Mass, Kähler Manifolds, &

Symplectic Geometry

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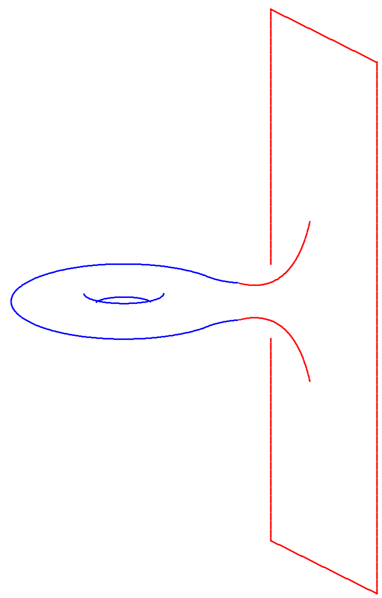
Builds on previous paper

Mass in Kähler Geometry
Comm. Math. Phys. 347 (2016) 621–653.

(Joint with Hans-Joachim Hein)

Definition. *A complete, non-compact Riemannian n-manifold* (M^n, g)



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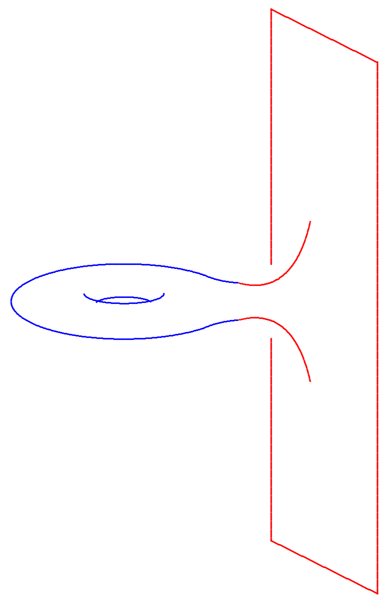
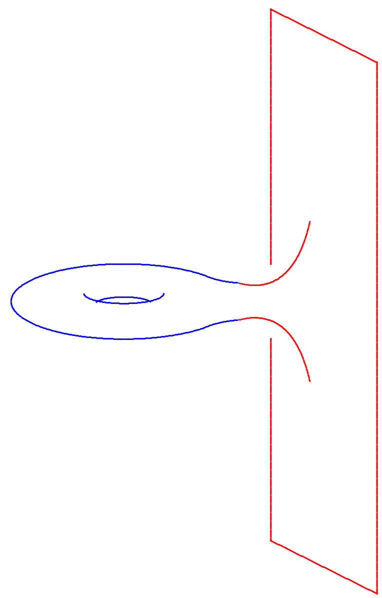


$$\boxed{n \geq 3}$$

Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g) *is called* asymptotically Euclidean



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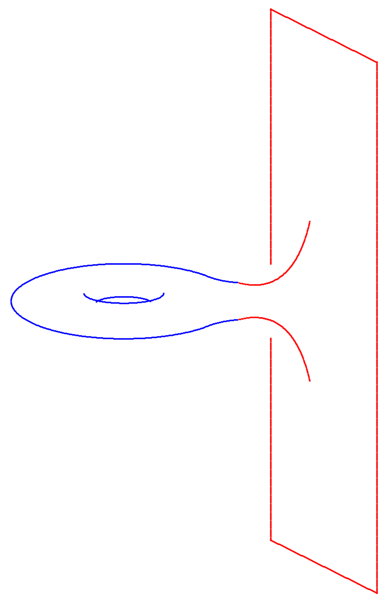

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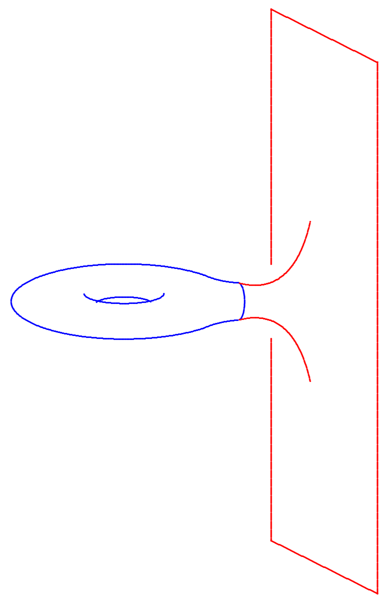
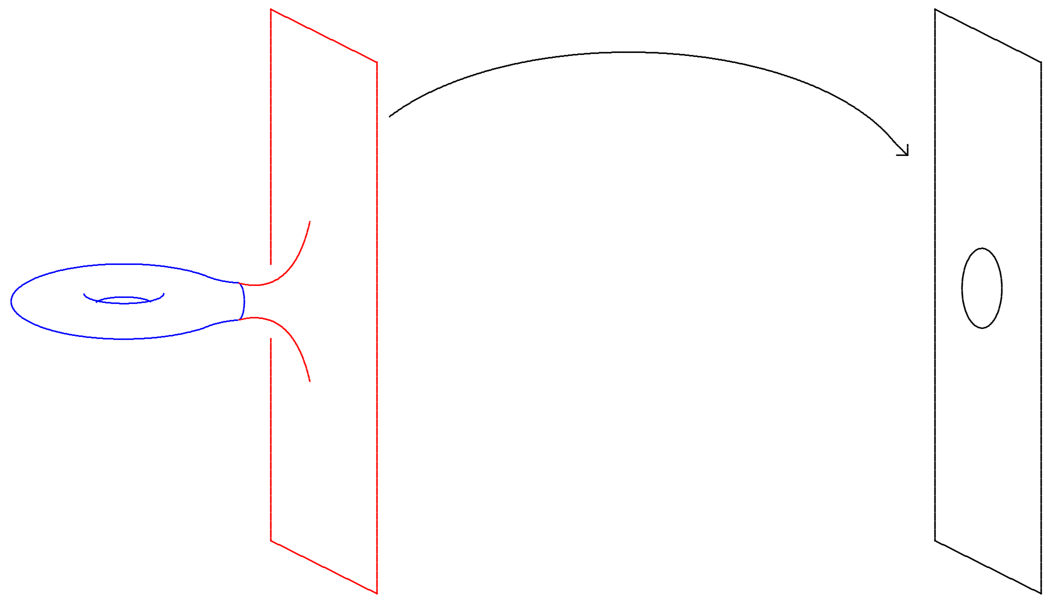


Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g) *is called* asymptotically Euclidean *(AE) if there is a compact set* $K \subset M$ *such that* $M - K$ *is diffeomorphic to* $\mathbb{R}^n - D^n$



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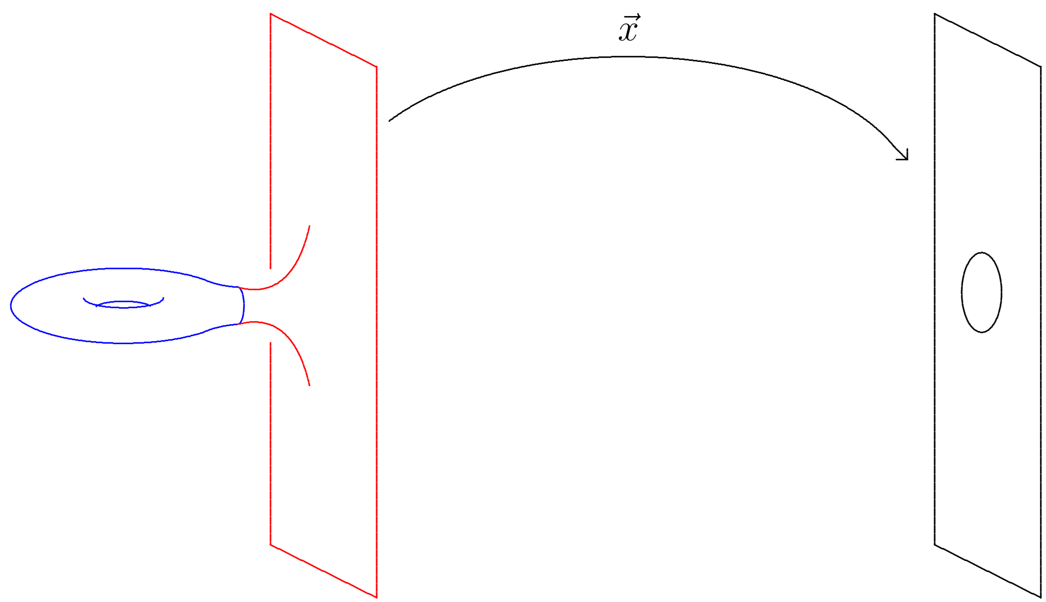

$$g_{jk} = \delta_{jk} + \text{terms that fall-off at infinity}$$

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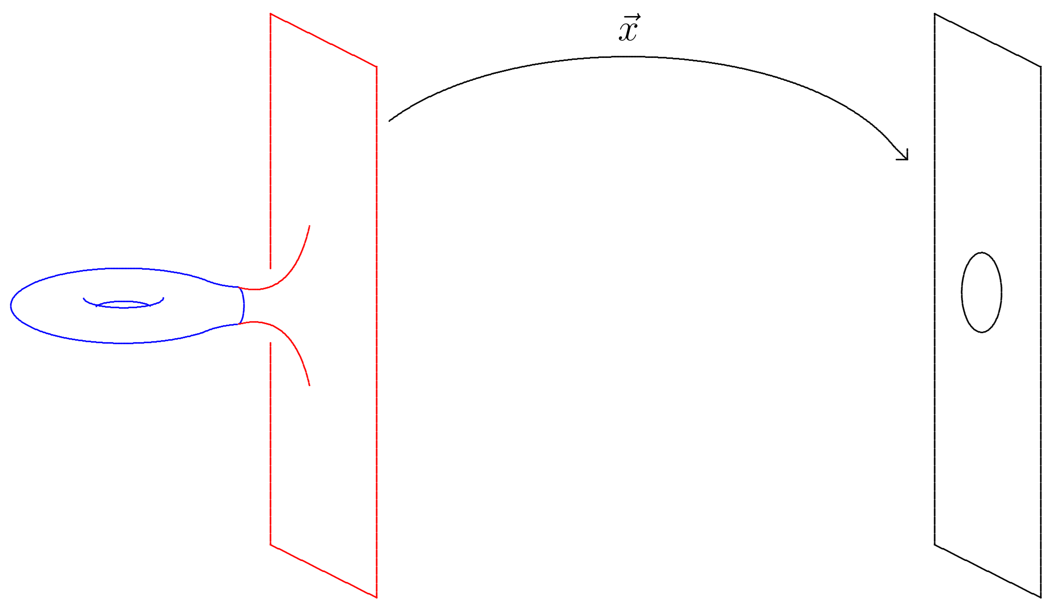

Weakest reasonable assumption:

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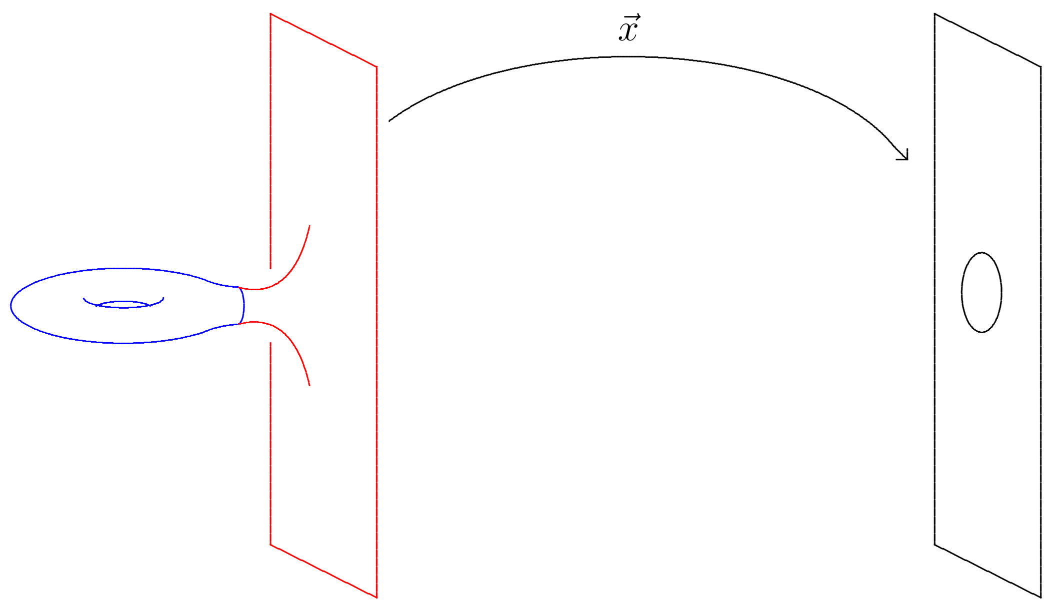

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Chruściel-type fall-off:

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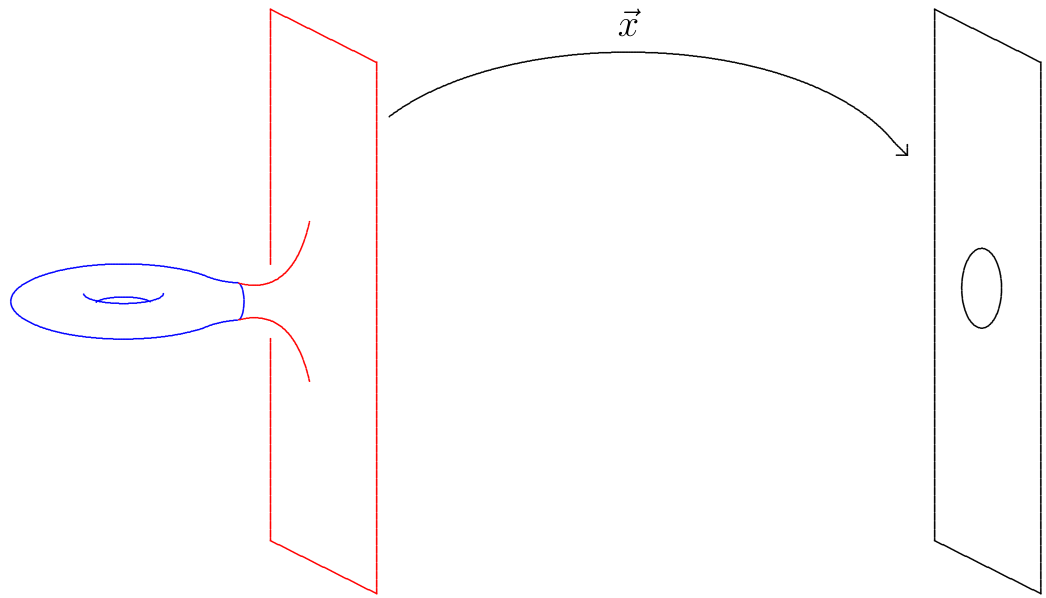

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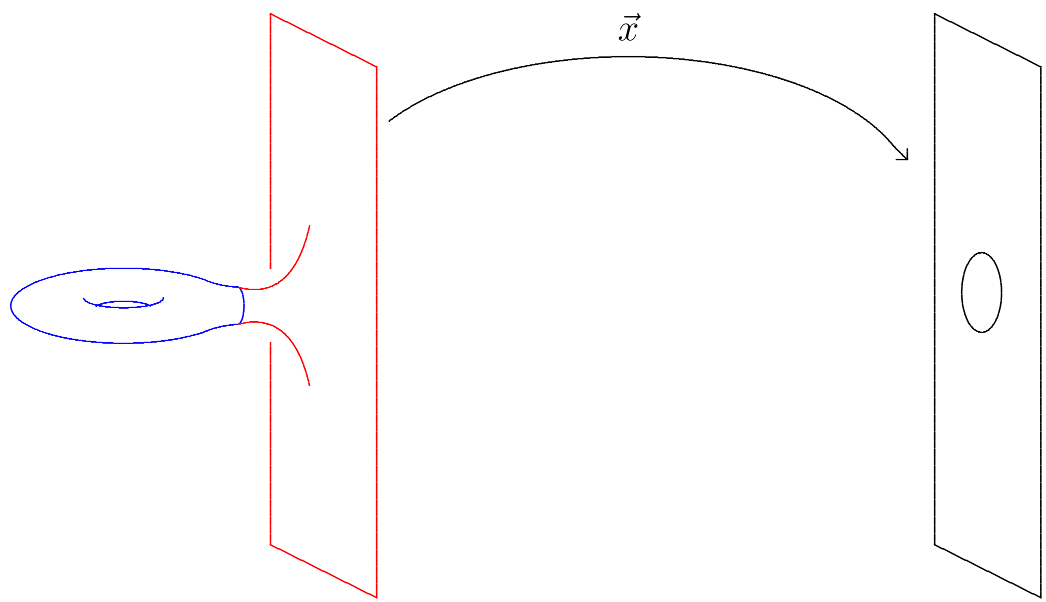

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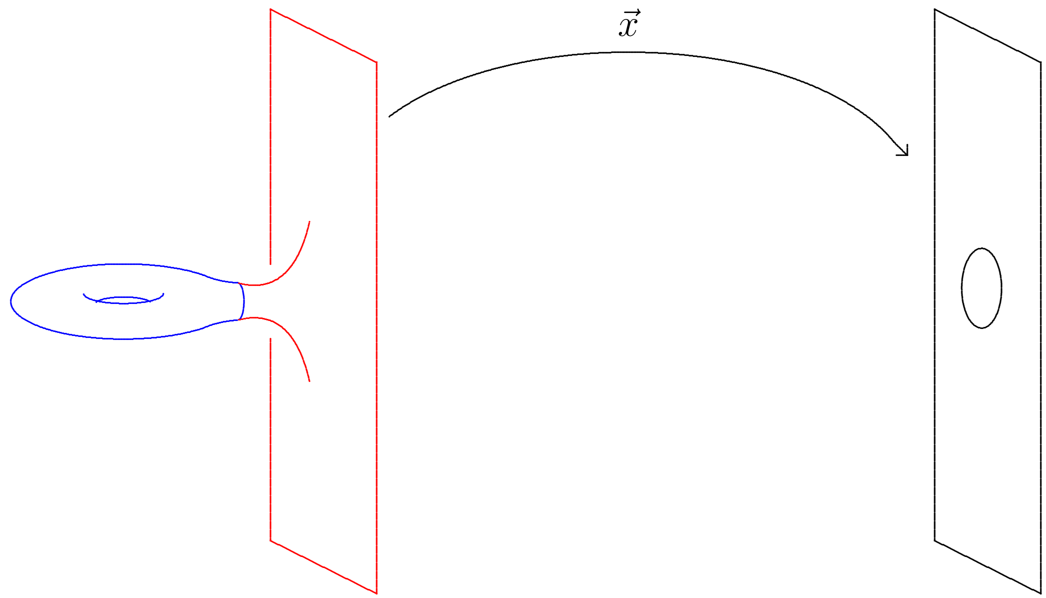

