Curvature Functionals,

Kähler Metrics, &

the Geometry of 4-Manifolds V

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IHP, December 7, 2012

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$$c_1(M^4, J) = \lambda[\omega].$$

Moreover, this metric is unique, up to isometry, if $\lambda \neq 0$.

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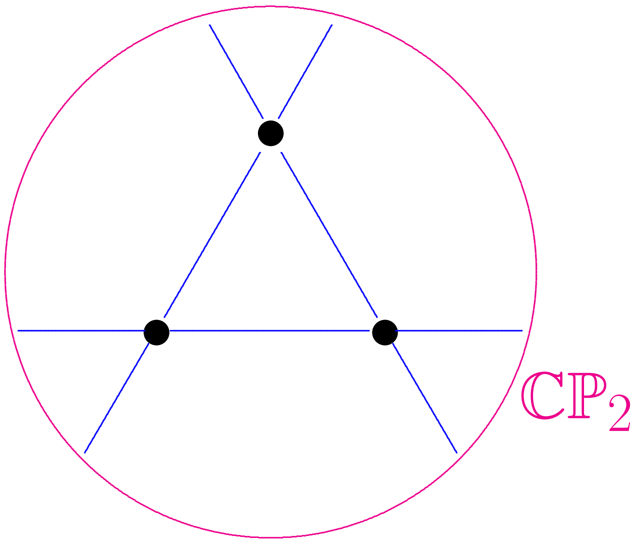
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



Theorem. *Any Del Pezzo surface* (M^4, J) *admits an Einstein metric* h *which is conformal to a* J*-compatible Kähler metric* g*. In particular, this Einstein metric* h *is Hermitian with respect to* J*.*

Rough strategy of proof:

Find Kähler metric which minimizes

$$g \mapsto \int_M s^2 d\mu_g$$

among all Kähler metrics g.

Here s = scalar curvature.

Note that Kähler class $[\omega]$ of g allowed to vary!

Corresponding problem with $[\omega]$ fixed:
Calabi's extremal Kähler metrics.
So minimize among extremal Kähler metrics.

Minimizer g has $s > 0$.

Einstein metric is $h = s^{-2}g$.

Theorem A. *There is a conformally Kähler, Einstein metric h on $M = \mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$ for which the conformally related Kähler g minimizes the functional*

$$g \longmapsto \int_M s^2 d\mu_g$$

among all Kähler metrics on M. Consequently, h is an absolute minimizer of the functional

$$h \longmapsto \int_M |W|_h^2 \, d\mu_h.$$

among all conformally Kähler metrics on M.

Theorem B. *This minimizing Kähler metric* g *on* $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$ *is conformal to an Einstein metric. Moreover, there is a* 1*-parameter family*

$$[0,1) \ni t \longmapsto g_t$$

of extremal Kähler metrics on $\mathbb{CP}_2 \# 3\overline{\mathbb{CP}}_2$ *s.t.*

- g_0 *is Kähler-Einstein, and such that*
- $g_{t_j} \to g$ *in the Gromov-Hausdorff sense*

for some $t_j \nearrow 1$.

$$0 = 12B = s\mathring{r} + 2\mathrm{Hess}_0(s)$$

$\Longrightarrow$ the conformal rescaling $h = s^{-2}g$ is Einstein courtesy of transformation rule

$$\mathring{r}(u^2 g) = \mathring{r}(g) + (n-2)u\mathrm{Hess}_0(u^{-1}) \ .$$

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Lemma. *For any extremal Kähler g on any Del Pezzo M, scalar curvature $s > 0$ everywhere.*

Explicit lower bound:

Any Kähler (M^4, g, J) satisfies

$$\frac{1}{32\pi^2}\int s^2 d\mu_g \geq \mathcal{A}([\omega])$$

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$$\mathcal{A}([\omega]) := \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2$$

where $\mathcal{F}$ is Futaki invariant.

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Lemma. *For all* $[\omega]$ *on any Del Pezzo* M*,*

$$\mathcal{B}([\omega]) < \frac{1}{4}$$

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Then there is an extremal Kähler metric g on M with Kähler form $\omega \in [\omega]$.

