega, \omega) + \frac{s}{2}|\omega|^2 \right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla \omega|^2 + \frac{3}{2}\langle \omega, \left(\nabla^*\nabla - 2W^+ + \frac{s}{3}\right)\omega\rangle\right] d\mu$$

$$0 \geq \int_M \left[\frac{1}{2}|\nabla\omega|^2 + \frac{3}{2}\langle\omega, \quad (d+d^*)^2 \quad \omega\rangle\right] d\mu$$

Because

$$(d+d^*)^2 = \nabla^*\nabla - 2W^+ + \frac{s}{3}$$

on $\Gamma\Lambda^+$.

$$0 \geq \frac{1}{2} \int_M |\nabla \omega|^2 \, d\mu + 3 \int_M |d\omega|^2 \, d\mu$$

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So $\nabla\omega \equiv 0$, and g is Kähler!

Theorem B. *Let* (M, h) *be a compact oriented Riemannian* 4*-manifold with harmonic self-dual Weyl curvature:*

$$\delta W^+ := -\nabla \cdot W^+ = 0.$$

Suppose that $b_+(M) \neq 0$, *and that* h *satisfies* $\det(W^+) > 0$ *at every point of* M. *Then* M *admits an orientation-compatible Kähler metric* g *of scalar curvature* $s > 0$ *such that* $h = s^{-2} g$.

Theorem A. *Let* (M, h) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W^+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W^+) > 0$$

at every point of M*. Then* h *is conformal to an orientation-compatible Bach-flat extremal Kähler metric* g *with scalar curvature* $s > 0$ *on* M*.*

Proof can also be made to work just assuming

$$\beta \leq \frac{1}{4}\alpha \neq 0.$$

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Produces harmonic ω with $W^+(\omega, \omega) > 0$.

Now use my earlier result!

Theorem C. *Let* (M, h) *be a compact oriented Riemannian* 4*-manifold with* $\delta W^+ = 0$*. If*

$$W^+ \neq 0 \quad \textit{and} \quad \det(W^+) \geq -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3$$

everywhere on M*, then actually* $\det(W^+) > 0$*. Thus, after at worst passing to a double cover* $\hat{M} \to M$*,* h *becomes conformally Kähler, in the manner described by* **Theorem B***. In particular, if* (M, h) *is a simply-connected Einstein manifold, it actually falls under the purview of* **Theorem A***.*