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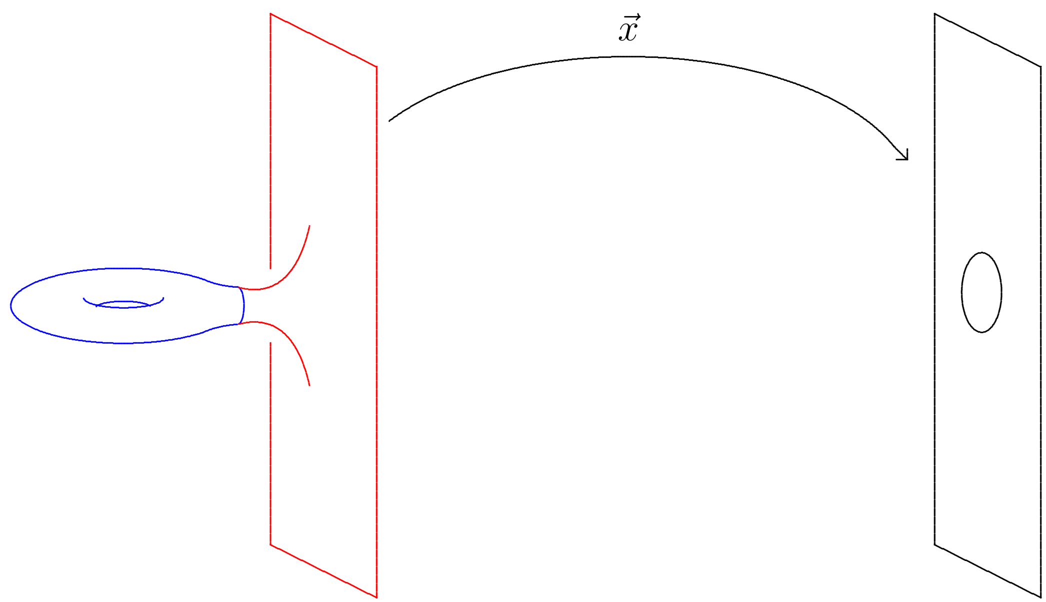

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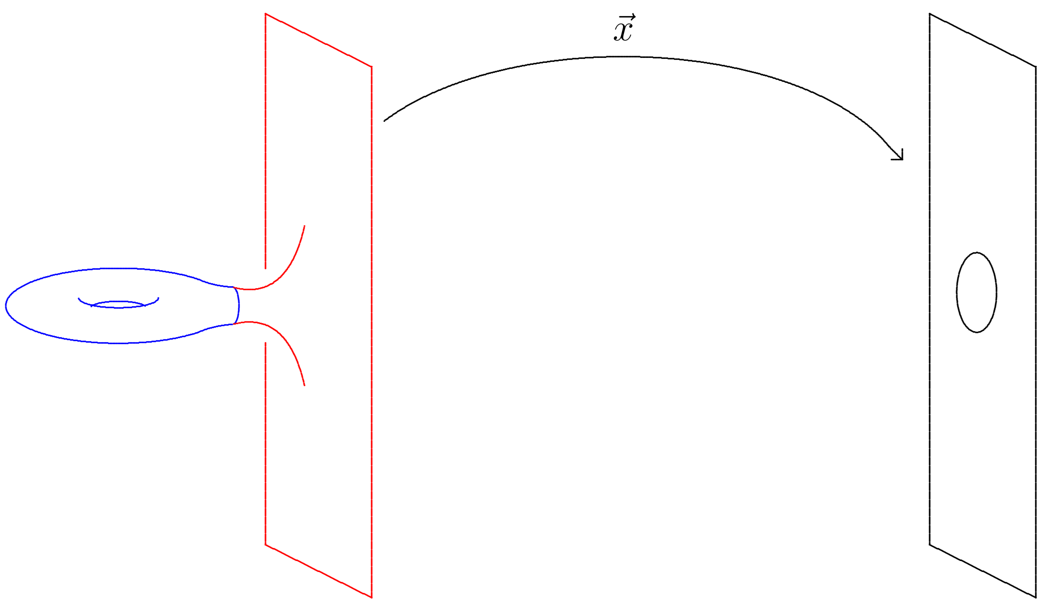

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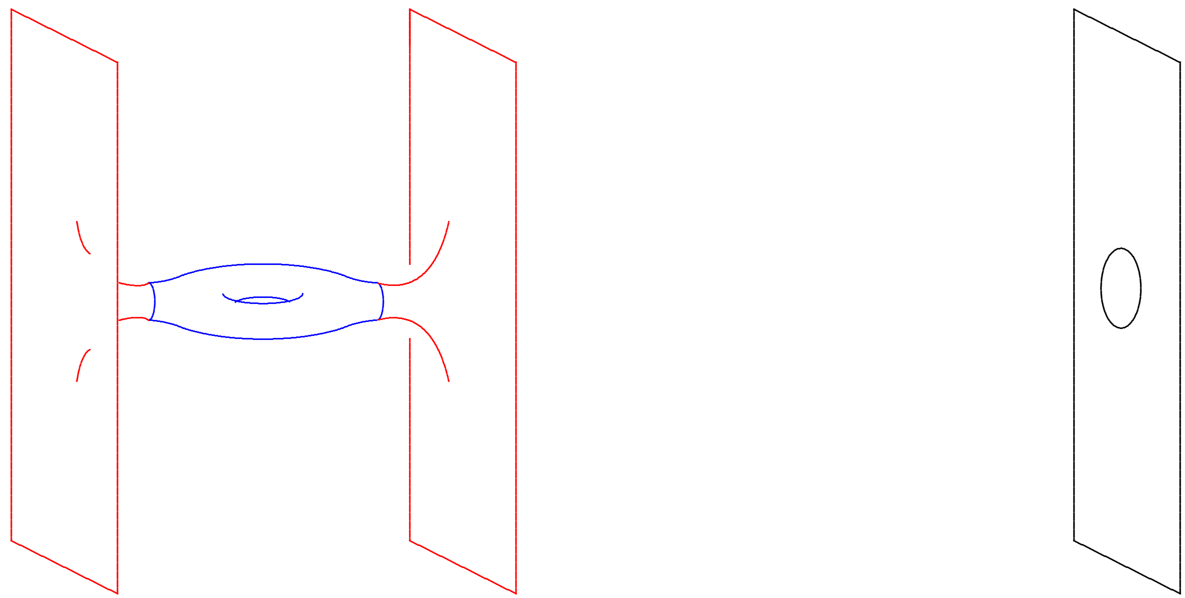

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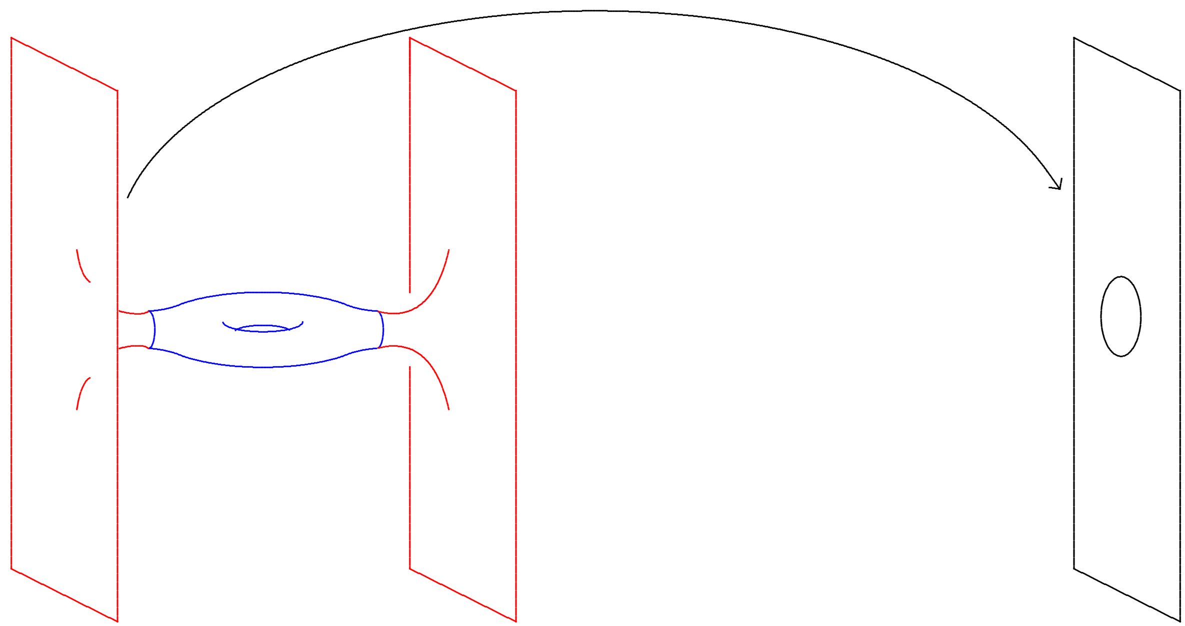

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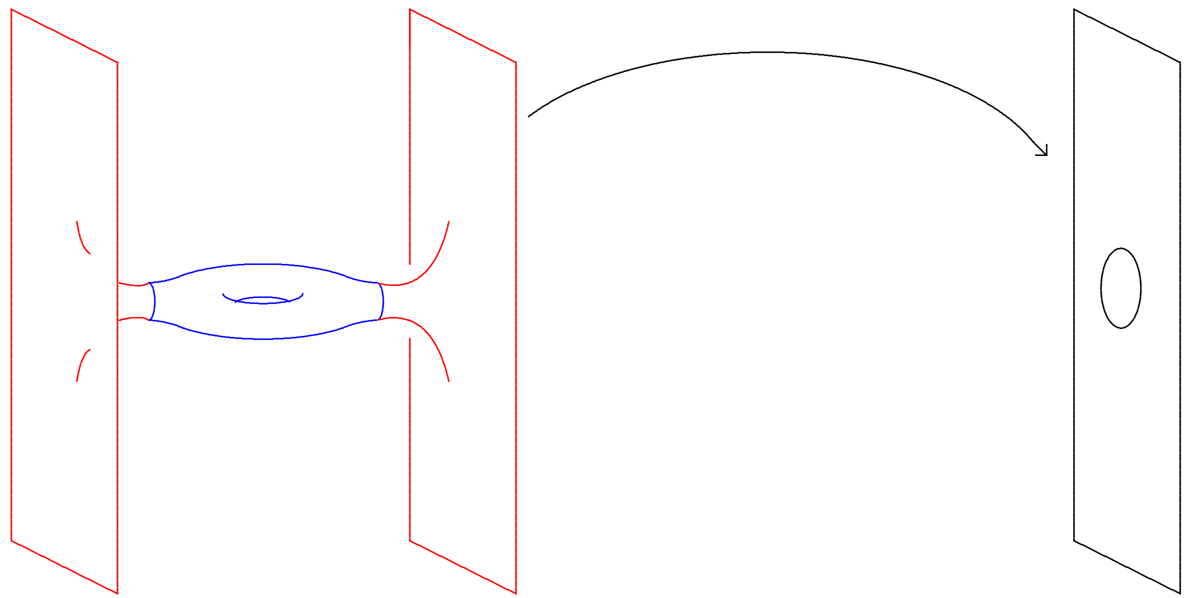

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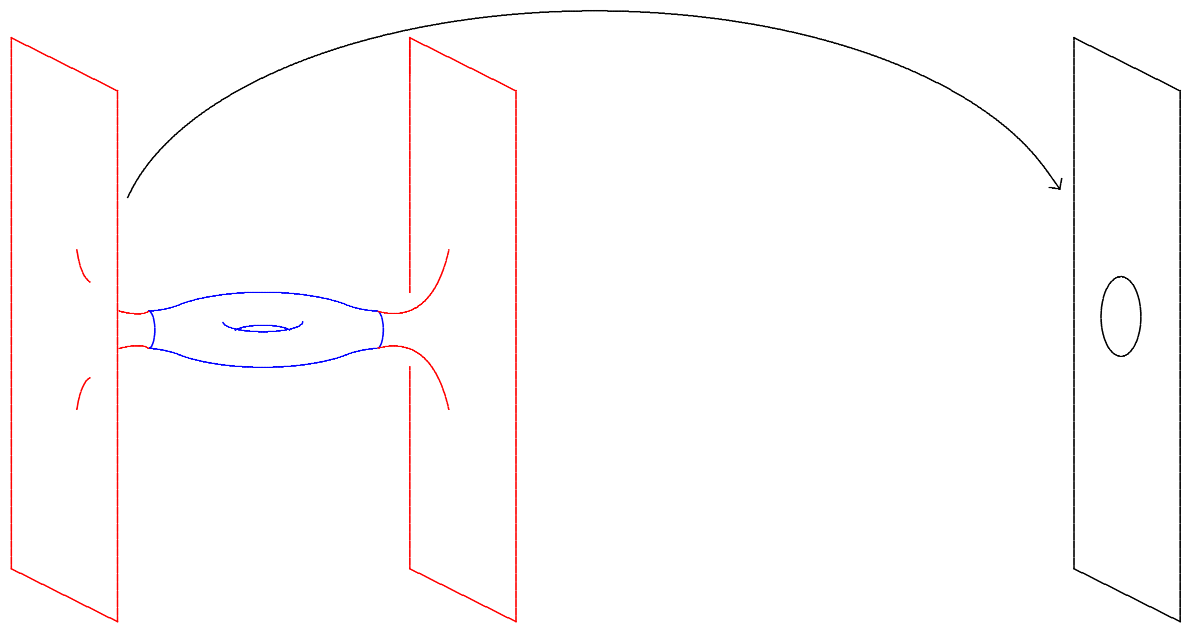

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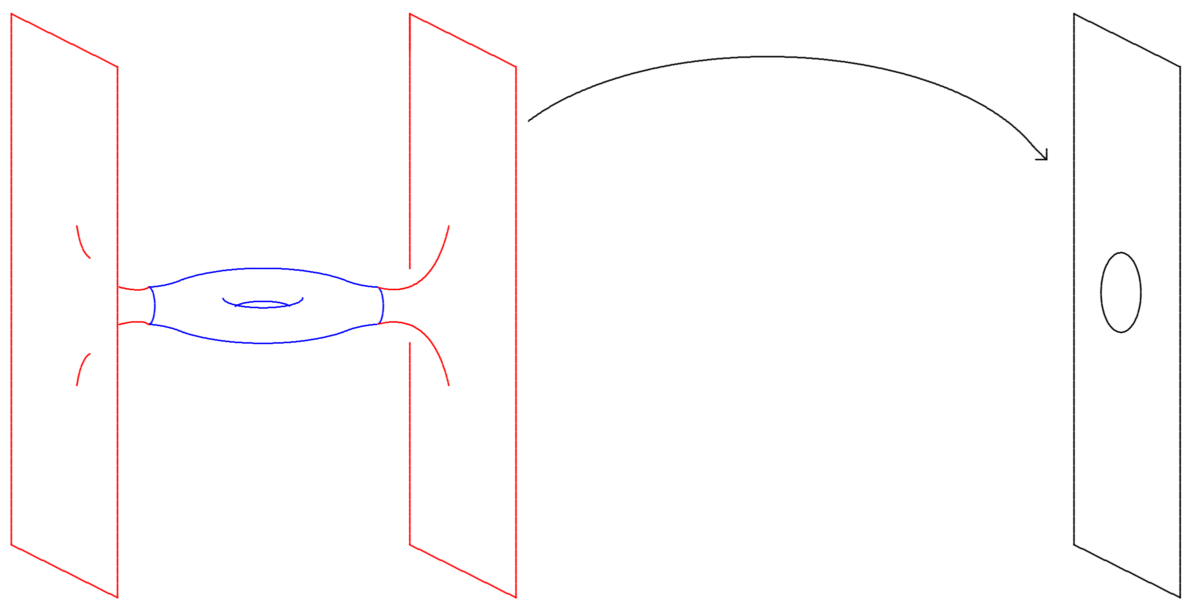

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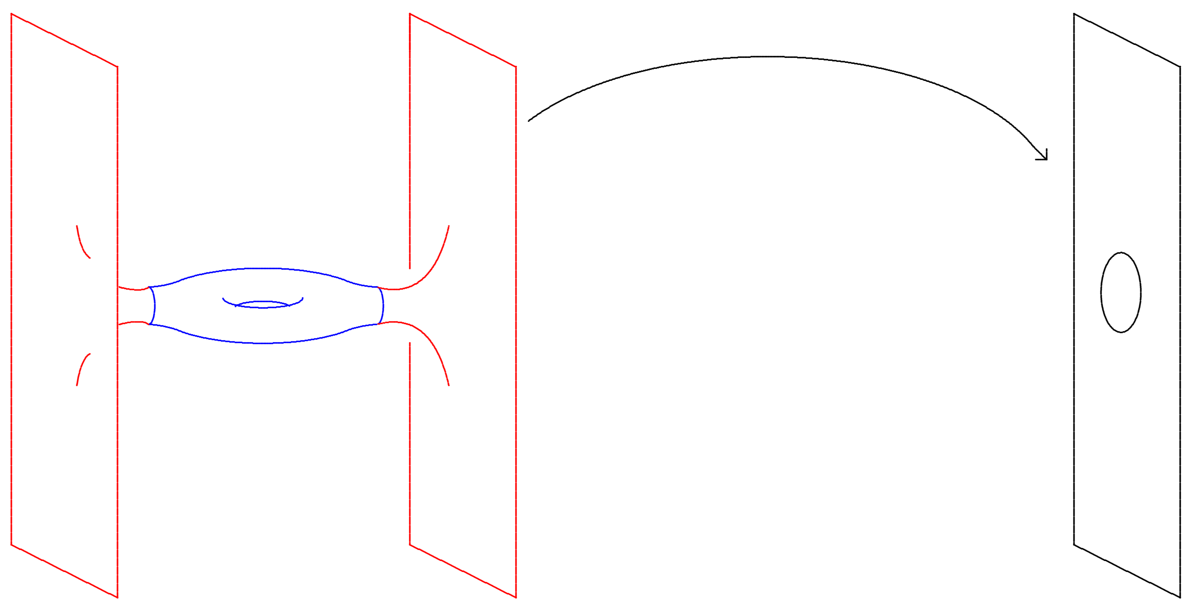

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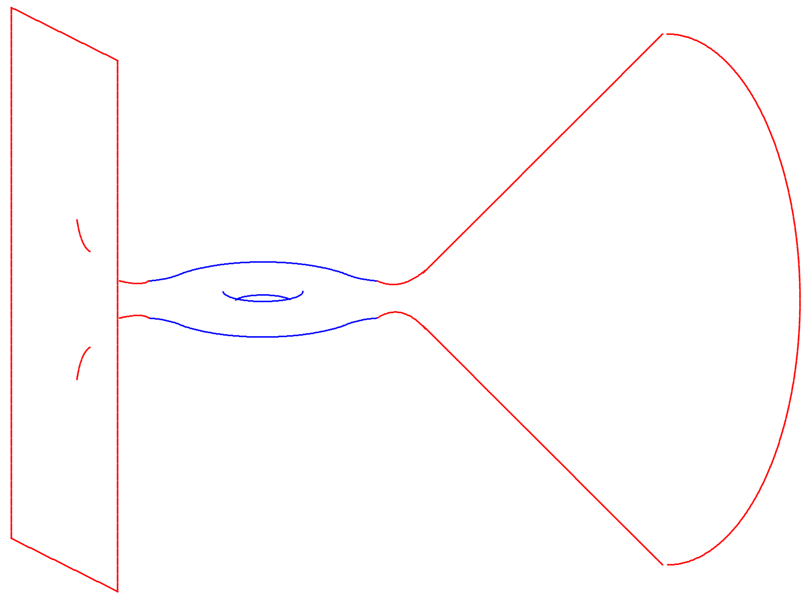

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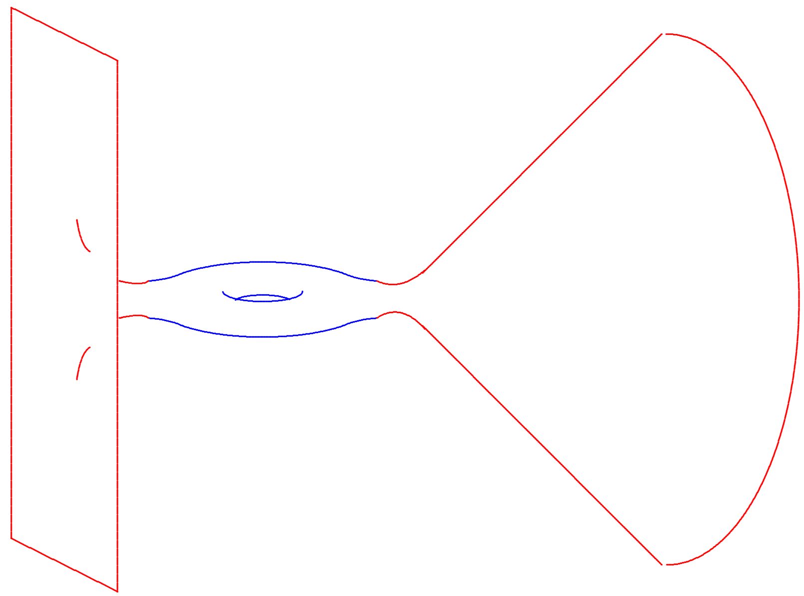