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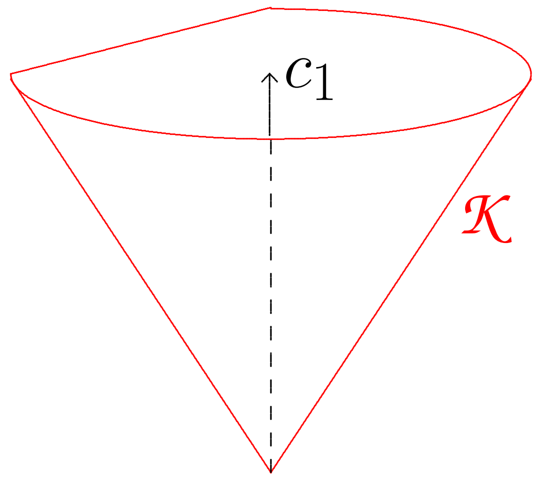
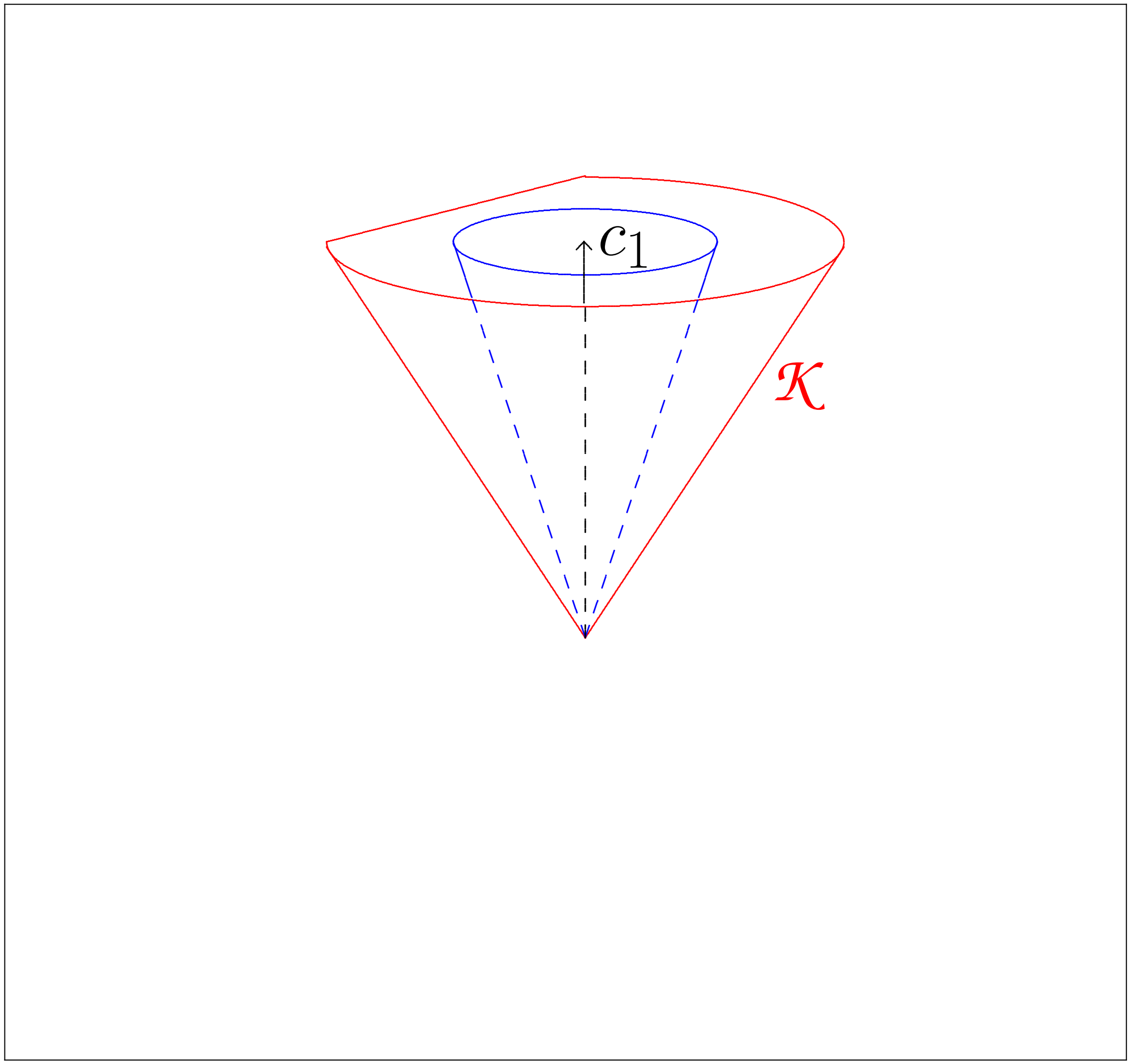
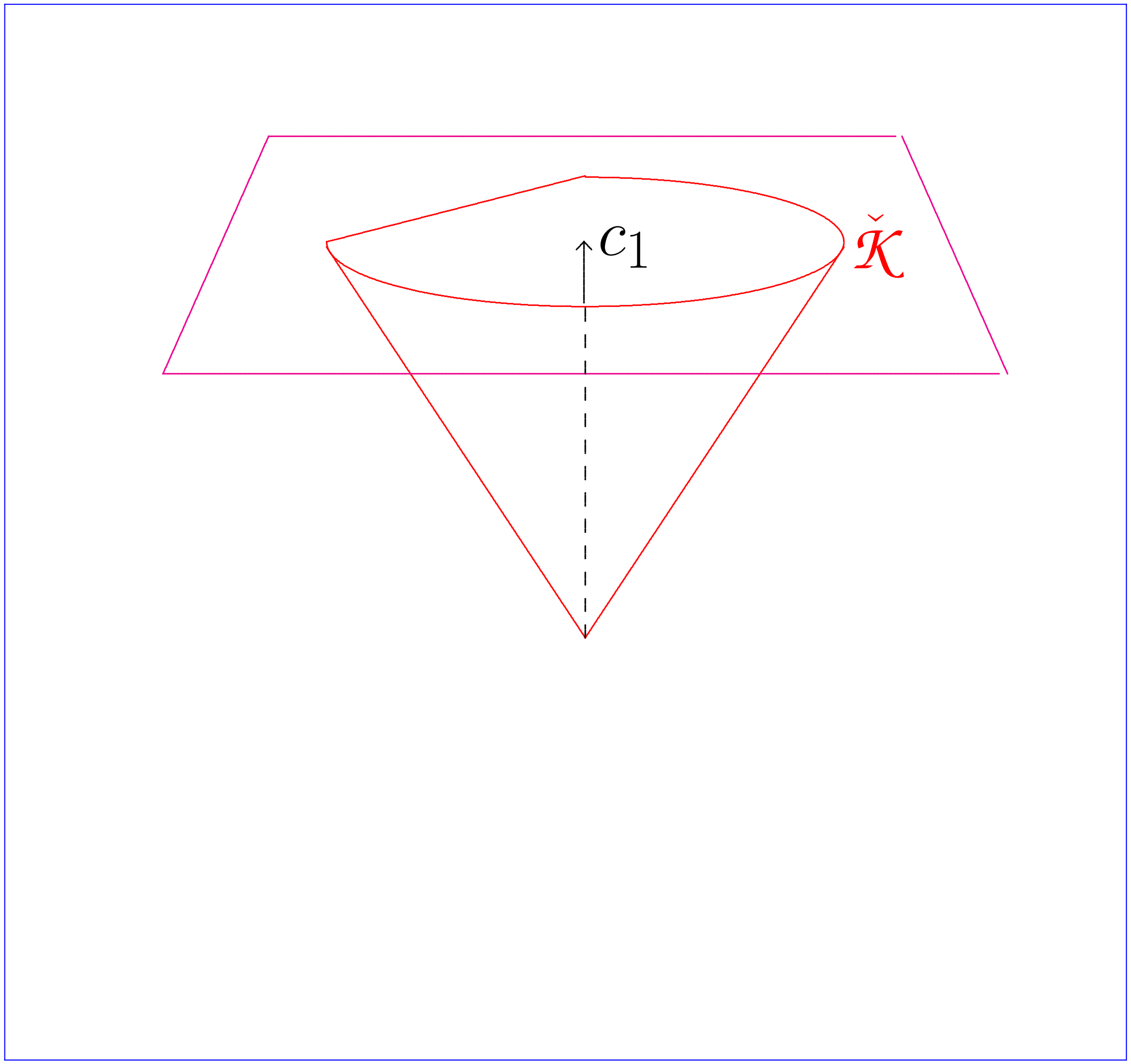


$$H^{1,1}(M, \mathbb{R}) = H^2(M, \mathbb{R})$$



$$\mathcal{T}([\omega]) = \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} \leq const$$



$$\check{\mathcal{K}} = \mathcal{K}/\mathbb{R}^+$$

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Generalizes work of Anderson/Tian-Viaclovsky

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Recall:

$$(2\chi+3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

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have Sobolev bound on convex cone

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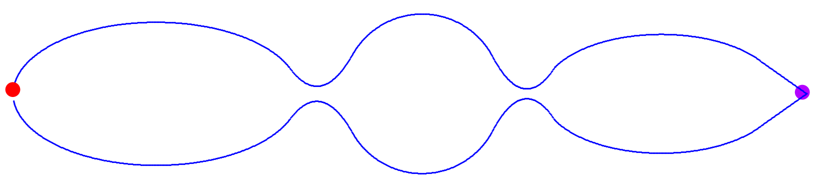

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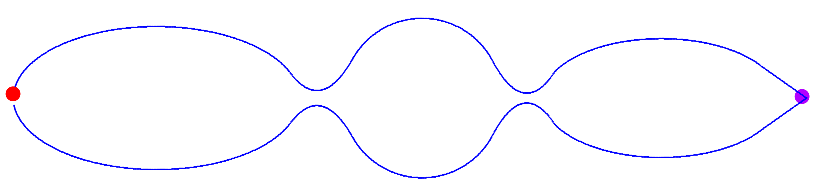