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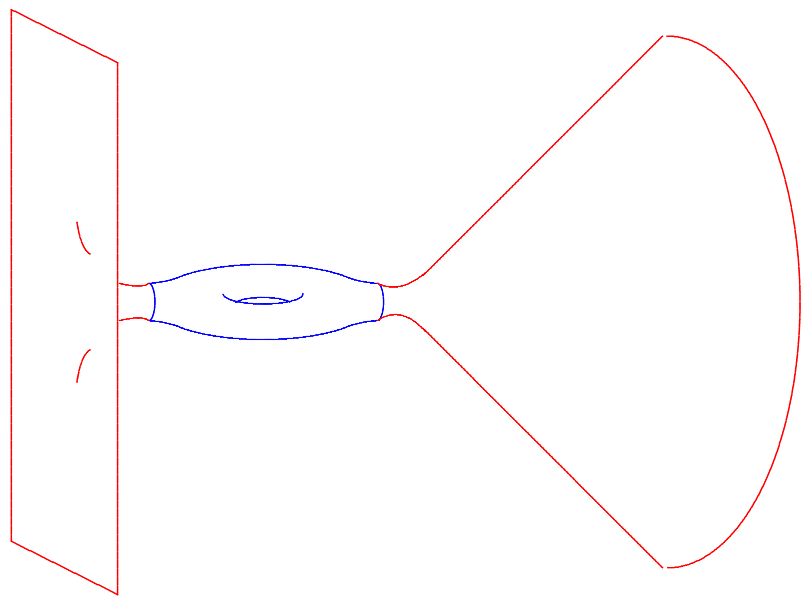

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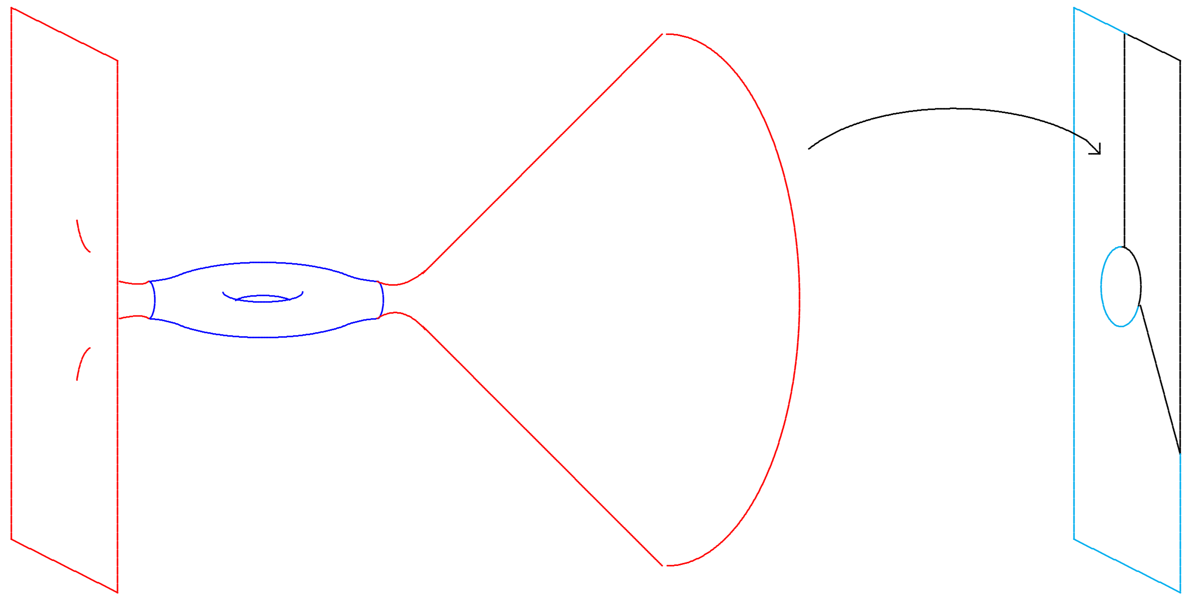

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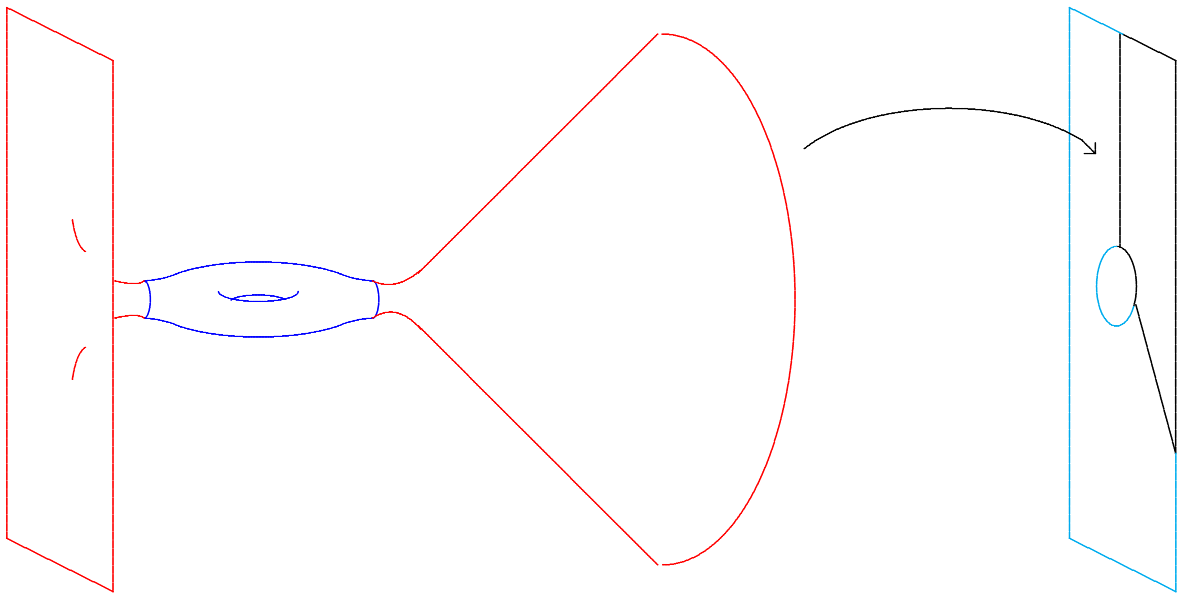

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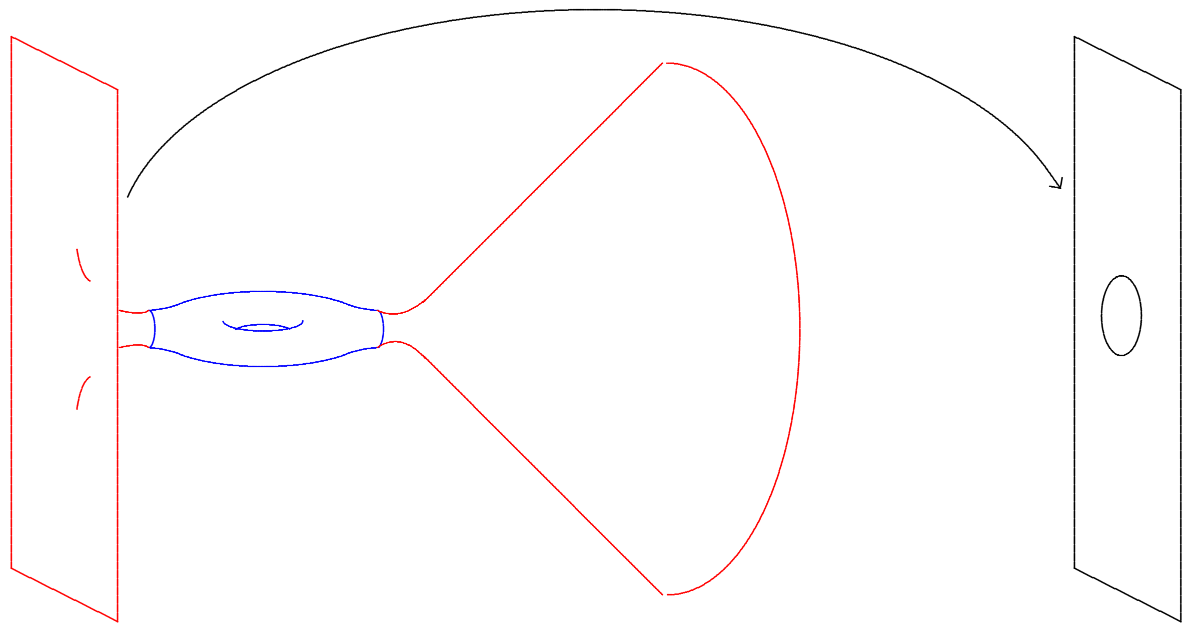

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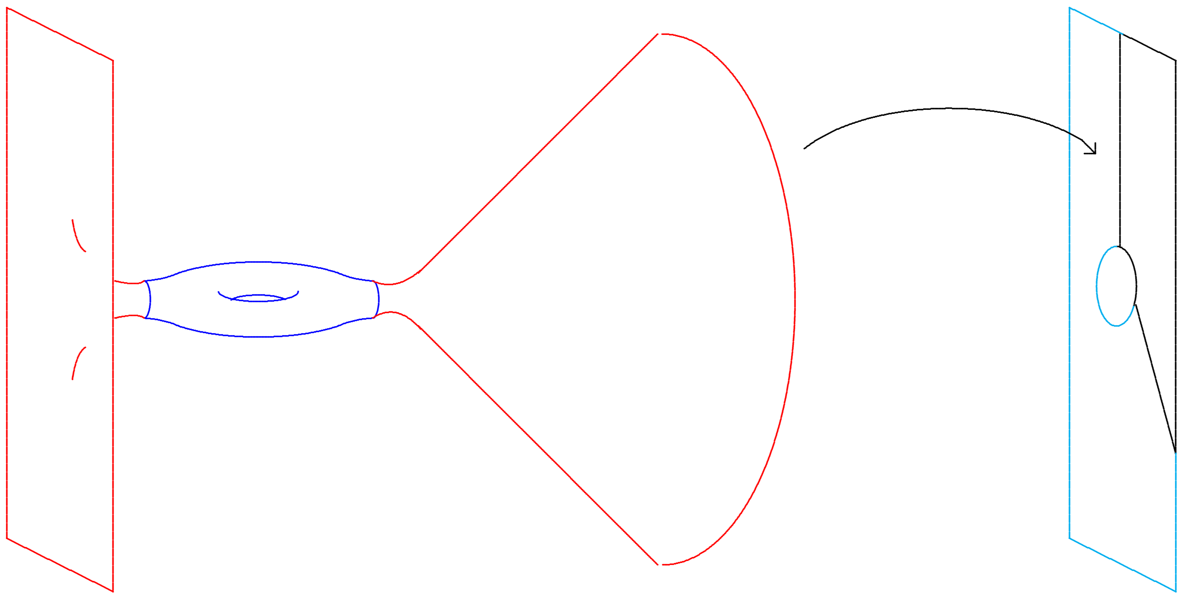

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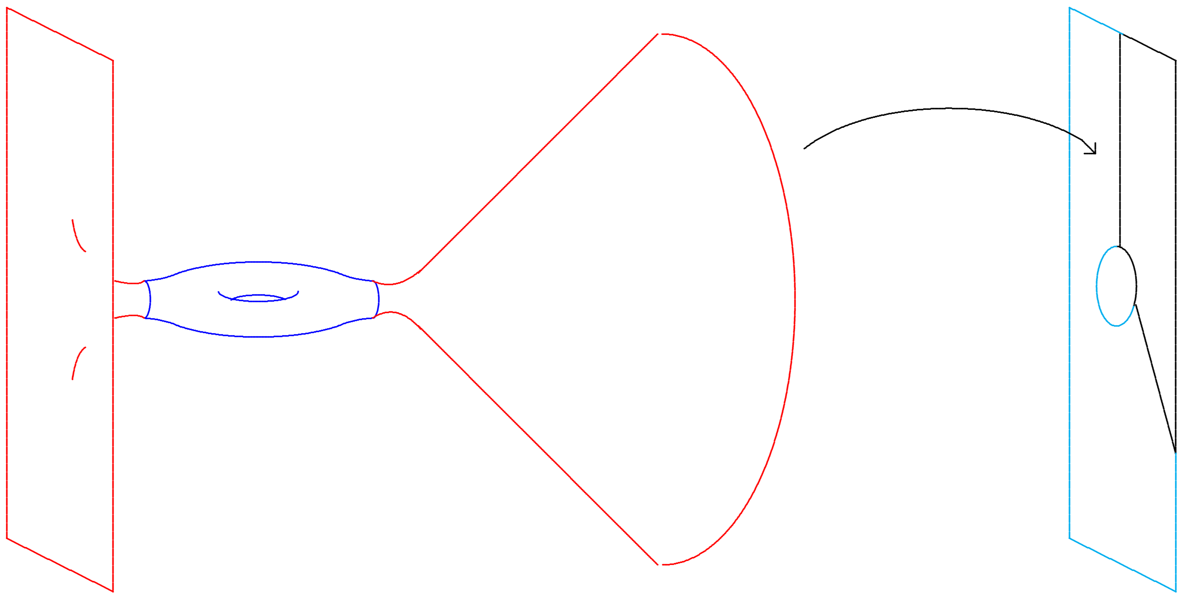

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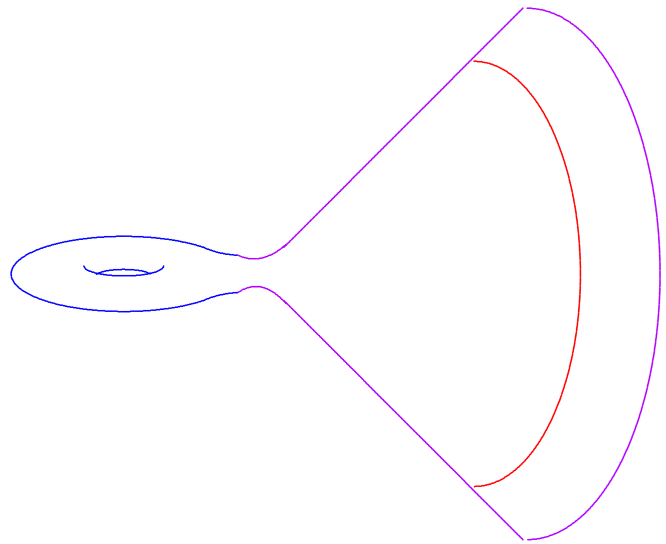

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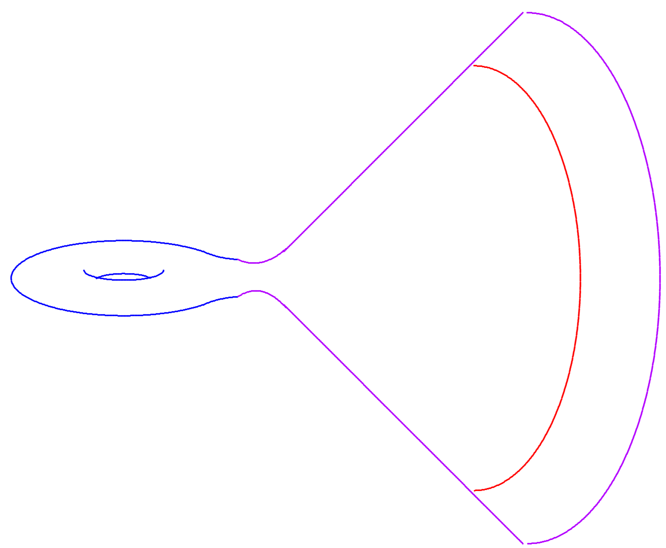

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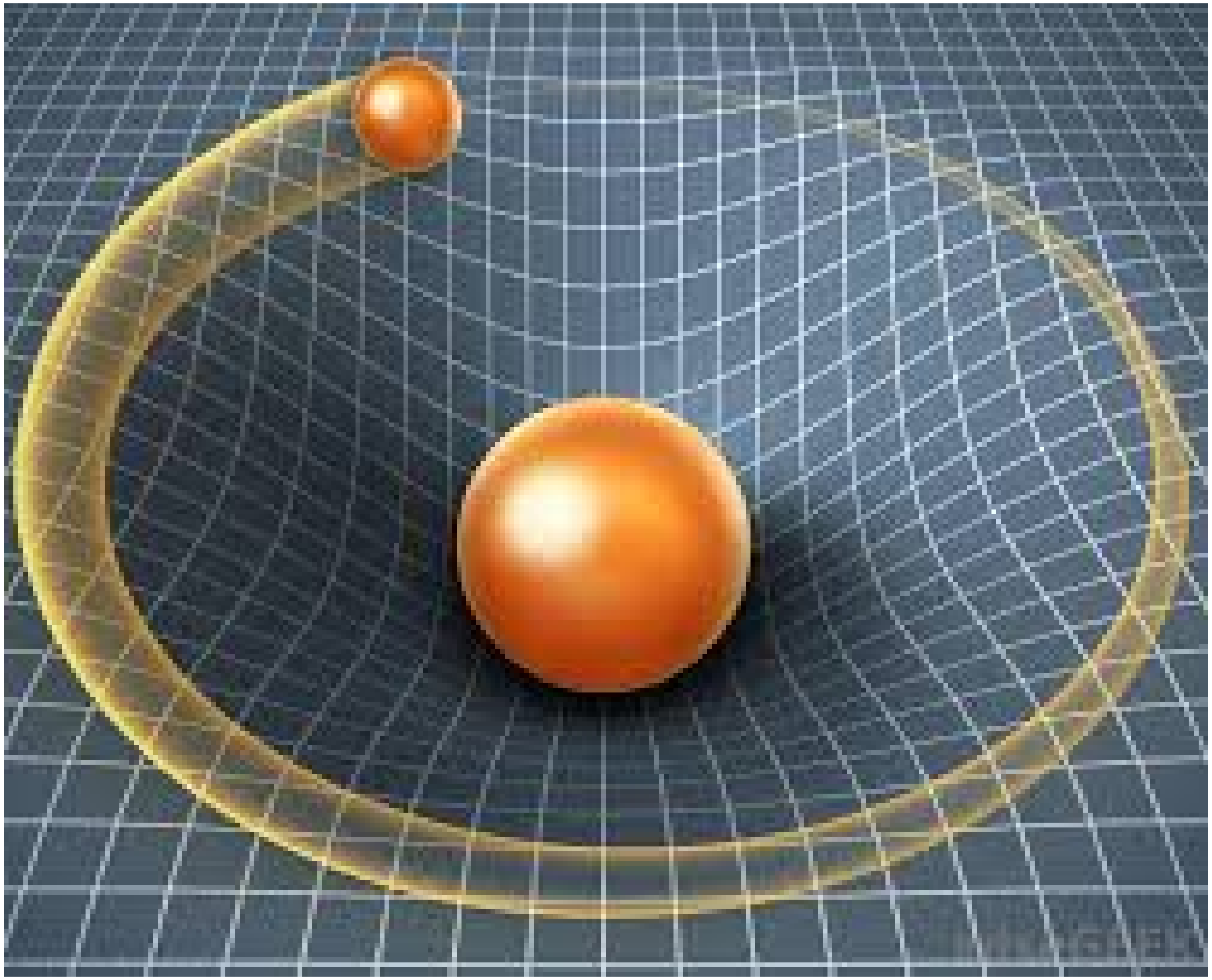

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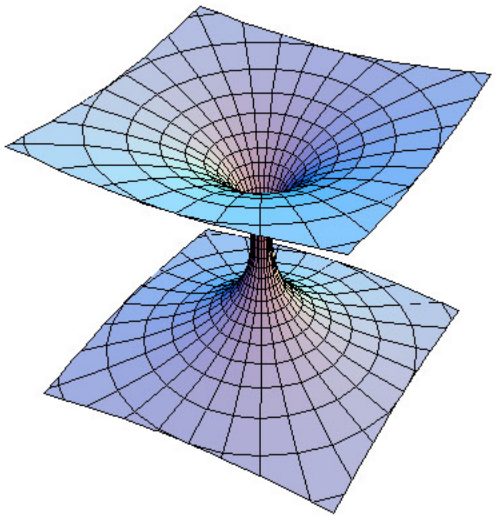

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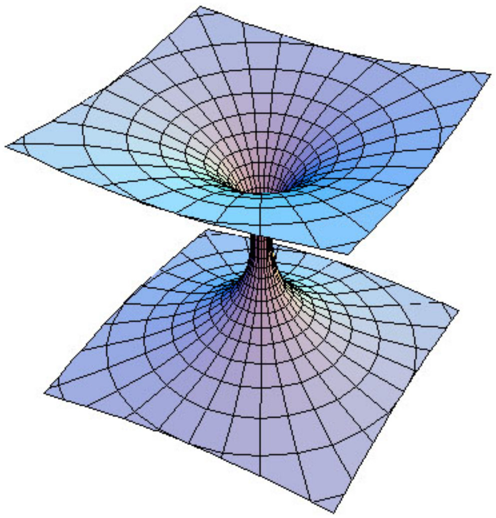

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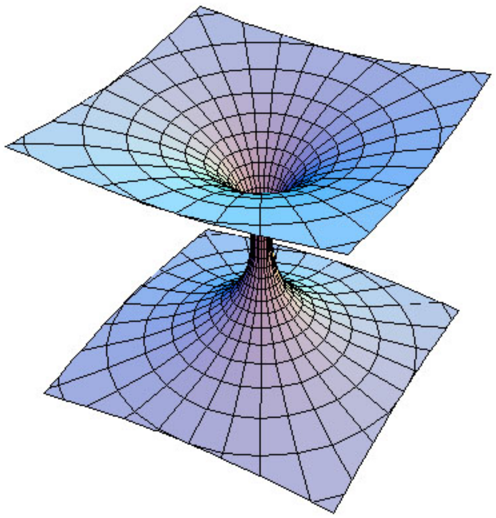

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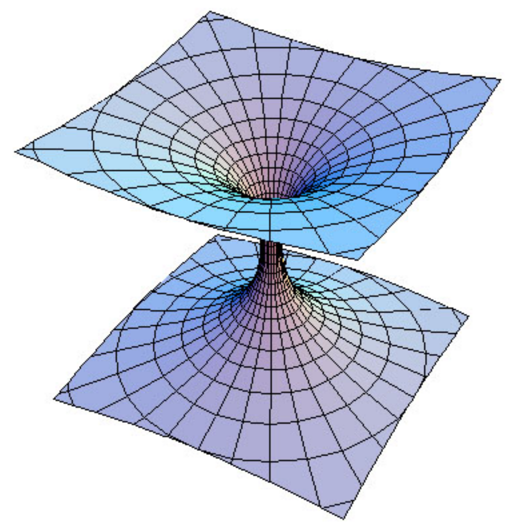

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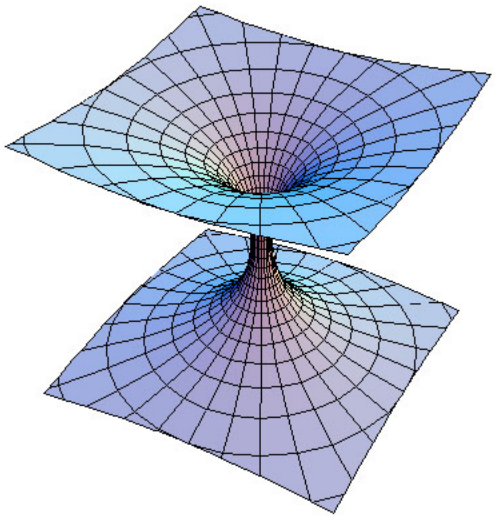

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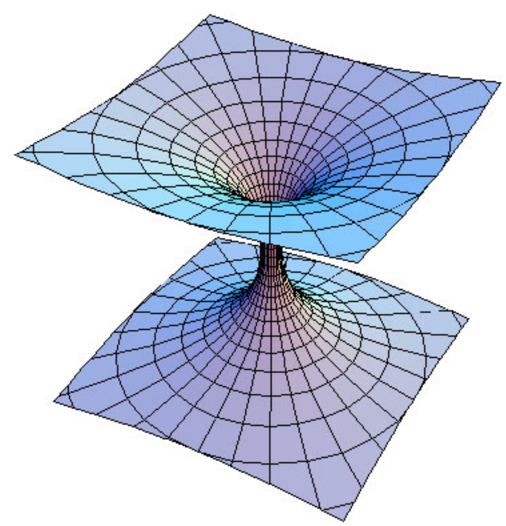

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But notice that this crude model for the mass in particular assumes faster metric fall-off!

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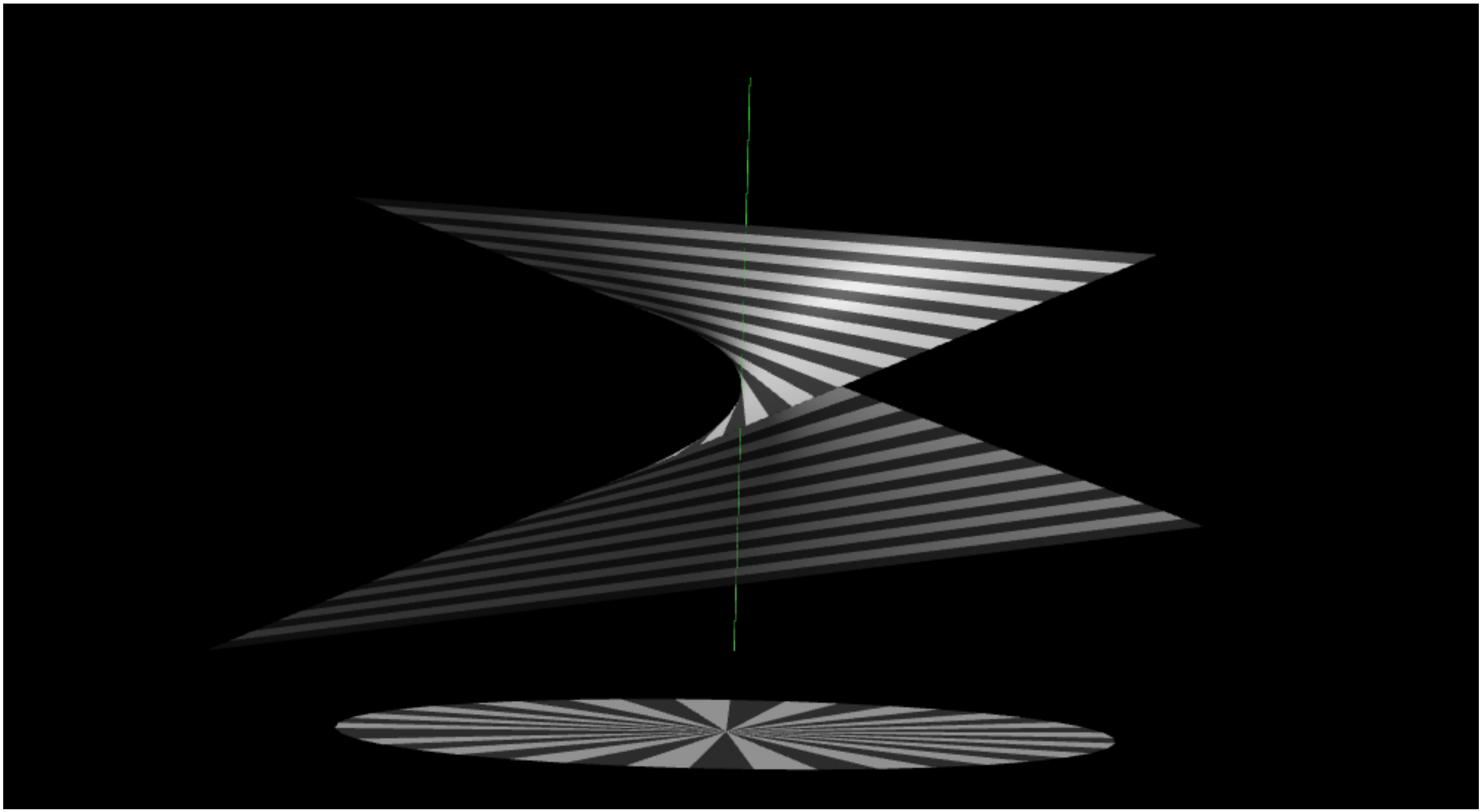

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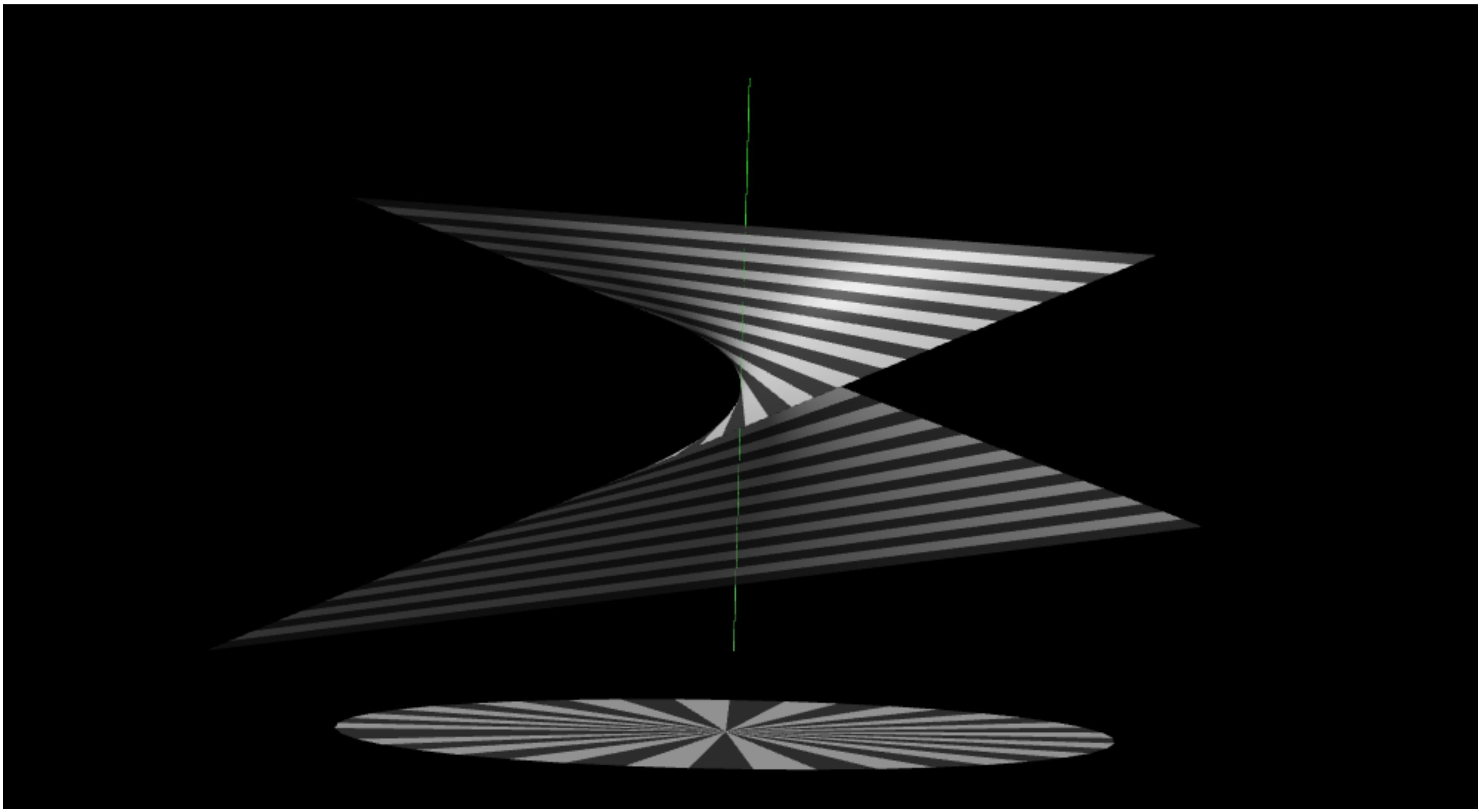

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Comm. Math. Phys. 347 (2016) 621–653.

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unless $\varepsilon > \frac{1}{2}$, when Chruściel fall-off sufficed.

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In particular, Chruściel fall-off suffices to imply all the following results:

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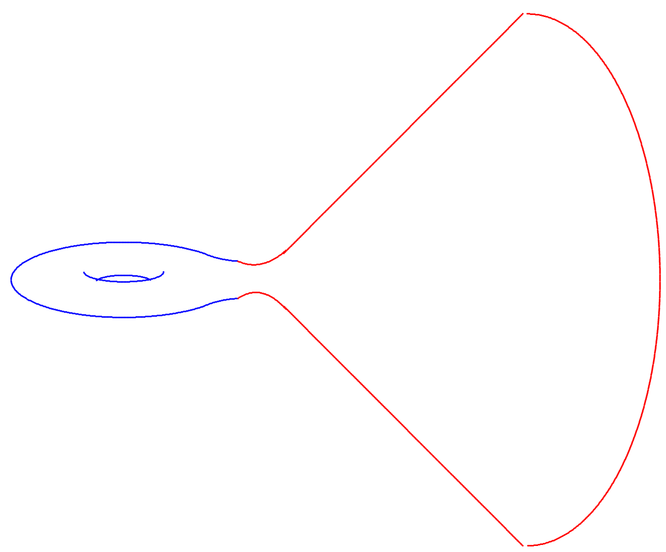

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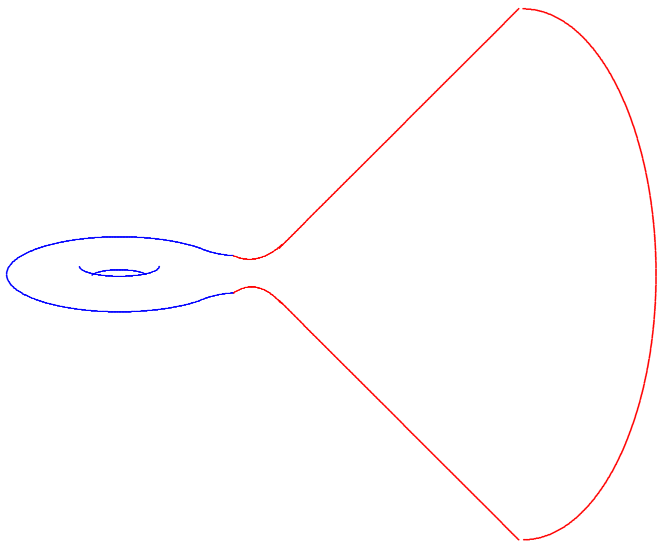

$$\boxed{n = 2m \geq 4}$$

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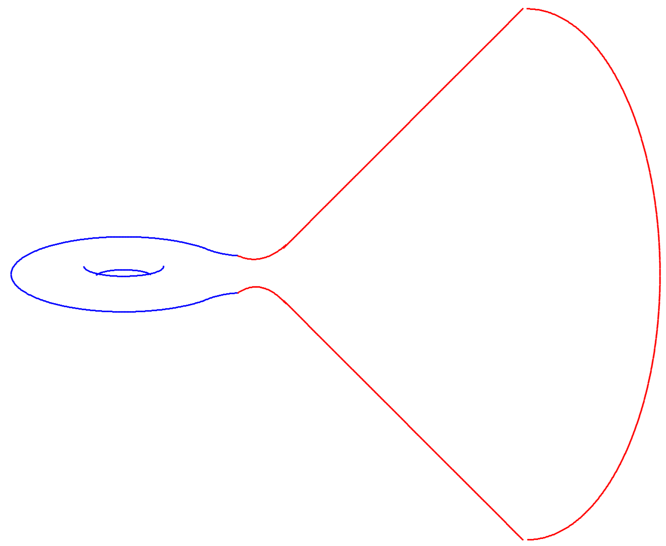

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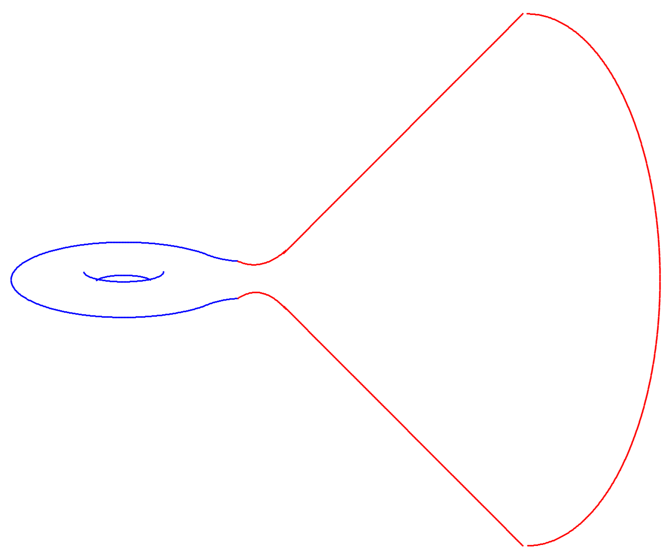

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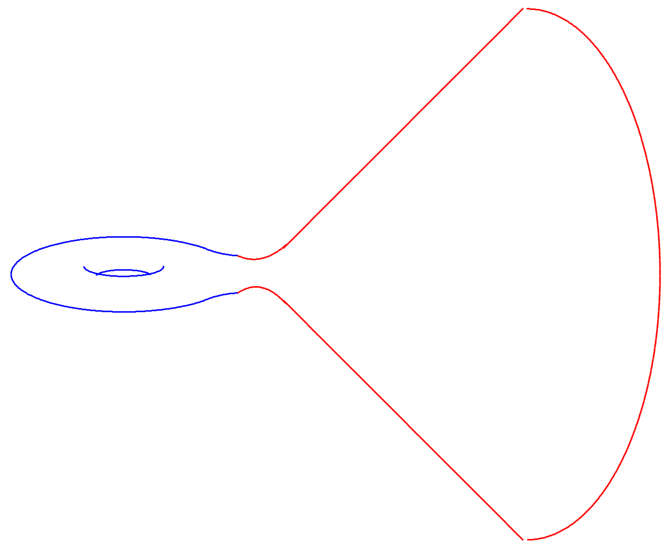

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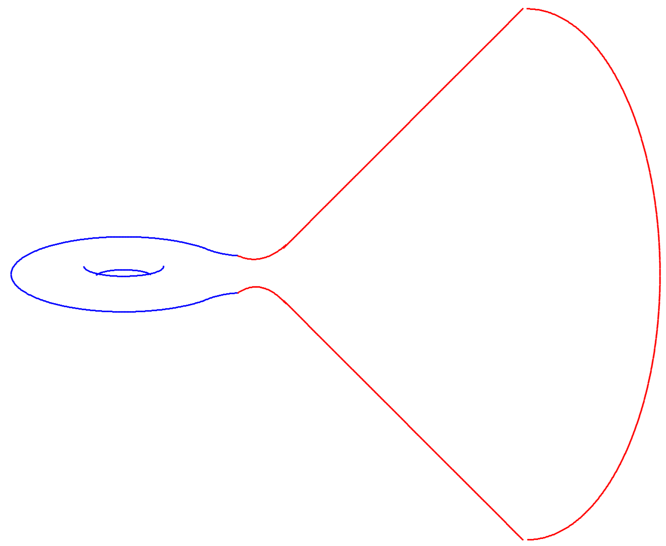

Upshot:

Mass of an ALE Kähler manifold is unambiguous.