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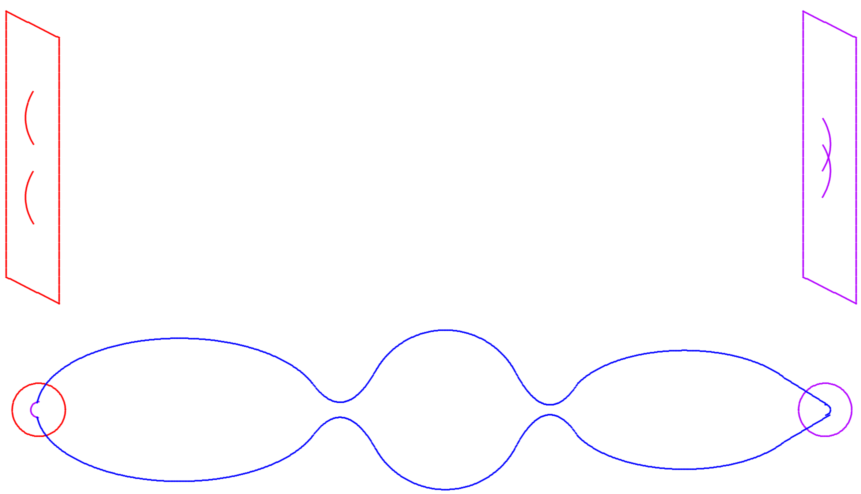

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Suggests continuity method...

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Difficulty: rule out deepest bubbles.

Lemma. *If M is toric, any deepest bubble (X, g_∞) must be toric, too, with $H_2(X, \mathbb{R}) \neq 0$ generated by holomorphic $\mathbb{CP}_1$'s.*

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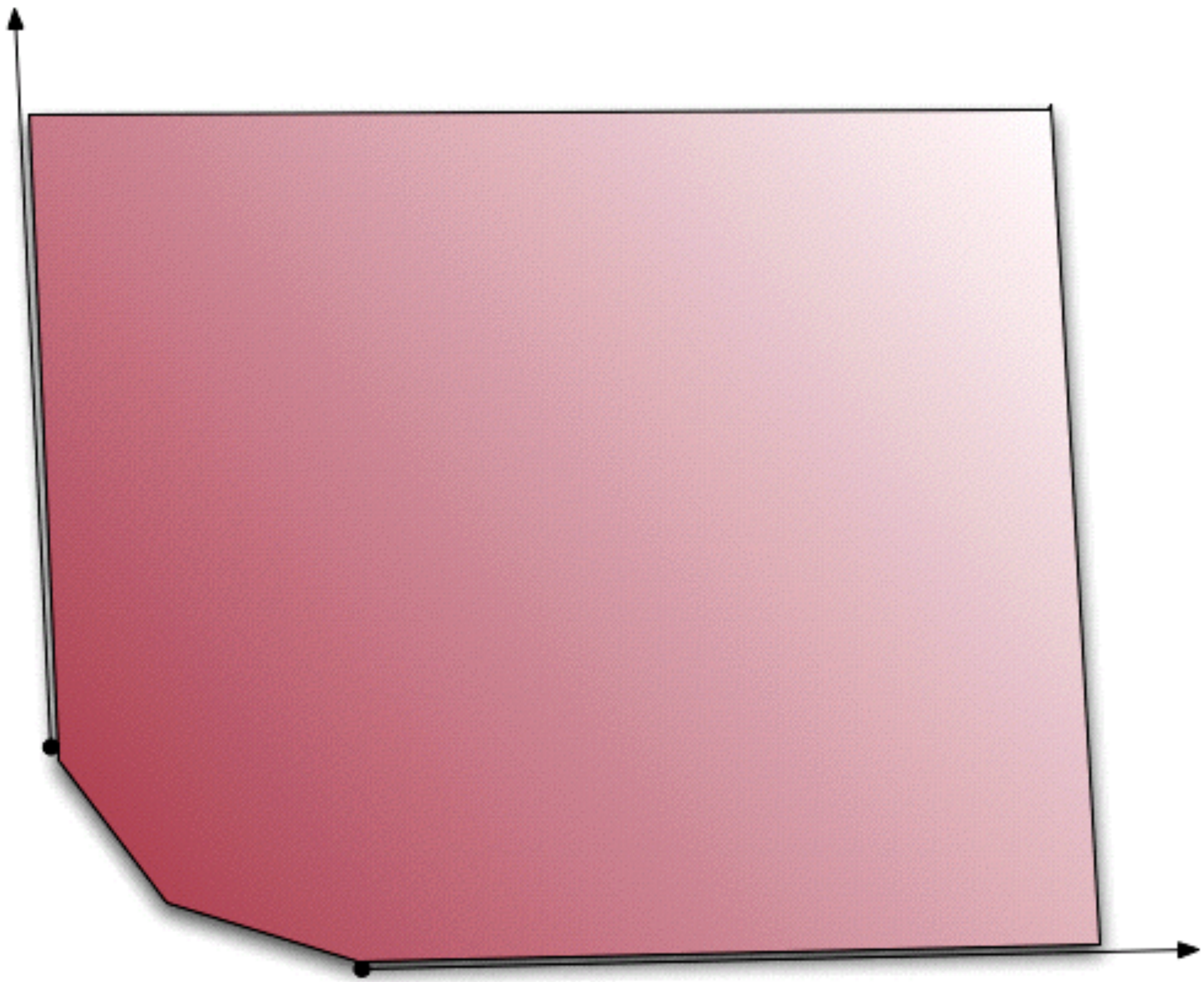
Moment map profile:



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Now suppose that $c_1 \in \mathcal{K}$ on toric Del Pezzo M represented by extremal Kähler metric. Suppose that $[\omega]$ satisfies

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$$\begin{aligned} [\omega_{t_j}] \cdot [S] &> 0 \\ [S] \cdot [S] &= -k < 0 \\ c_1 \cdot [S] &= 2 - k \\ \Omega \cdot [S] &= 0 \end{aligned}$$

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$$\begin{aligned}
(u_j c_1 + v_j \Omega) \cdot [S] &> 0, \quad \exists u_j, v_j > 0 \\
[S] \cdot [S] &= -k < 0 \\
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\end{aligned}$$

Now suppose that $c_1 \in \mathcal{K}$ on toric Del Pezzo M represented by extremal Kähler metric. Suppose that $[\omega]$ satisfies

$$\mathcal{T}([\omega]) \leq \frac{3}{2}{c_1}^2 - \frac{1}{4}$$

Consider line segment

$$[\omega_t] = (1-t)c_1 + t[\omega]$$

of Kähler classes, and suppose extremal metric exists for $t \in [0, \mathfrak{t})$. If bubbling occurred for $t_j \nearrow \mathfrak{t}$, then, setting $\Omega = [\omega_{\mathfrak{t}}]$, would have $[S] \in H_2(M, \mathbb{Z})$ with

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It follows that bubbling off cannot occur!

Theorem 2. *Let $M = \mathbb{CP}_2 \# 3\overline{\mathbb{CP}_2}$ be the blow-up of $\mathbb{CP}_2$ at three non-collinear points, and let $[\omega]$ be a Kähler class on M for which*

$$\mathcal{T}([\omega]) := \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} \leq \frac{3}{2}{c_1}^2 - \frac{1}{4} = {c_1}^2 + 2.75.$$

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Then there is an extremal Kähler metric g *on* M *with Kähler form* $\omega \in [\omega]$.