Does not depend on the choice of an end!

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New proof shows this follows from Chruściel fall-off.

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Theorems A & B are corollaries concerning scalar-flat Kähler metrics.

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Why?

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

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Another key consequence…

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This has an interesting corollary...

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Now use Bishop-Gromov inequality.

Some applications . . .

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Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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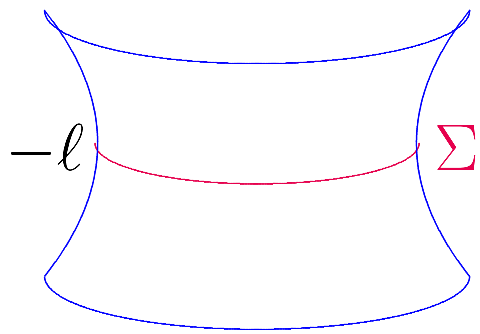
Example. Blow up Chern class $-\ell$ line bundle over $\mathbb{CP}_1$ at k points on zero section Σ.



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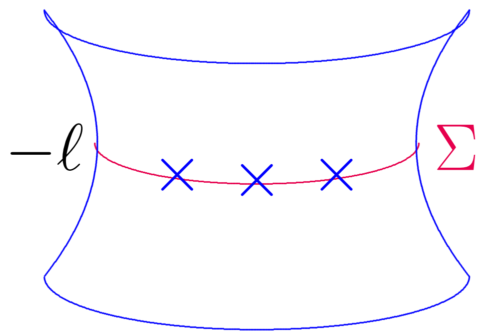

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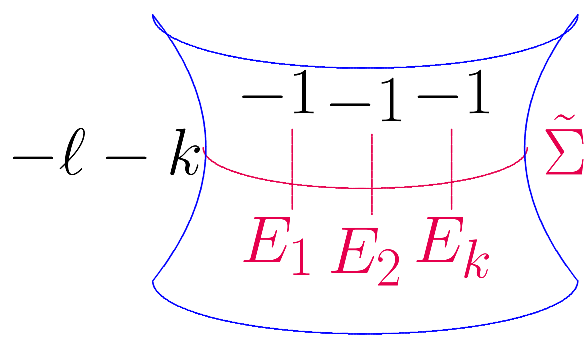

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L, Calderbank-Singer, Lock-Viaclovsky...

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That is, $\mathfrak{m}(M, g, J)$ *is completely determined by*

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How does one prove main results?

In high dimensions, the complex structure J of an ALE Kähler manifold is always standard at infinity.

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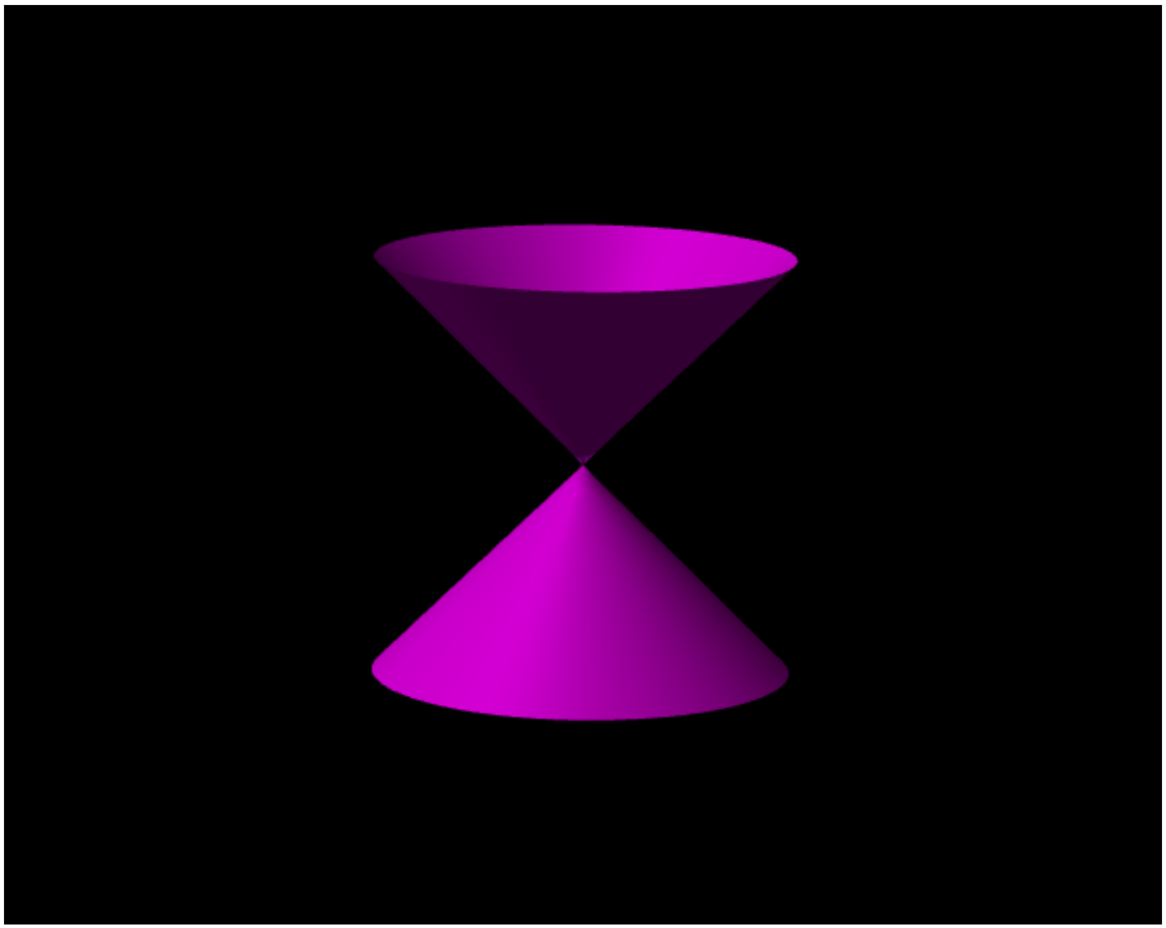

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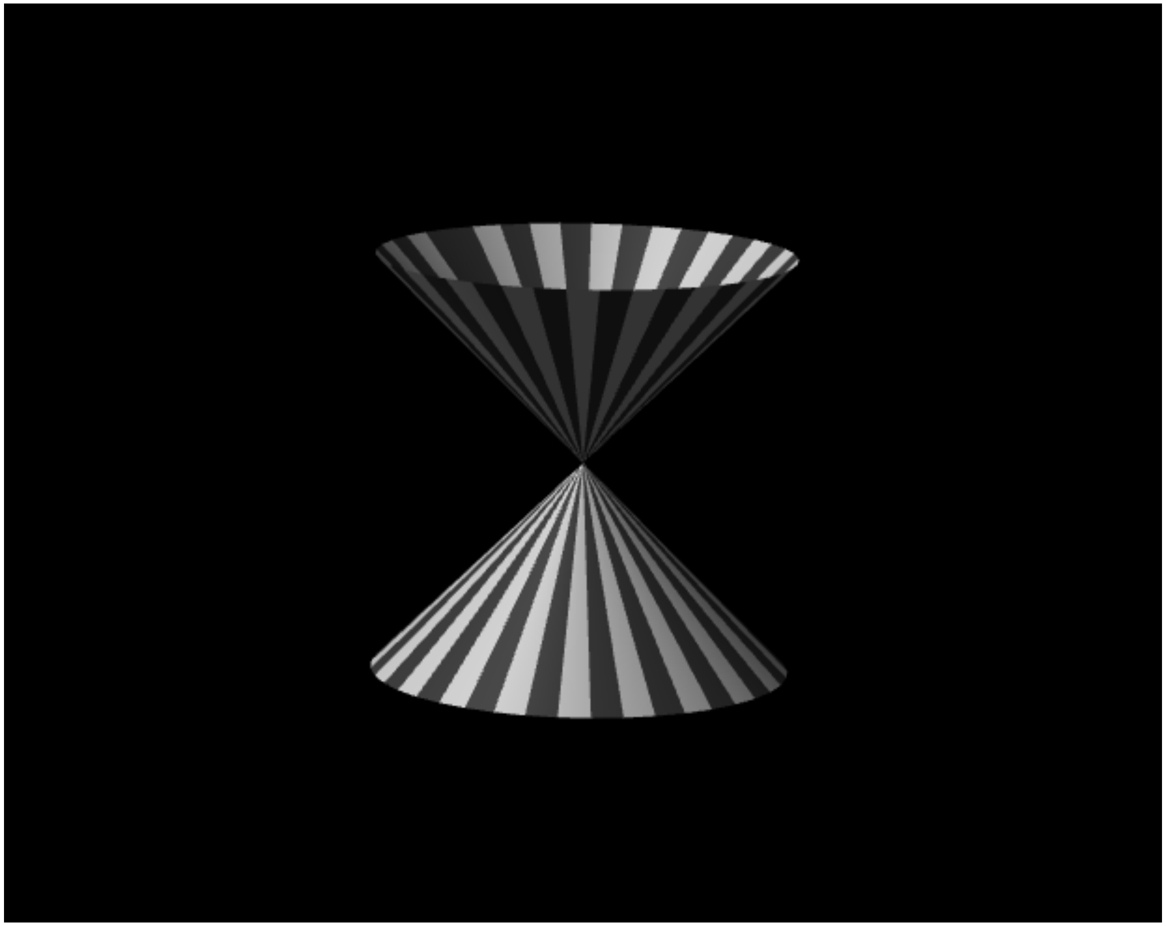

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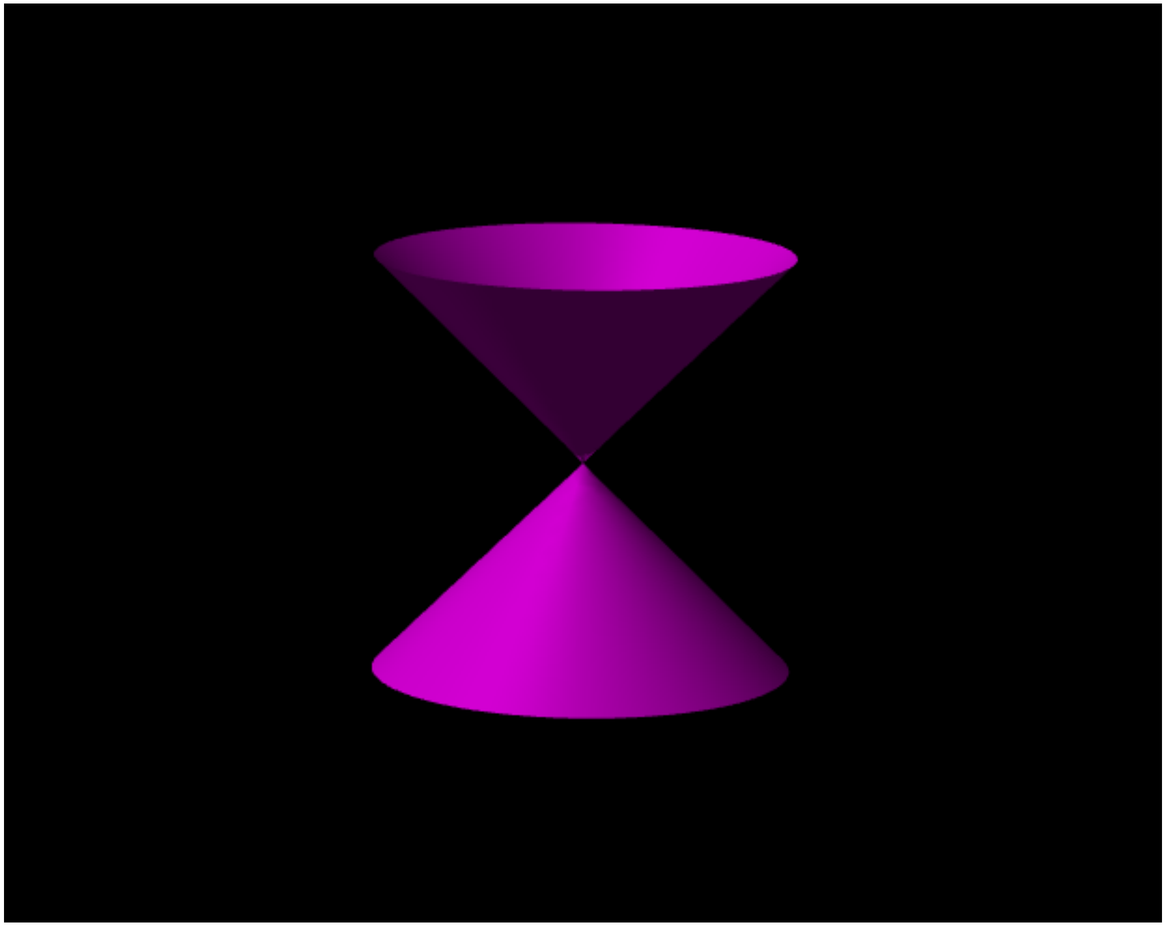

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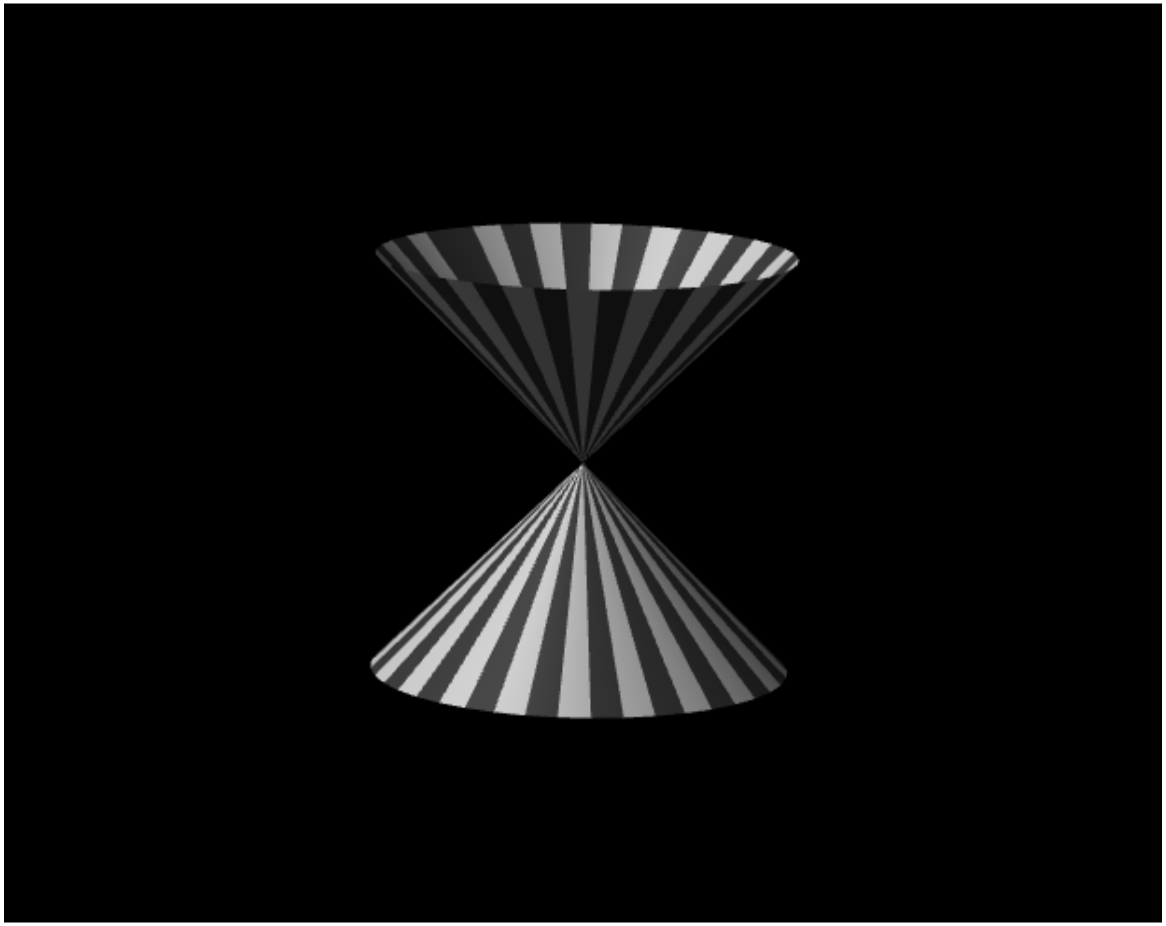

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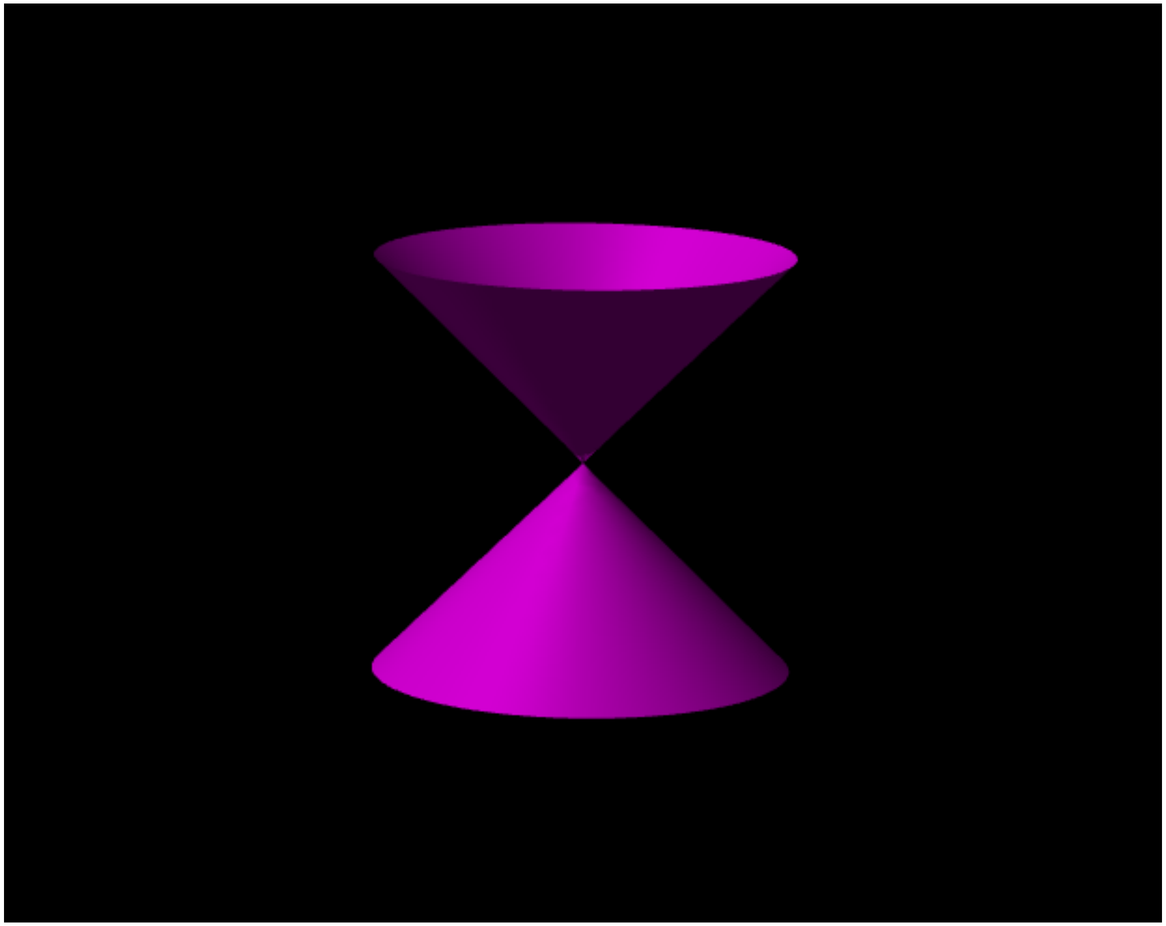

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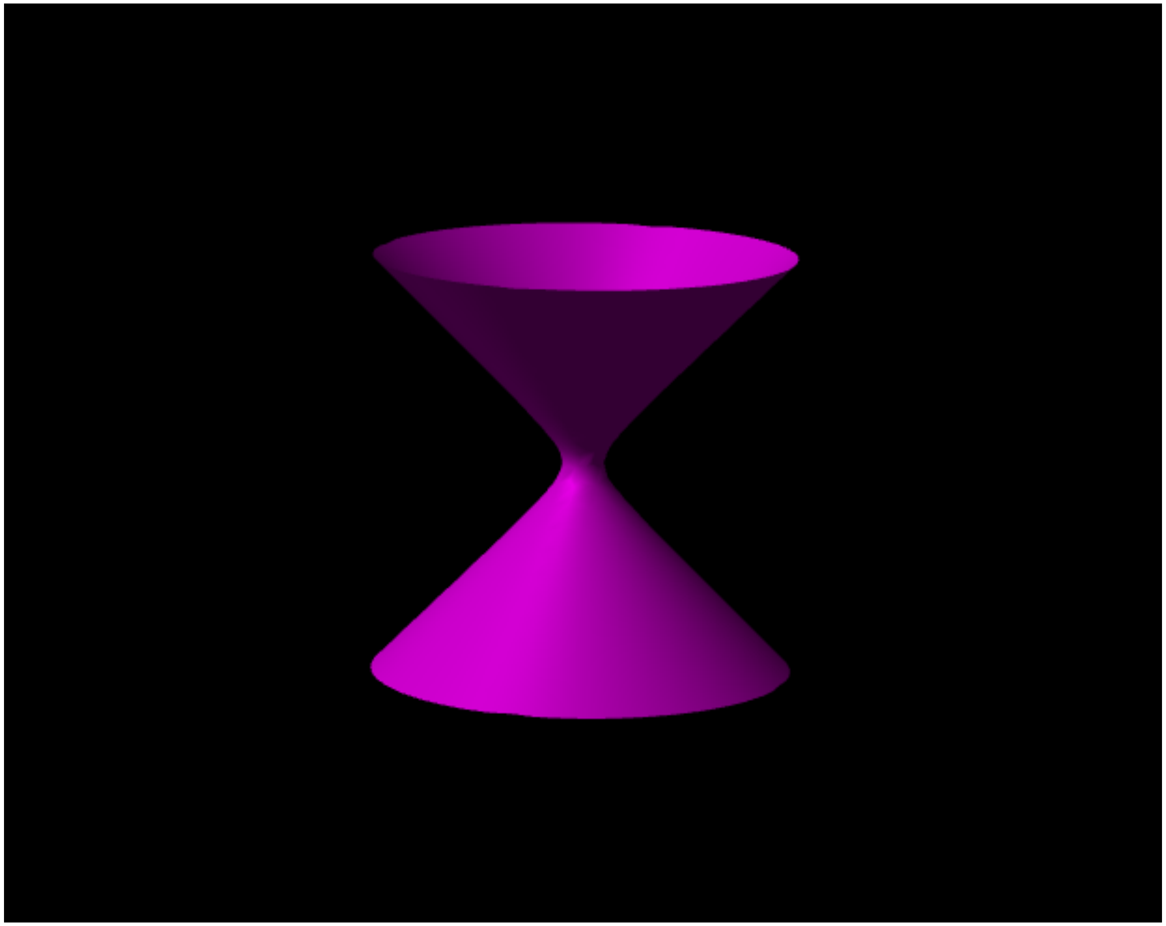

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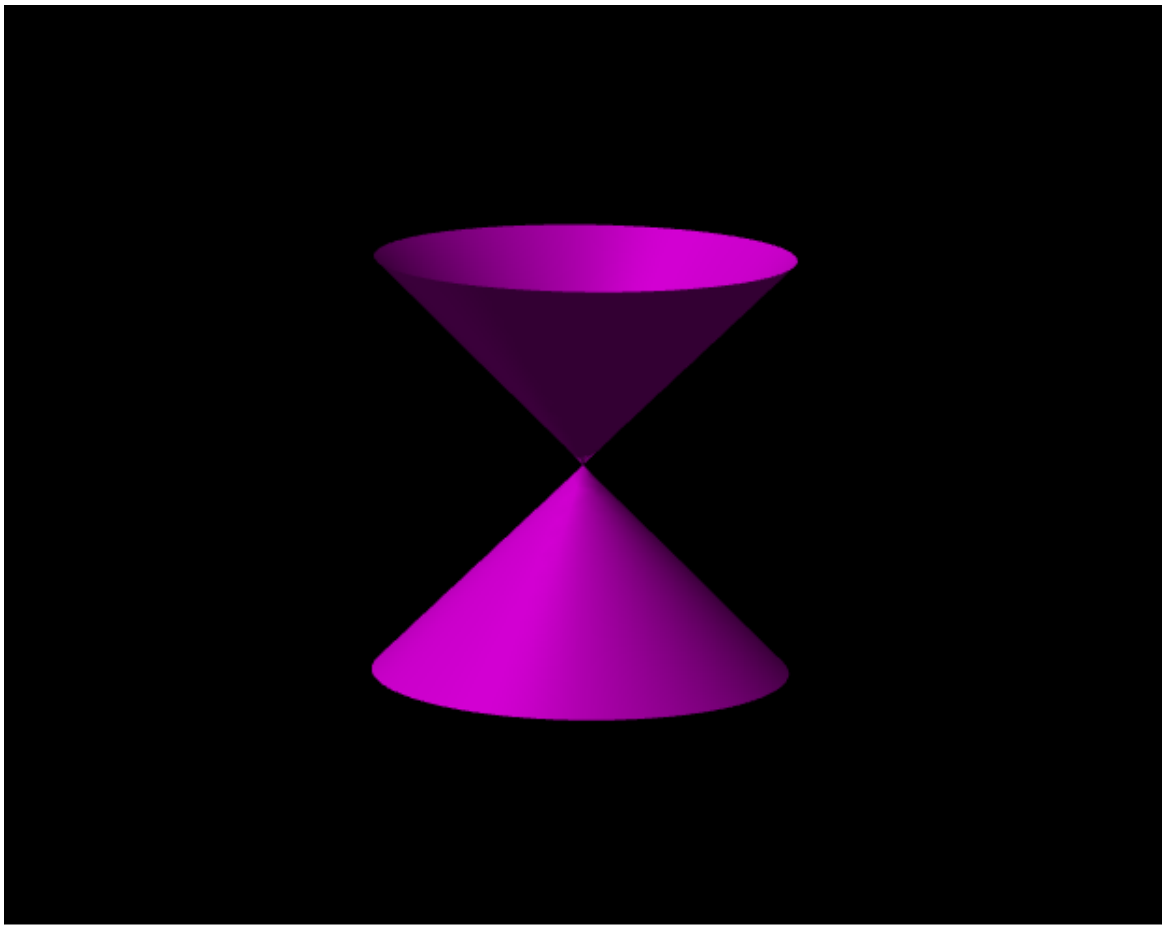

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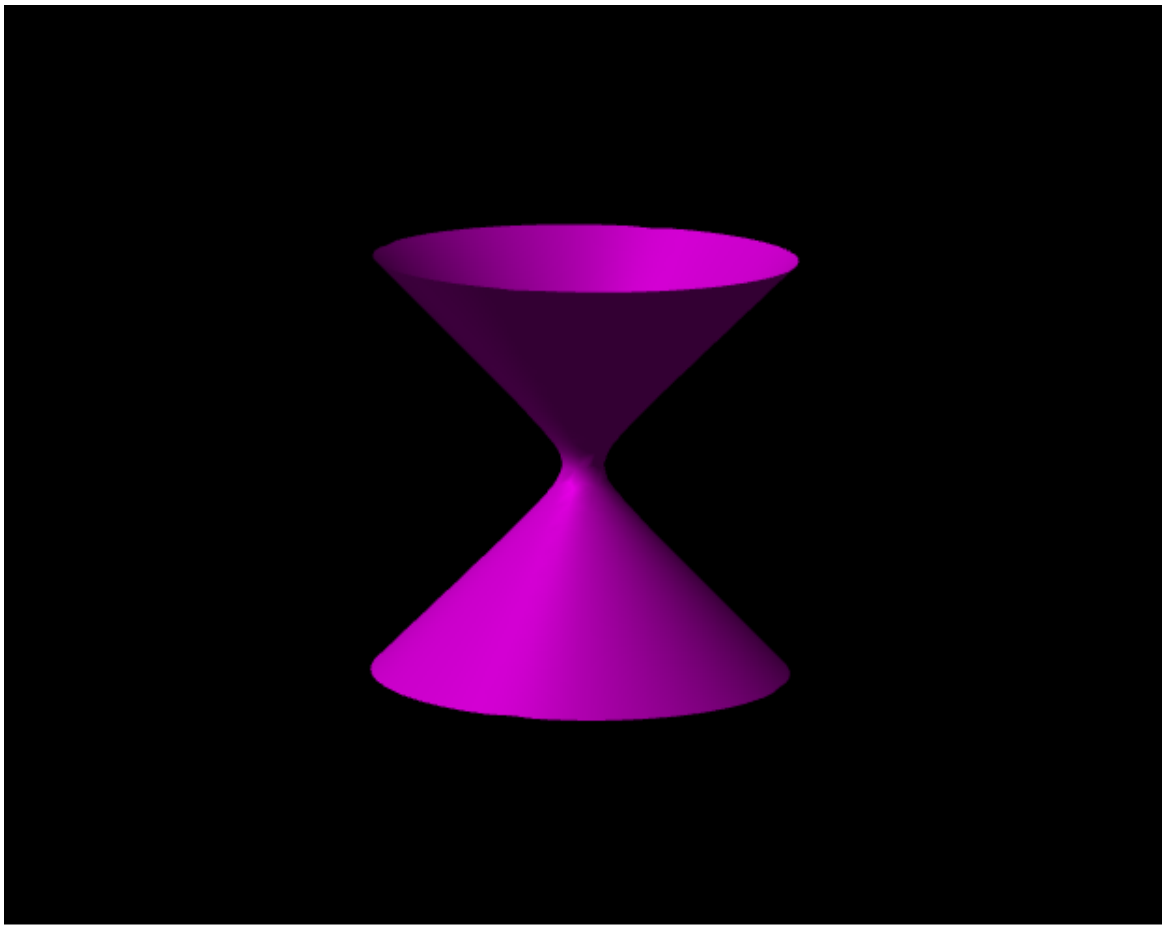

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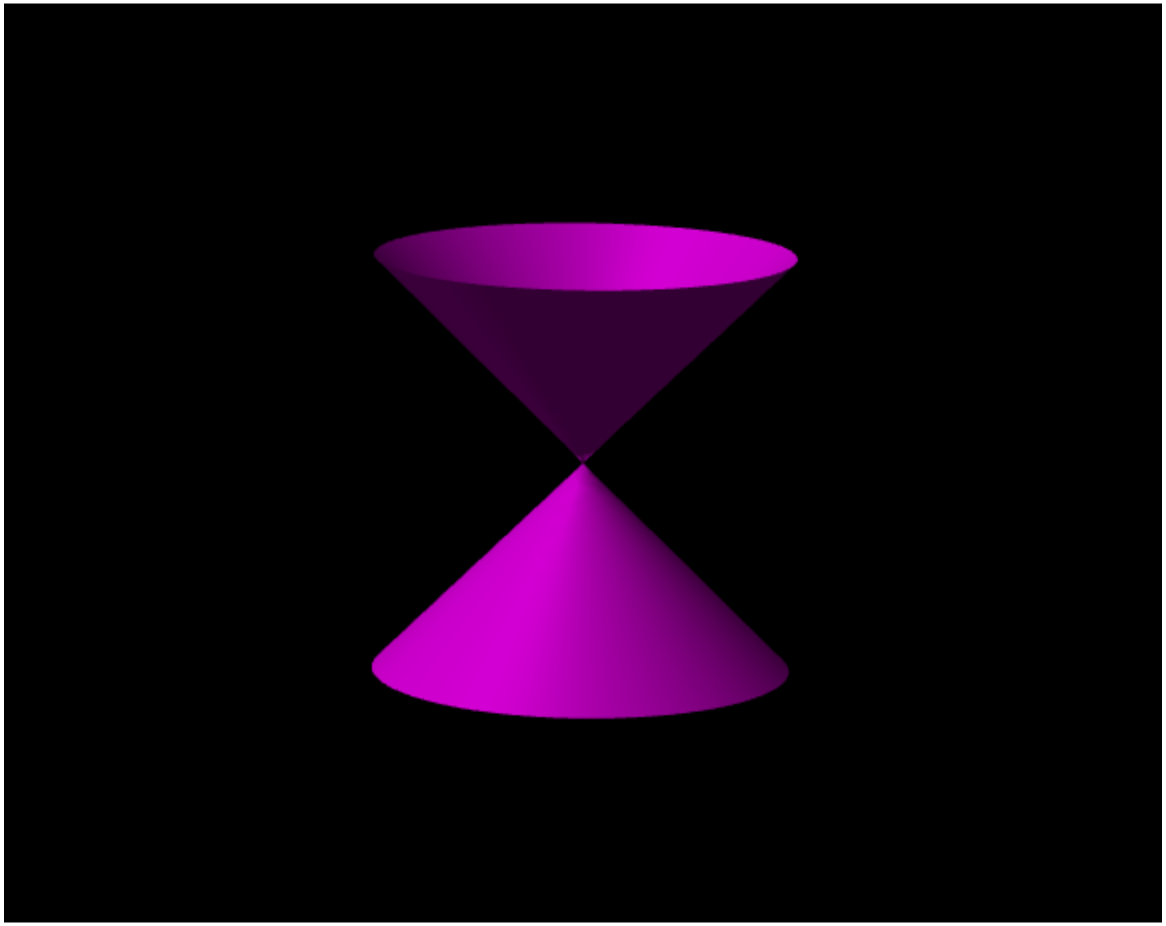

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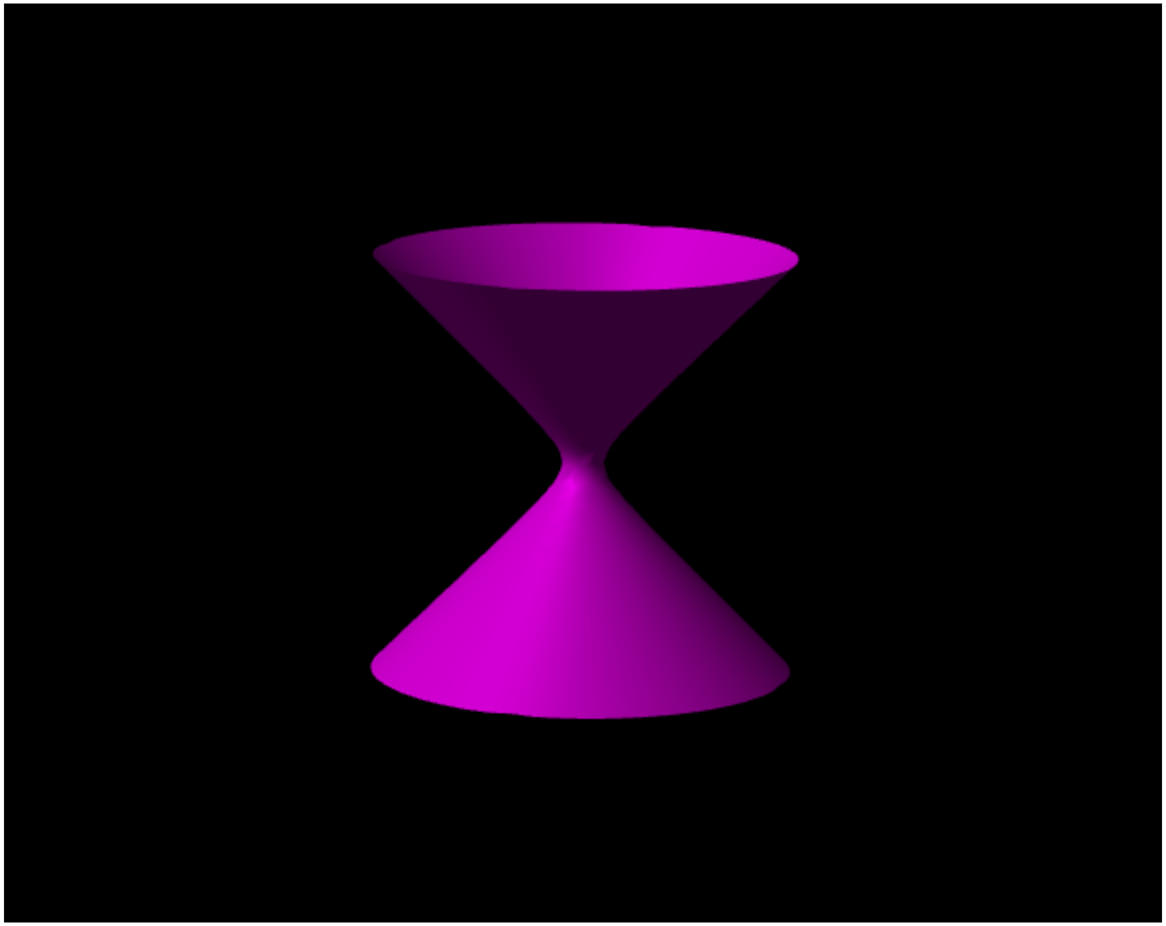

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Fortunately, however, the symplectic structure is always is standard at infinity!

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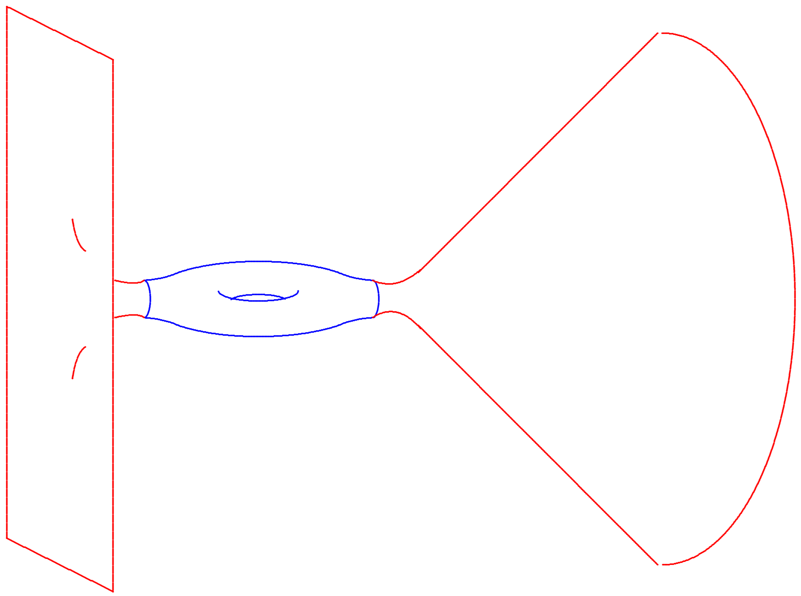
Quantitative version of Moser stability argument. . .