Also works when approaching boundary of Kähler cone, but can bubble off (-1)-curves.

Theorem 3. *Let Ω be any Kähler class on $\mathbb{CP}_2\#2\overline{\mathbb{CP}_2}$ for which*

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- g_0 *is Kähler-Einstein, and such that*
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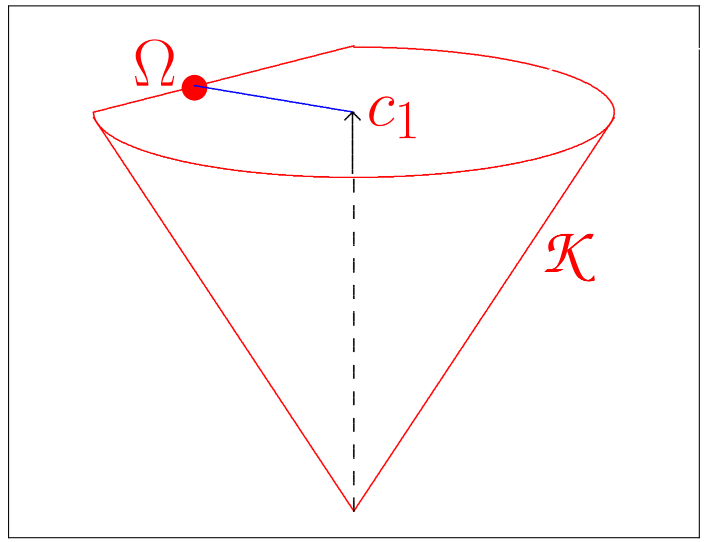

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Uniform bound $\mathcal{B}([\omega]) < 1/4$ now implies that

$$\mathcal{A} = \mathcal{T} + \mathcal{B}$$

has minimizer $[\omega]$ represented by conformally Einstein Kähler metric.

Theorem C. *Let* (M^4, J) *be a compact complex surface, and suppose that* h *is an Einstein metric on* M *which is Hermitian with respect to* J*:*

$$h(J\cdot, J\cdot) = h.$$

Then either

- (M, J, h) *is Kähler-Einstein; or*
- $M \approx \mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$*, and* h *is a constant times the Page metric; or*
- $M \approx \mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$ *and* h *is a constant times the Chen-LeBrun-Weber metric.*

Uniqueness:

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Exceptional cases: $\mathbb{CP}_2$ blown up at 1 or 2 points.

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Warning: when h is non-Kähler, its relation to ω is surprisingly complicated!

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In other words,

$$h = fg$$

$\exists$ Kähler metric g, smooth function $f : M \to \mathbb{R}^+$.

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Similarly for $S^{2n+1} \times S^{2m+1}$.

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Actually, g must be an extremal Kähler metric.

May normalize so that either $f = s^{-2}$ or $f = 1$.

Calabi:

Extremal Kähler metrics = critical points of

$$g \mapsto \int_M s^2 d\mu_g$$

where $g = g_\omega$ for J and $[\omega] \in H^2(M, \mathbb{R})$ fixed.

Euler-Lagrange equations $\Longleftrightarrow$

$\nabla^{1,0} s$ is a holomorphic vector field.

Donaldson/Mabuchi/Chen-Tian:
unique in Kähler class, modulo bihomorphisms.

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature *(conformally invariant)*

W_- = anti-self-dual Weyl curvature ″

Kähler case:

$$\Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^{-}$$

$$\Lambda^{+} = \mathbb{R}\omega \oplus \Re e(\Lambda^{2,0})$$

$$\nabla J = 0 \Longrightarrow \mathcal{R} \in \mathrm{End}(\Lambda^{1,1}) \Longrightarrow$$

$$W_{+} = \begin{pmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{pmatrix}$$

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Notice that W_{+} has a repeated eigenvalue.

Lemma. *Let* (M^4, J) *be a compact complex surface, and suppose that* h *is an Einstein metric on* M *which is Hermitian with respect to* J*:*

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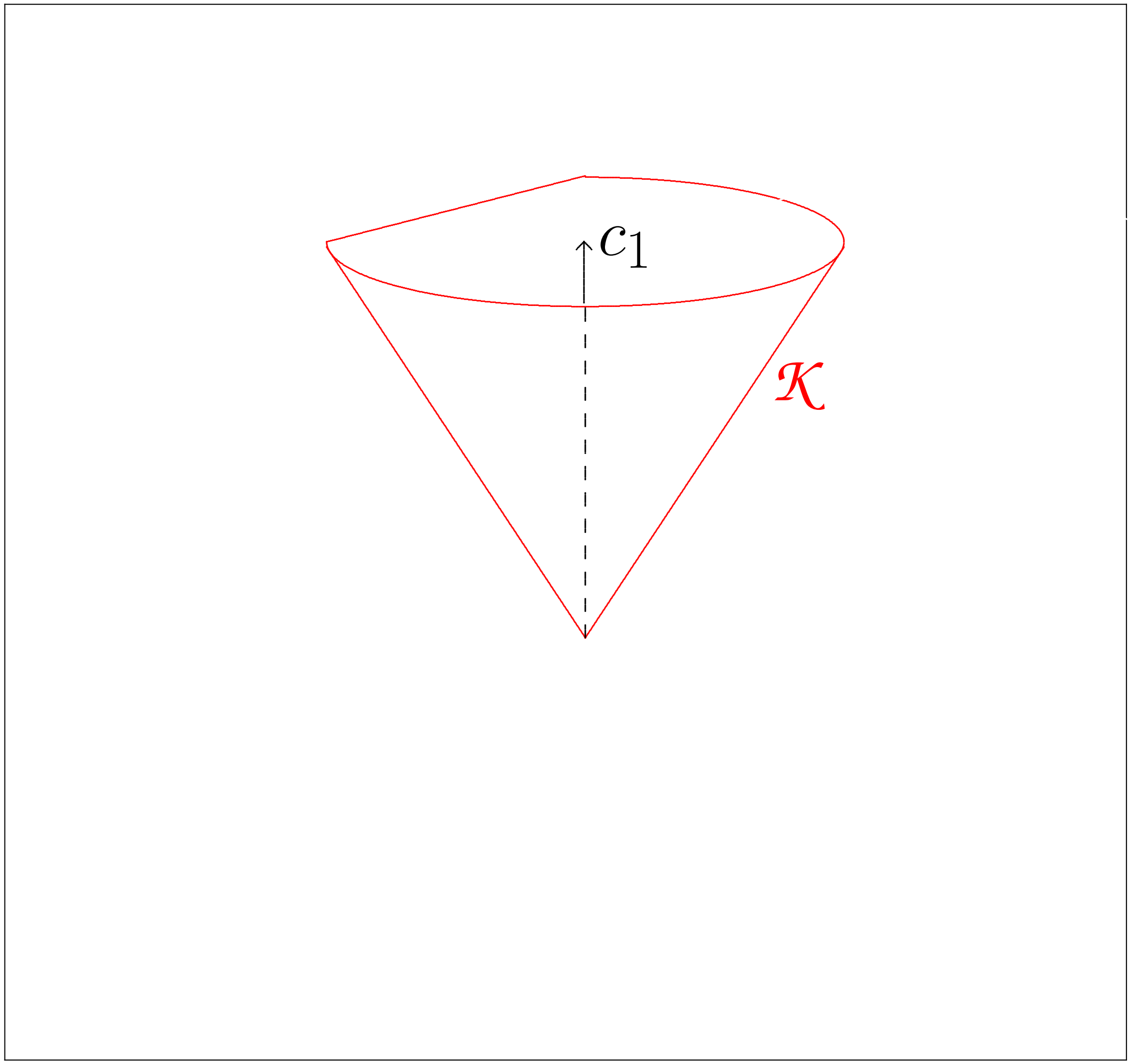