Symplectically compactify any ALE Kähler surface:



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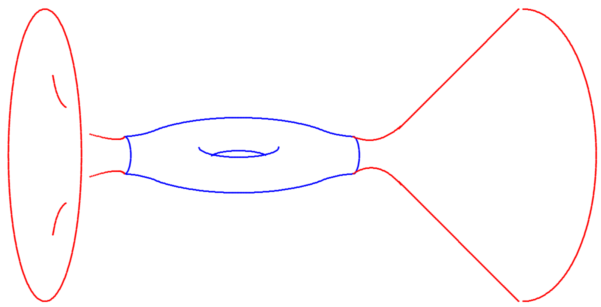

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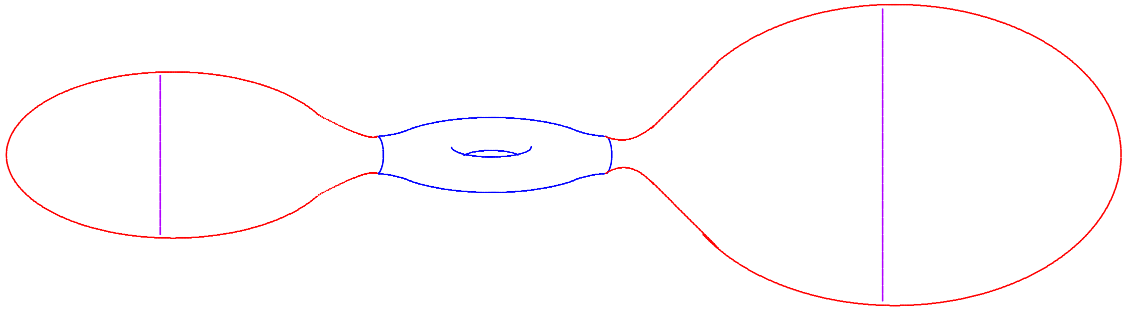

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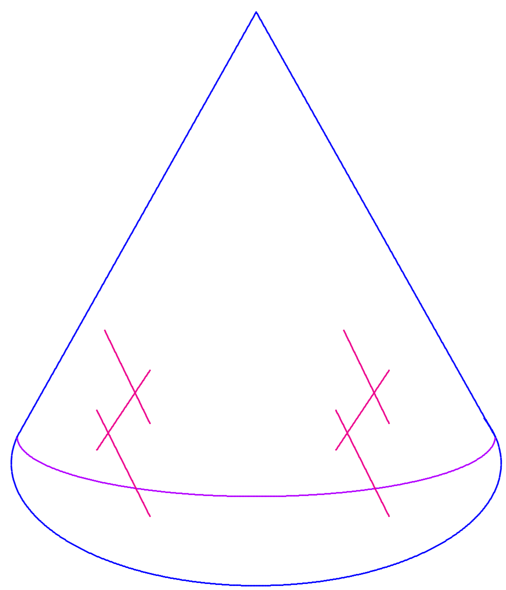

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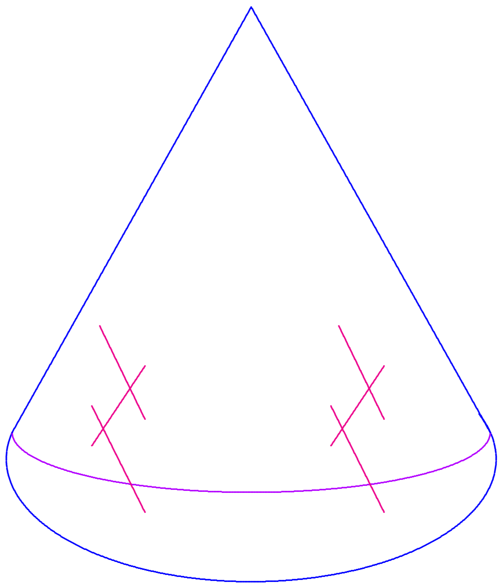

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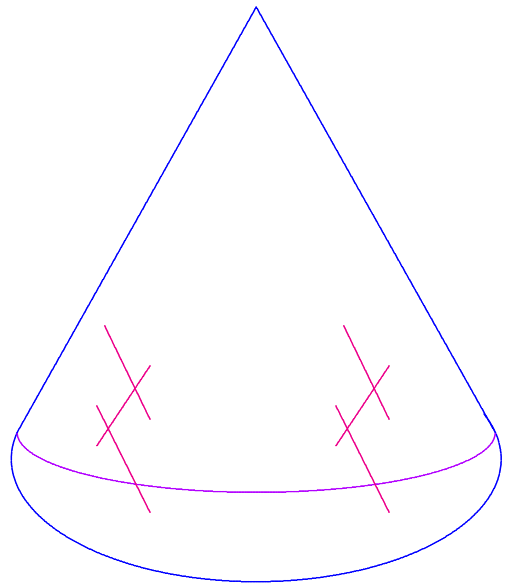

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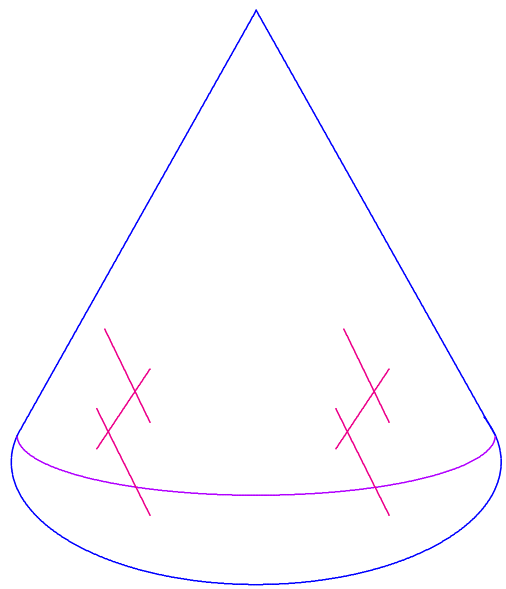

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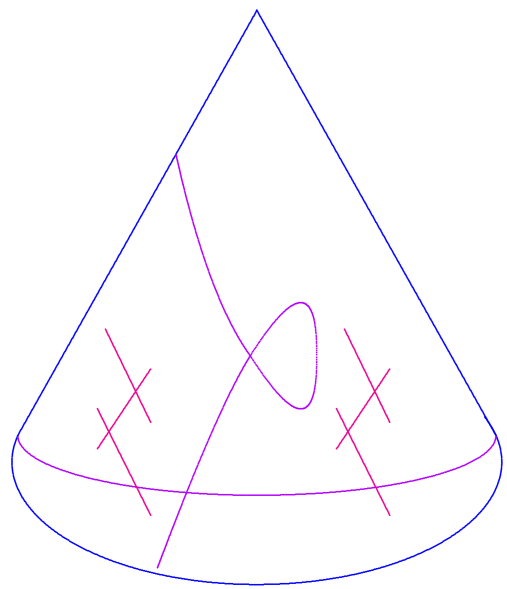

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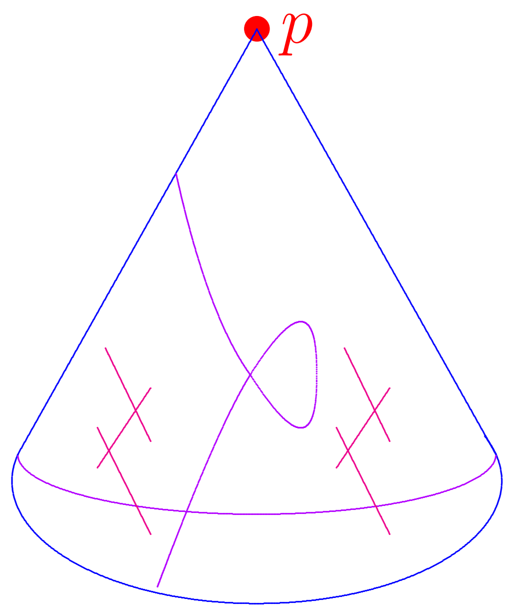

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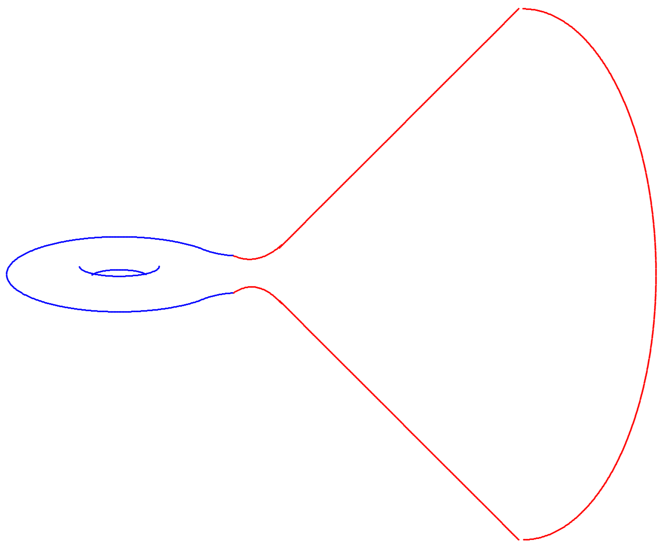
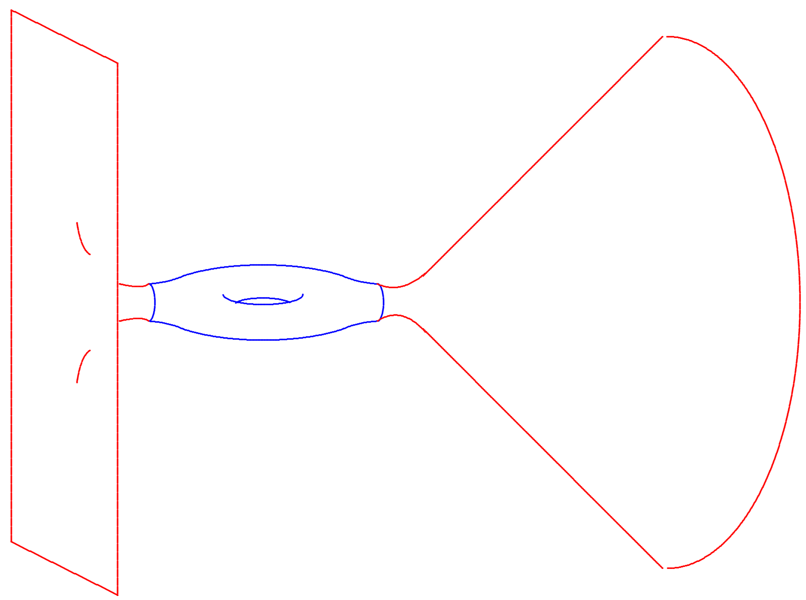
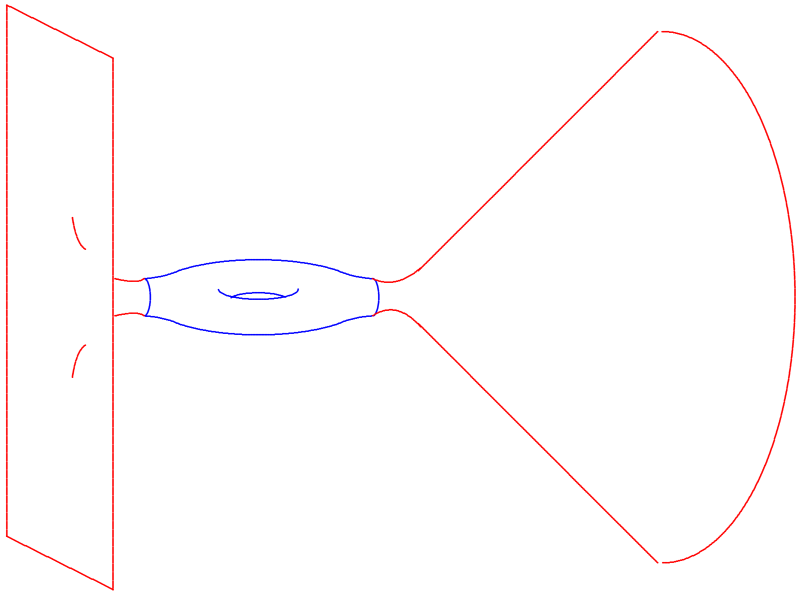
Lemma. *Any ALE Kähler manifold has only one end.*

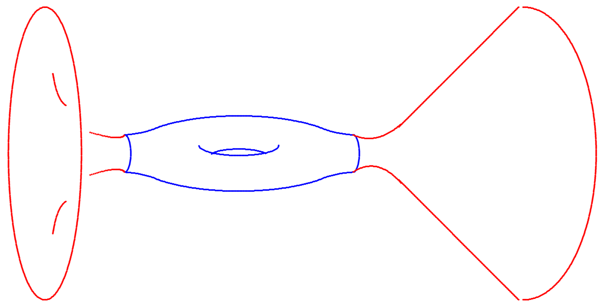




What if M^4 has more than one end?



Compactify M as symplectic 4-manifold $\widehat{M}$.



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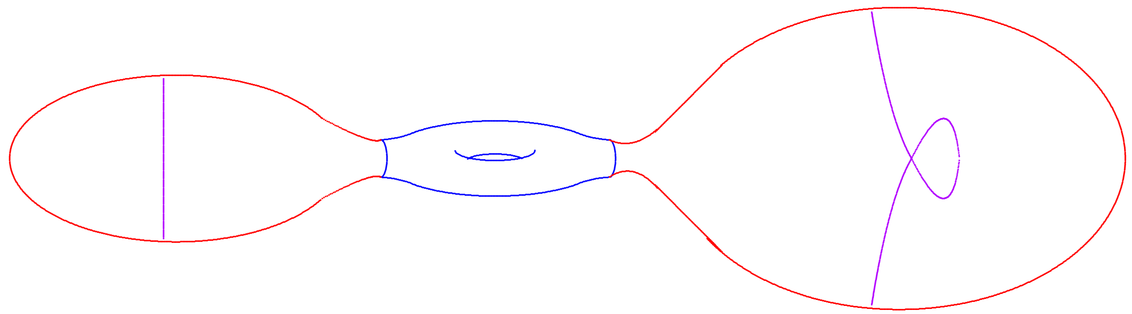

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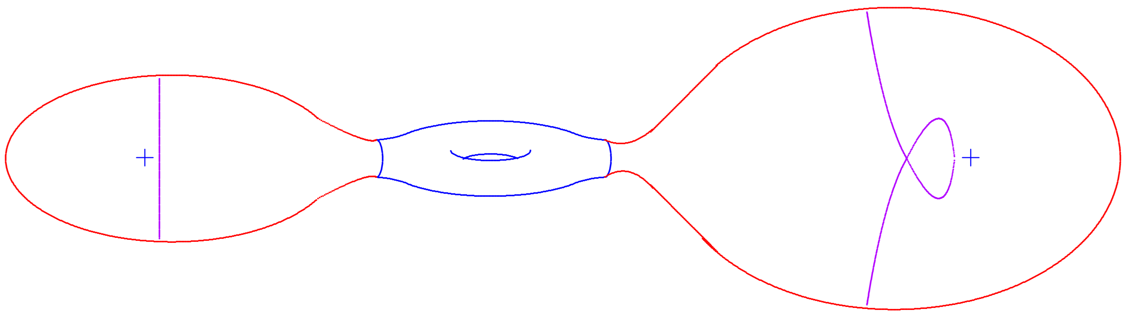

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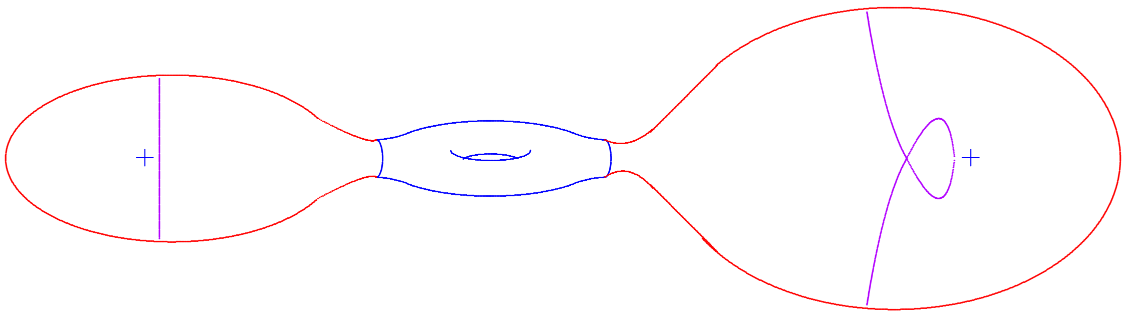
Each end-plug contains immersed symplectic 2-sphere of positive normal bundle.



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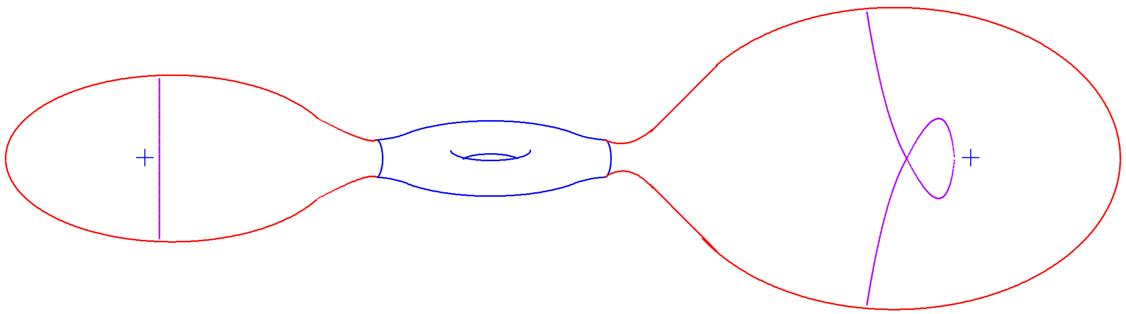
McDuff $\Longrightarrow$ $\widehat{M} \approx$ rational complex surface.



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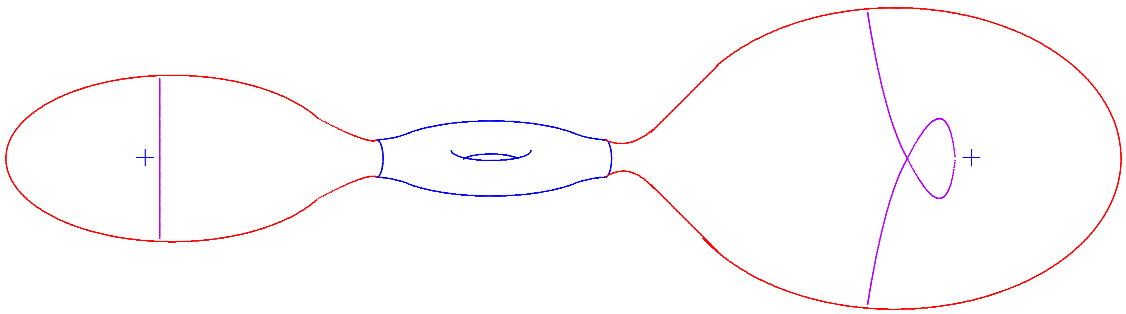
McDuff $\Longrightarrow$ intersection form $(+ - \cdots -)$.