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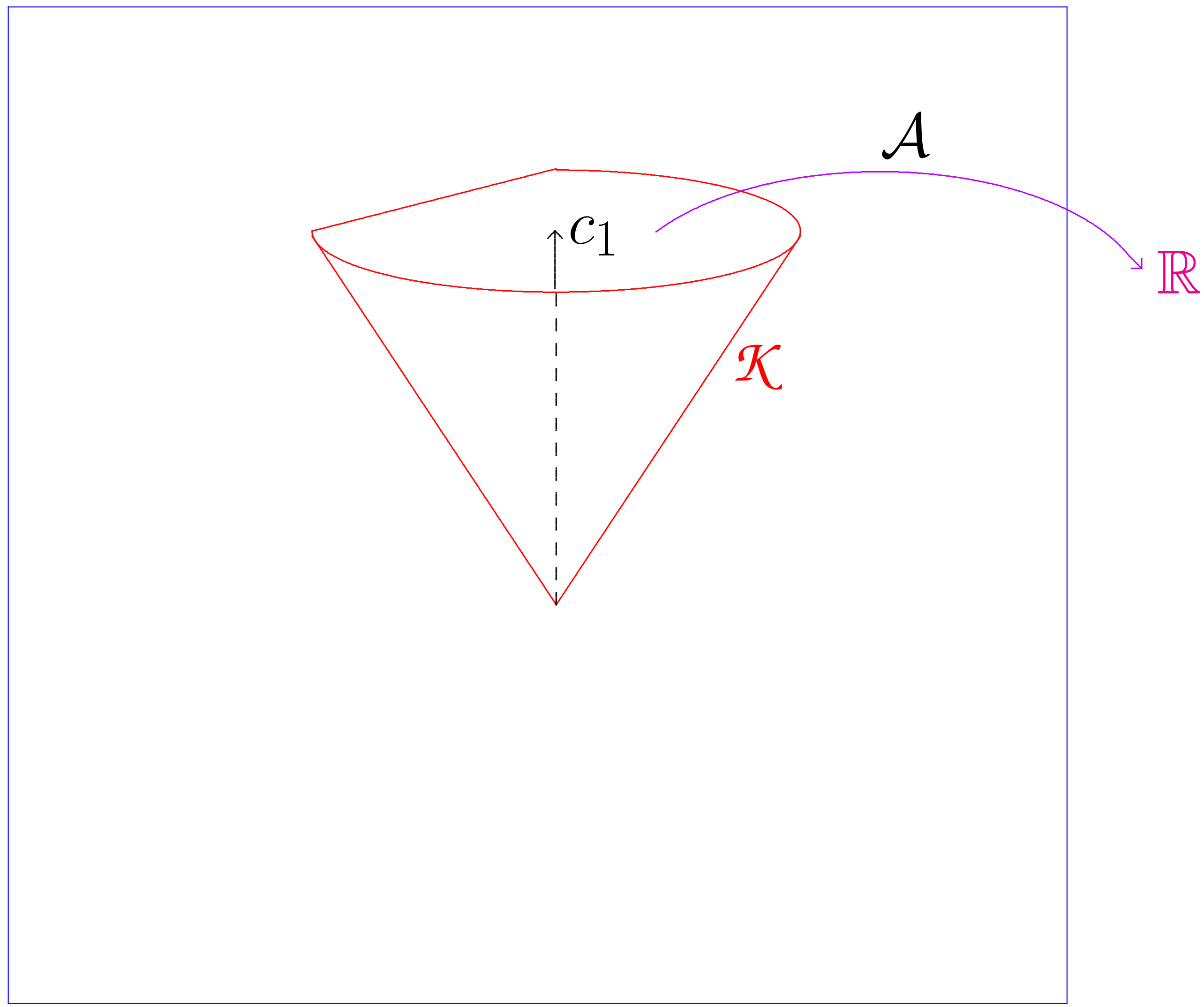
(M Del Pezzo)



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$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.

Lemma. *If g is a Kähler metric on a compact complex surface (M^4, J), with Kähler class $[\omega]$, then g satisfies $B = 0 \Longleftrightarrow$*

- *g is an extremal Kähler metric; and*

Action Function on Kähler Cone

For any extremal Kähler (M^4, g, J),

$$\frac{1}{32\pi^2}\int s^2 d\mu_g = \frac{(c_1\cdot[\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2$$
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$$0 = 12s^{-1}B = \mathring{r} + 2s^{-1}\mathrm{Hess}_0(s)$$

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$$\rho + 2i\partial\bar{\partial}\log s > 0.$$

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Hermitian, Einstein metric then given by

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and uniqueness Theorem A follows.

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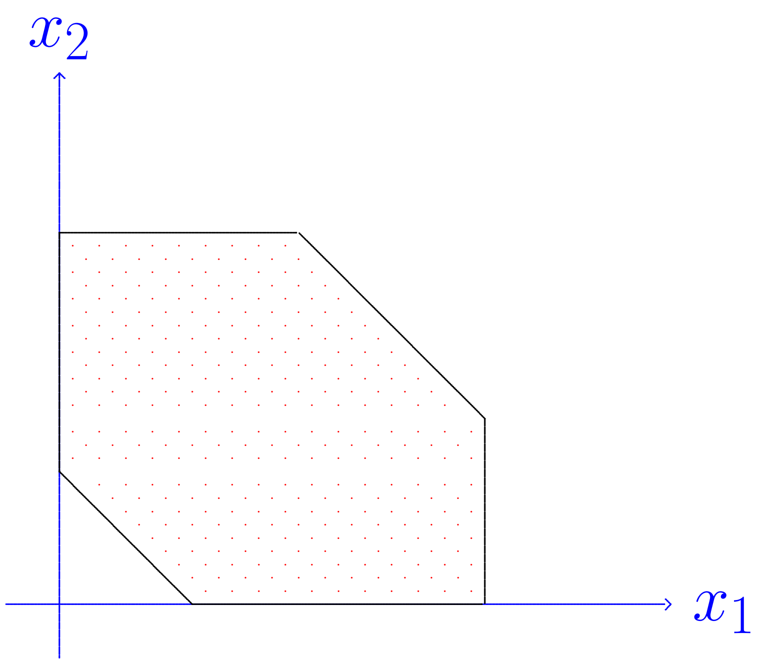
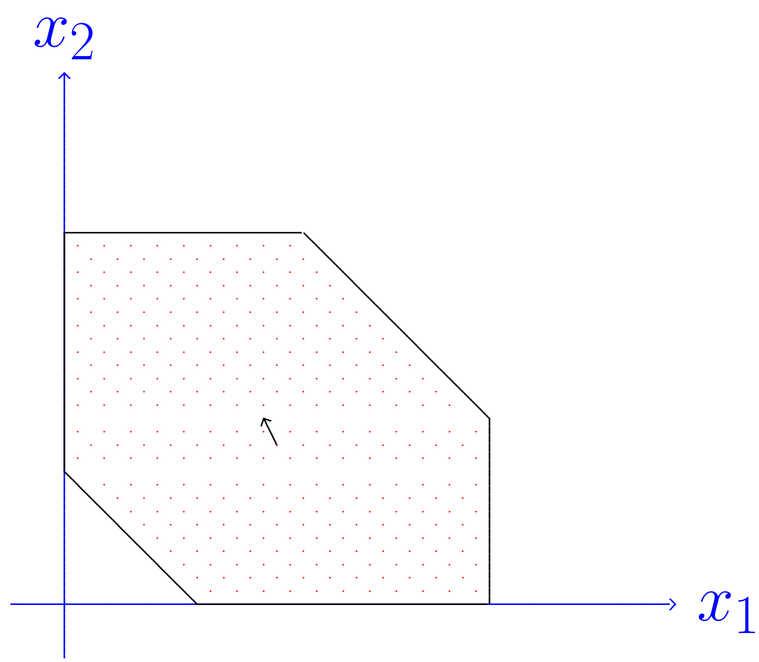
Only three cases are non-trivial:

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \qquad k = 1, 2, 3.$$

The non-trivial cases are toric, and the action $\mathcal{A}$ can be directly computed from moment polygon.



The non-trivial cases are toric, and the action $\mathcal{A}$ can be directly computed from moment polygon. Formula involves barycenters, moments of inertia.



$$\mathcal{A}([\omega]) = \frac{|\partial P|^2}{2}\left(\frac{1}{|P|} + \vec{\mathfrak{D}} \cdot \Pi^{-1}\vec{\mathfrak{D}}\right)$$

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$\mathcal{A}$ is explicit rational function —

$$3[3+28\gamma+96\gamma^2+168\gamma^3+164\gamma^4+80\gamma^5+16\gamma^6+16\beta^6(1+\gamma)^4+16\alpha^6(1+\beta+\gamma)^4+16\beta^5(5+24\gamma+43\gamma^2+37\gamma^3+15\gamma^4+2\gamma^5)+4\beta^4(41+228\gamma+478\gamma^2+496\gamma^3+263\gamma^4+$$

$$60\gamma^5+4\gamma^6)+8\beta^3(21+135\gamma+326\gamma^2+392\gamma^3+248\gamma^4+74\gamma^5+8\gamma^6)+4\beta(7+58\gamma+176\gamma^2+270\gamma^3+228\gamma^4+96\gamma^5+16\gamma^6)+4\beta^2(24+176\gamma+479\gamma^2+652\gamma^3+478\gamma^4+$$

$$172\gamma^5+24\gamma^6)+16\alpha^5(5+2\beta^5+24\gamma+43\gamma^2+37\gamma^3+15\gamma^4+2\gamma^5+\beta^4(15+14\gamma)+\beta^3(37+70\gamma+30\gamma^2)+\beta^2(43+123\gamma+108\gamma^2+30\gamma^3)+\beta(24+92\gamma+123\gamma^2+70\gamma^3+$$

$$14\gamma^4))+4\alpha^4(41+4\beta^6+228\gamma+478\gamma^2+496\gamma^3+263\gamma^4+60\gamma^5+4\gamma^6+\beta^5(60+56\gamma)+\beta^4(263+476\gamma+196\gamma^2)+8\beta^3(62+169\gamma+139\gamma^2+35\gamma^3)+2\beta^2(239+876\gamma+1089\gamma^2+$$

$$556\gamma^3+98\gamma^4)+4\beta(57+263\gamma+438\gamma^2+338\gamma^3+119\gamma^4+14\gamma^5))+8\alpha^3(21+135\gamma+326\gamma^2+392\gamma^3+248\gamma^4+74\gamma^5+8\gamma^6+8\beta^6(1+\gamma)+2\beta^5(37+70\gamma+30\gamma^2)+4\beta^4(62+$$

$$169\gamma+139\gamma^2+35\gamma^3)+4\beta^3(98+353\gamma+428\gamma^2+210\gamma^3+35\gamma^4)+2\beta^2(163+735\gamma+1179\gamma^2+856\gamma^3+278\gamma^4+30\gamma^5)+\beta(135+736\gamma+1470\gamma^2+1412\gamma^3+676\gamma^4+140\gamma^5+$$

$$8\gamma^6))+4\alpha(7+58\gamma+176\gamma^2+270\gamma^3+228\gamma^4+96\gamma^5+16\gamma^6+16\beta^6(1+\gamma)^3+4\beta^5(24+92\gamma+123\gamma^2+70\gamma^3+14\gamma^4)+4\beta^4(57+263\gamma+438\gamma^2+338\gamma^3+119\gamma^4+14\gamma^5)+$$

$$2\beta^3(135+736\gamma+1470\gamma^2+1412\gamma^3+676\gamma^4+140\gamma^5+8\gamma^6)+4\beta^2(44+278\gamma+645\gamma^2+735\gamma^3+438\gamma^4+123\gamma^5+12\gamma^6)+2\beta(29+210\gamma+556\gamma^2+736\gamma^3+526\gamma^4+184\gamma^5+$$

$$24\gamma^6))+4\alpha^2(24+176\gamma+479\gamma^2+652\gamma^3+478\gamma^4+172\gamma^5+24\gamma^6+24\beta^6(1+\gamma)^2+4\beta^5(43+123\gamma+108\gamma^2+30\gamma^3)+2\beta^4(239+876\gamma+1089\gamma^2+556\gamma^3+98\gamma^4)+4\beta^3(163+$$

$$735\gamma+1179\gamma^2+856\gamma^3+278\gamma^4+30\gamma^5)+4\beta(44+278\gamma+645\gamma^2+735\gamma^3+438\gamma^4+123\gamma^5+12\gamma^6)+\beta^2(479+2580\gamma+5058\gamma^2+4716\gamma^3+2178\gamma^4+432\gamma^5+24\gamma^6))]/$$

$$[1+10\gamma+36\gamma^2+64\gamma^3+60\gamma^4+24\gamma^5+24\beta^5(1+\gamma)^5+24\alpha^5(1+\beta+\gamma)^5+12\beta^4(1+\gamma)^2(5+20\gamma+23\gamma^2+10\gamma^3)+16\beta^3(4+28\gamma+72\gamma^2+90\gamma^3+57\gamma^4+15\gamma^5)+$$

$$12\beta^2(3+24\gamma+69\gamma^2+96\gamma^3+68\gamma^4+20\gamma^5)+2\beta(5+45\gamma+144\gamma^2+224\gamma^3+180\gamma^4+60\gamma^5)+12\alpha^4(1+\beta+\gamma)^2(5+20\gamma+23\gamma^2+10\gamma^3+10\beta^3(1+\gamma)+\beta^2(23+46\gamma+$$

$$16\gamma^2)+2\beta(10+30\gamma+23\gamma^2+5\gamma^3))+16\alpha^3(4+28\gamma+72\gamma^2+90\gamma^3+57\gamma^4+15\gamma^5+15\beta^5(1+\gamma)^2+3\beta^4(19+57\gamma+50\gamma^2+13\gamma^3)+3\beta^3(30+120\gamma+155\gamma^2+78\gamma^3+$$

$$13\gamma^4)+3\beta^2(24+120\gamma+206\gamma^2+155\gamma^3+50\gamma^4+5\gamma^5)+\beta(28+168\gamma+360\gamma^2+360\gamma^3+171\gamma^4+30\gamma^5))+12\alpha^2(3+24\gamma+69\gamma^2+96\gamma^3+68\gamma^4+20\gamma^5+20\beta^5(1+\gamma)^3+$$

$$\beta^4(68+272\gamma+366\gamma^2+200\gamma^3+36\gamma^4)+4\beta^3(24+120\gamma+206\gamma^2+155\gamma^3+50\gamma^4+5\gamma^5)+2\beta(12+84\gamma+207\gamma^2+240\gamma^3+136\gamma^4+30\gamma^5)+\beta^2(69+414\gamma+864\gamma^2+$$

$$824\gamma^3+366\gamma^4+60\gamma^5))+2\alpha(5+45\gamma+144\gamma^2+224\gamma^3+180\gamma^4+60\gamma^5+60\beta^5(1+\gamma)^4+12\beta^4(15+75\gamma+136\gamma^2+114\gamma^3+43\gamma^4+5\gamma^5)+12\beta^2(12+84\gamma+207\gamma^2+$$

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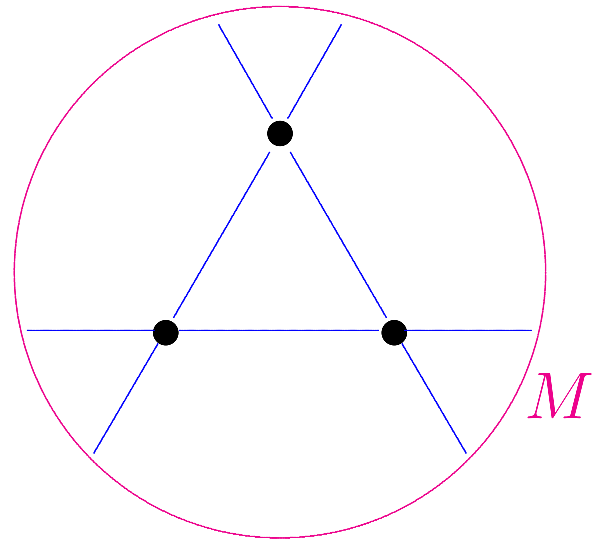
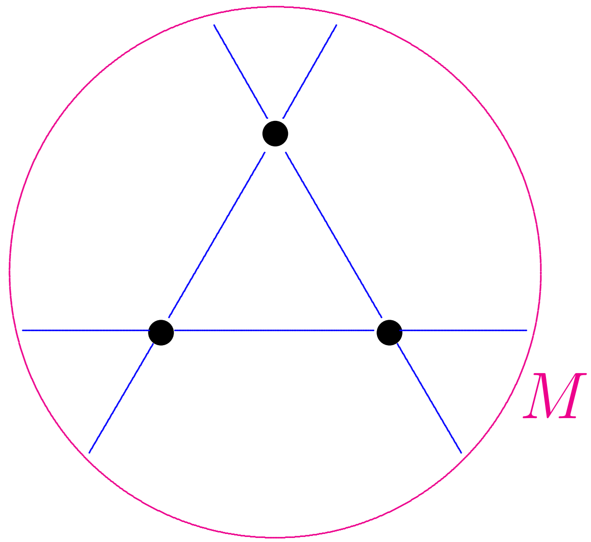

M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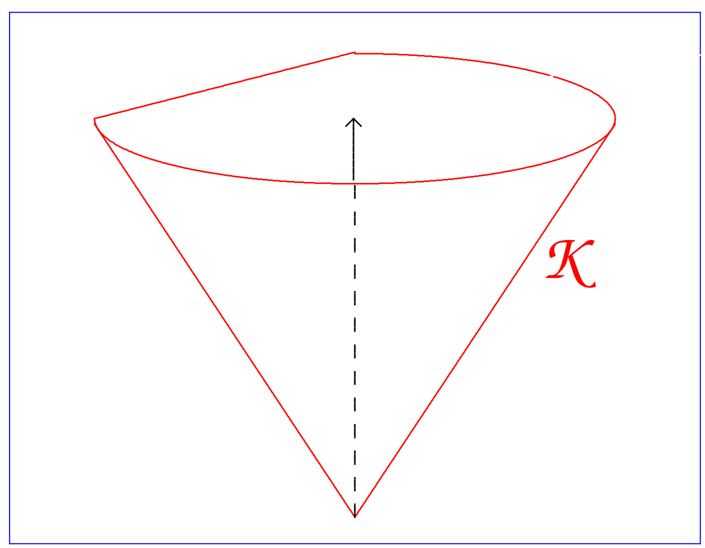
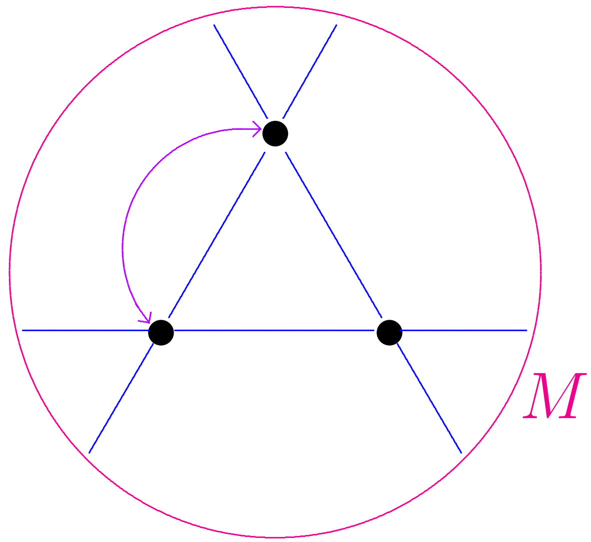
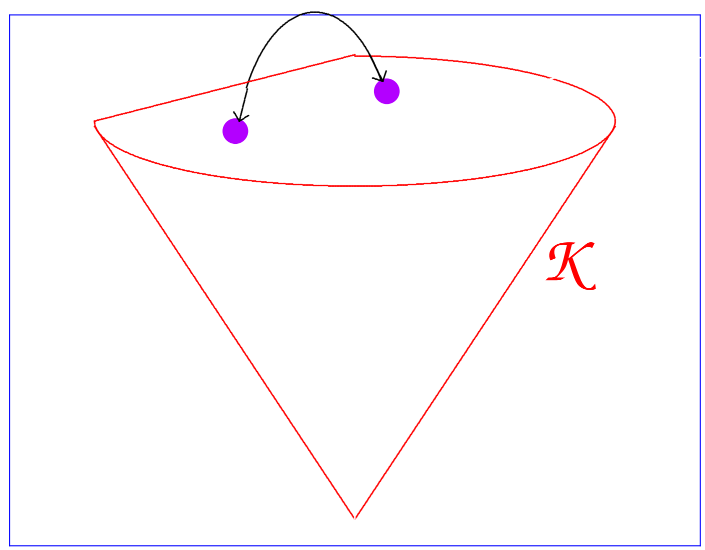
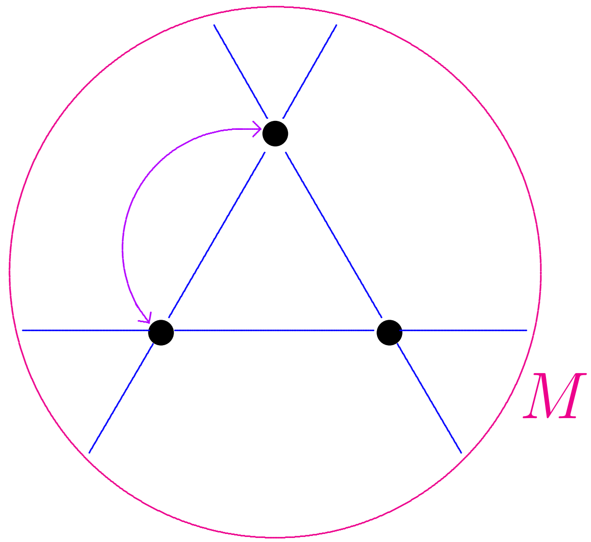
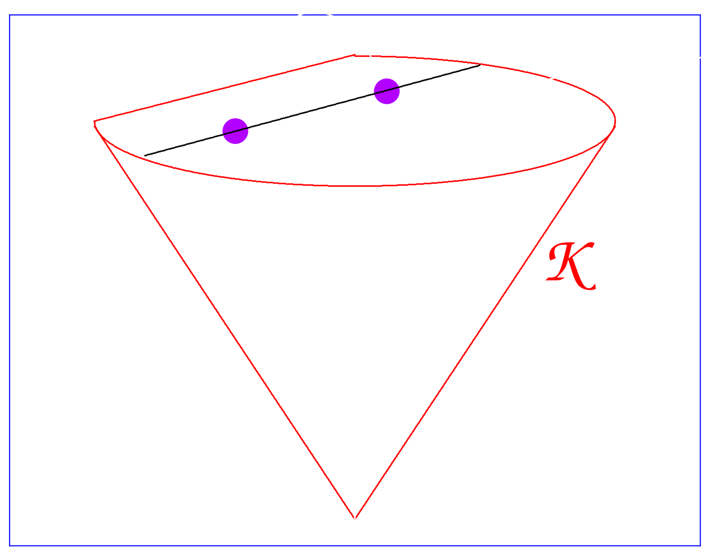

$\mathcal{K}$


M

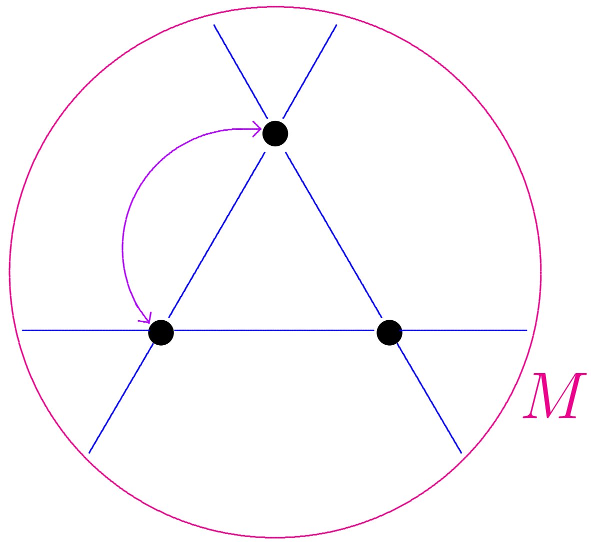

𝒦



M



$\mathcal{K}$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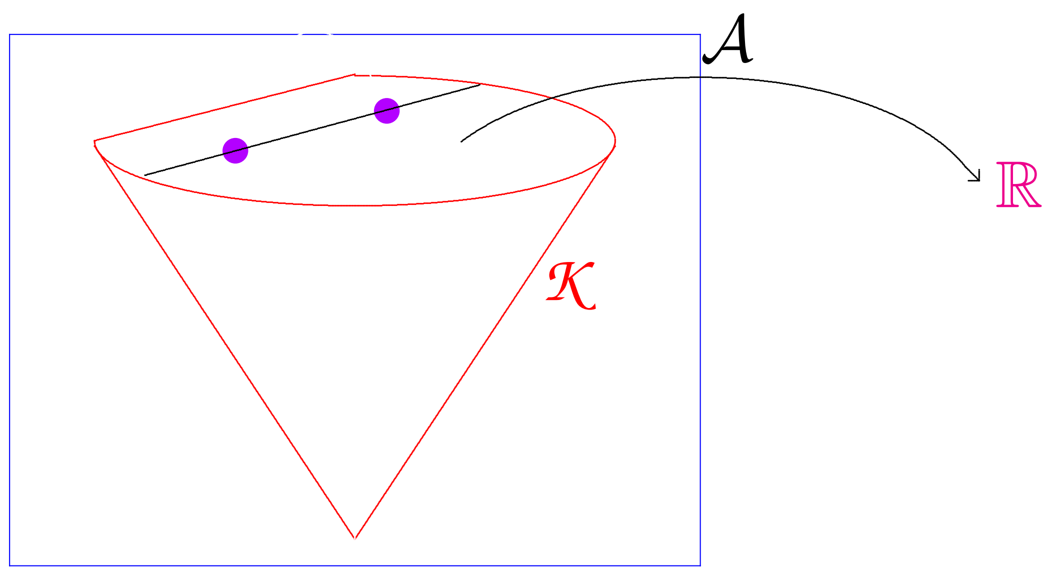
$\mathcal{A}$
$\mathbb{R}$
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Final step then just calculus in one variable. . .

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Proposition. *Modulo rescalings and biholomorphisms, there is only one conformally Kähler, Einstein metric* h *on* $M = \mathbb{CP}_2 \# 3\overline{\mathbb{CP}}_2$. *This metric is actually Kähler-Einstein, and is exactly the metric discovered by Siu.*

Theorem E. *Suppose that M is a smooth compact oriented 4-manifold which admits an integrable complex structure J. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \approx \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \\ \\ \\ \\ \end{cases}$$

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Same conclusion holds in symplectic case.

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Question. *Are there any non-minimal complex surfaces M of general type which actually admit Einstein metrics?*

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If so, quite different from Kähler-Einstein metrics!

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Question. *Are the constructed Einstein metrics on rational surfaces the only ones? For example, are there non-standard Einstein metrics on* $S^2 \times S^2$ *or on* $\mathbb{CP}_2$*?*

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Question. *When a* 4*-manifold* M *admits a Kähler-Einstein metric* g *with* $s > 0$, **Gursky** *has shown that, among all metrics with* $s > 0$, *it is a minimizes* $\int |W|^2 d\mu$, *and that the only minimizers are other K-E metrics. Are the Page and C-L-W metrics similarly minimizers for this problem? What happens if we consider metrics which do not have* $s > 0$*?*

End, Part V