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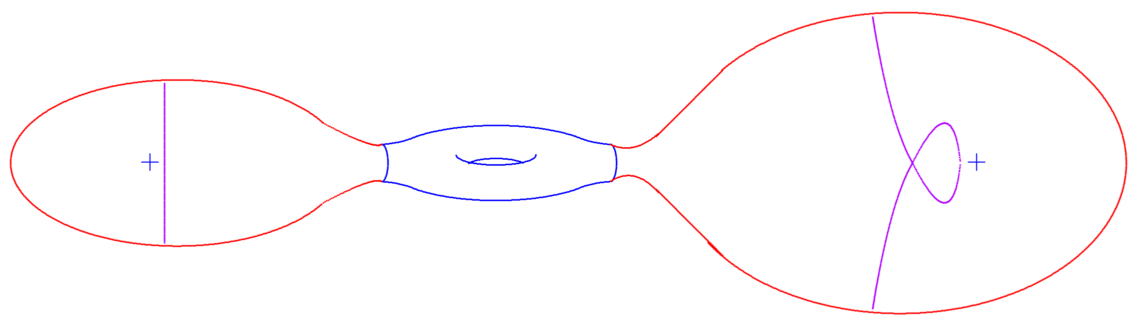
McDuff $\Longrightarrow b_+(M) = 1$.



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Since each end contributes positive direction. . .

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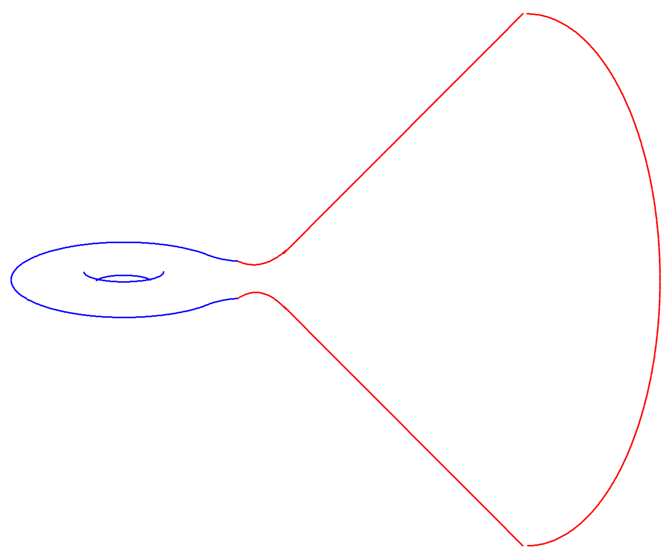

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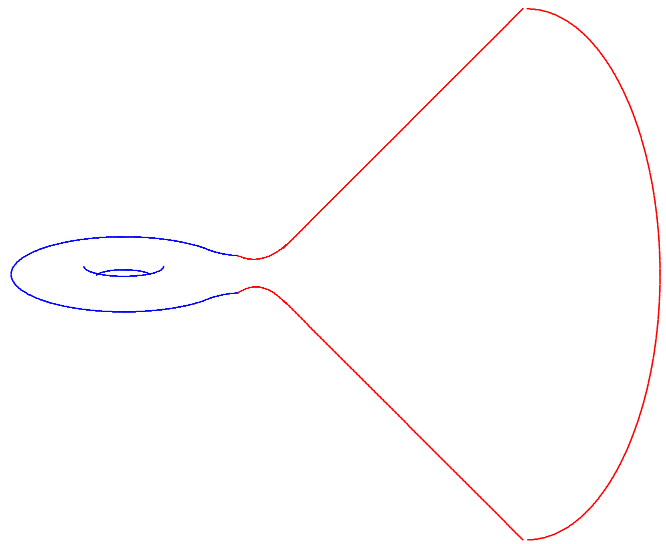

[In higher dimensions, one similarly shows that (M, J) can be compactified as Kähler orbifold. The Hodge theorem on intersection form instead tells one that form on $H^{1,1}(\widehat{M}, \mathbb{R})$ is of type $(+- \cdots -)$.]

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The following result provides the key. . .

Proposition. *Let g be a C^2 Kähler metric on $(\mathbb{R}^{2m} - \mathbf{D}^{2m})/\Gamma$, $m \geq 2$, where $\Gamma \subset \mathbf{SO}(2m)$ is some finite group that acts without fixed-points on S^{2m-1}.*

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We'll now deduce the mass formula. . .

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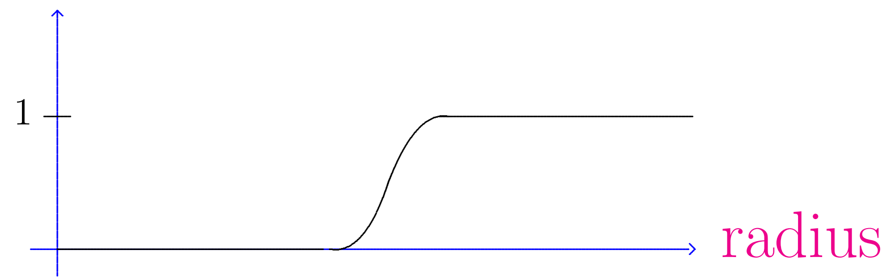

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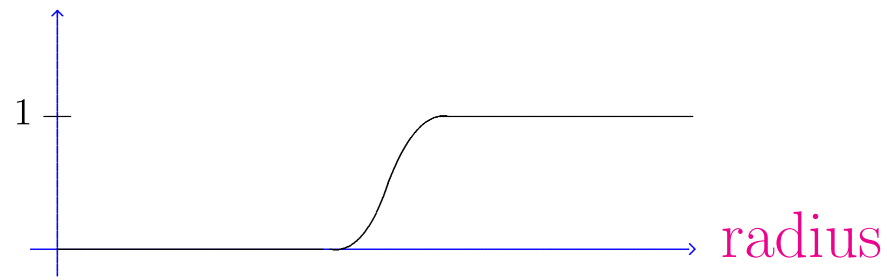

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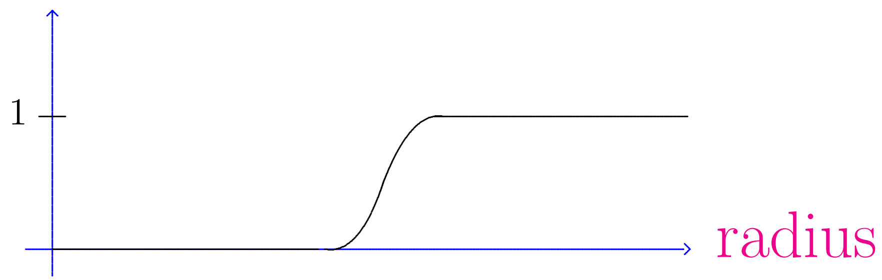

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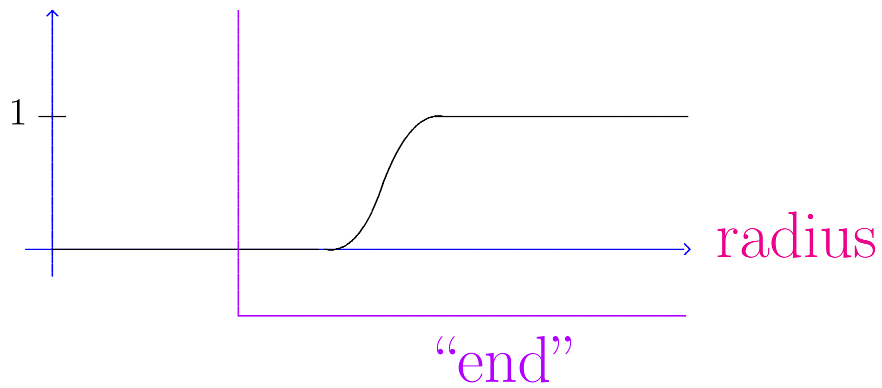

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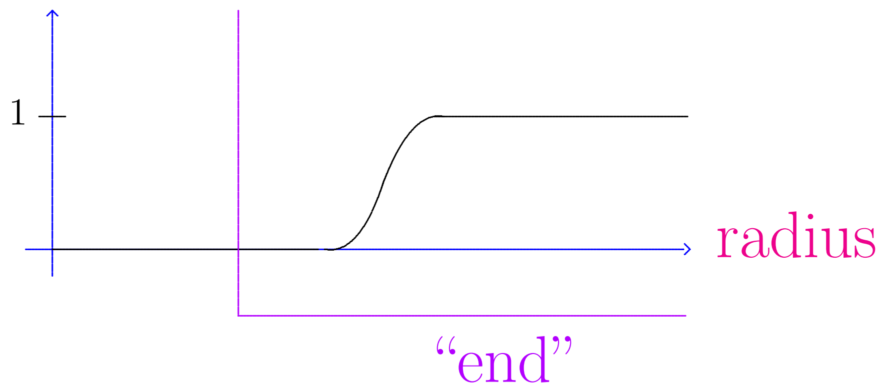

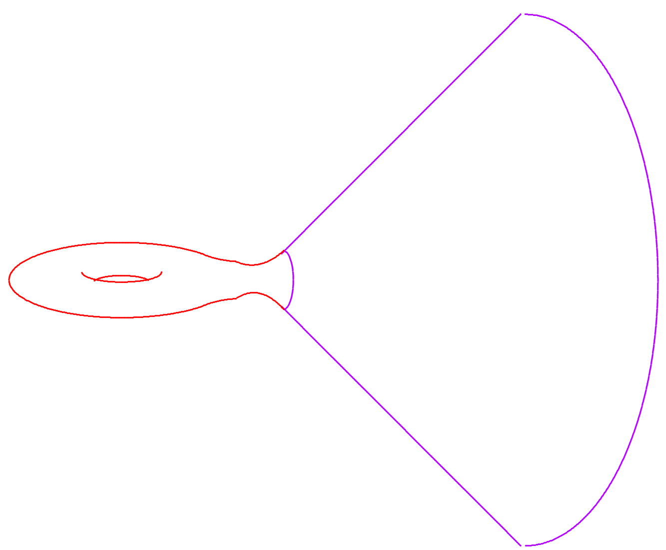

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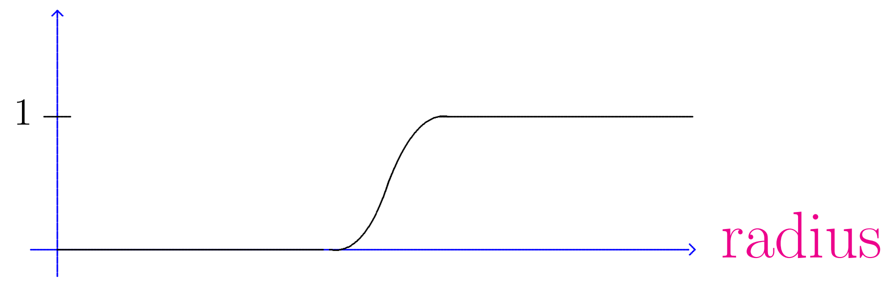

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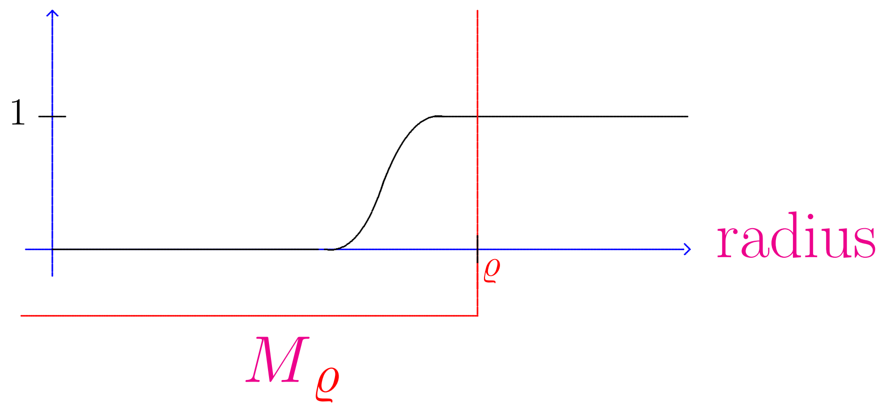

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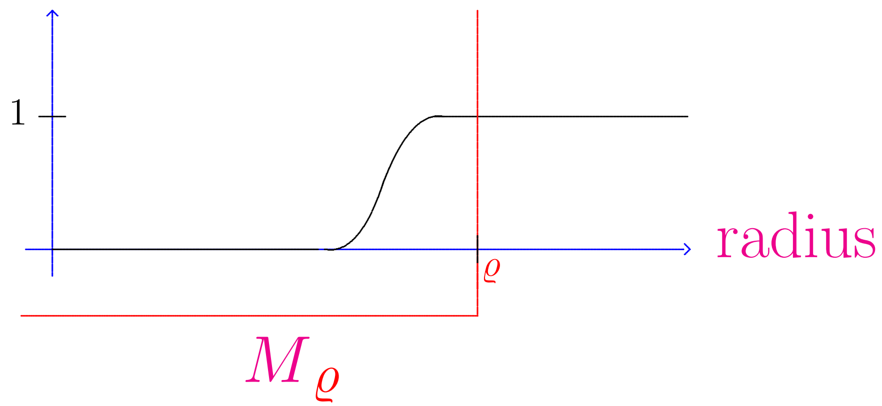

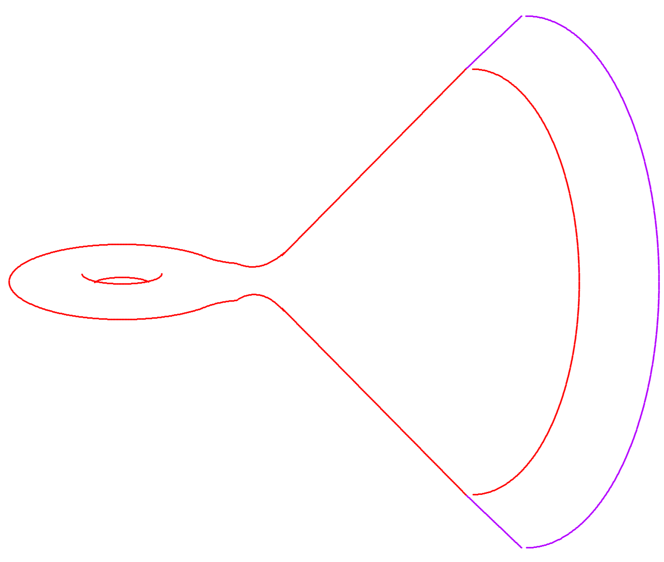

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Compactly supported, because $d\theta = \rho$ near infinity.

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where M_ϱ defined by radius $\leq \varrho$.

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$$[\psi] = \clubsuit([\rho]) = 2\pi\clubsuit(c_1) \in H^2_c(M)$$

$$\int_{S_\varrho/\Gamma} \left[g_{kj,k} - g_{kk,j}\right] \mathbf{n}^j d\mathfrak{a}_E =$$

$$-\frac{4\pi\langle\clubsuit(c_1), [\omega]^{m-1}\rangle}{(m-1)!} + \int_{M_\varrho} s \, d\mu + O(\varrho^{-2\varepsilon})$$

Let $f : M \to \mathbb{R}$ be smooth cut-off function:

$\equiv 0$ away from end,

$\equiv 1$ near infinity.

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Limit as $\varrho \to \infty$ now yields the mass formula.

$$\frac{4\pi^m(2m-1)}{(m-1)!}\mathfrak{m}(M,g) = -\frac{4\pi}{(m-1)!}\langle \clubsuit(c_1), [\omega]^{m-1}\rangle + \int_M s_g d\mu_g$$

$$\mathscr{m}(M,g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g\, d\mu_g$$

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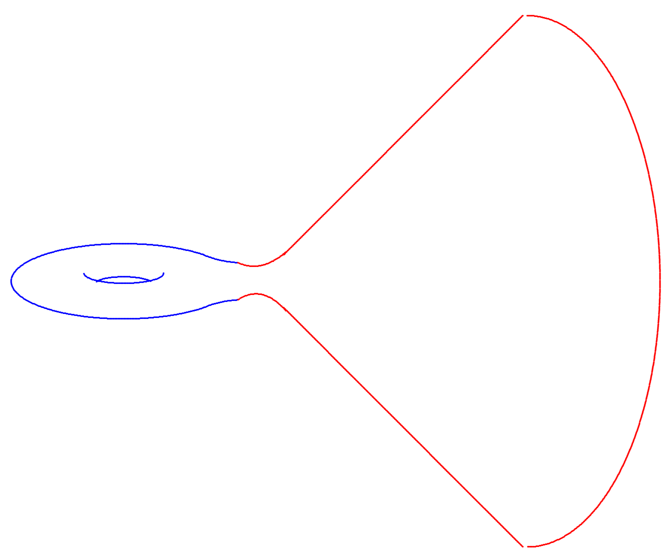

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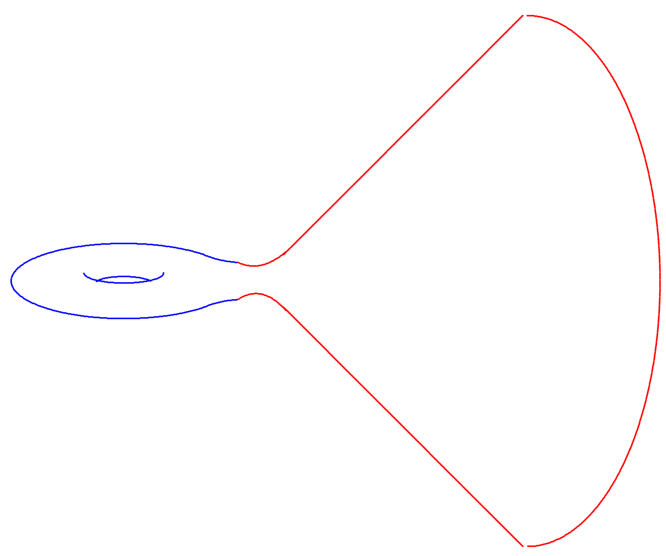

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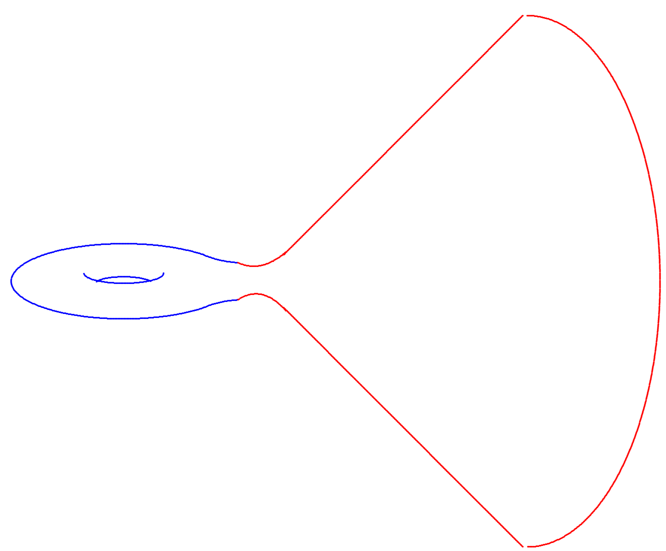
But the Penrose-type inequality is more subtle.

Theorem E (Penrose Inequality). *Let* (M^{2m}, g, J) *be an AE Kähler manifold with scalar curvature* $s \geq 0$. *Then* (M, J) *carries a canonical divisor* D *that is expressed as a sum* $\sum_j \mathbf{n}_j D_j$ *of compact complex hypersurfaces with positive integer coefficients, with the property that* $\bigcup_j D_j \neq \varnothing$ *whenever* $M \not\approx \mathbb{R}^{2m}$. *In terms of this divisor, we then have*

$$\mathfrak{m}(M, g) \geq \frac{(m-1)!}{(2m-1)\pi^{m-1}} \sum \mathbf{n}_j \, Vol(D_j)$$

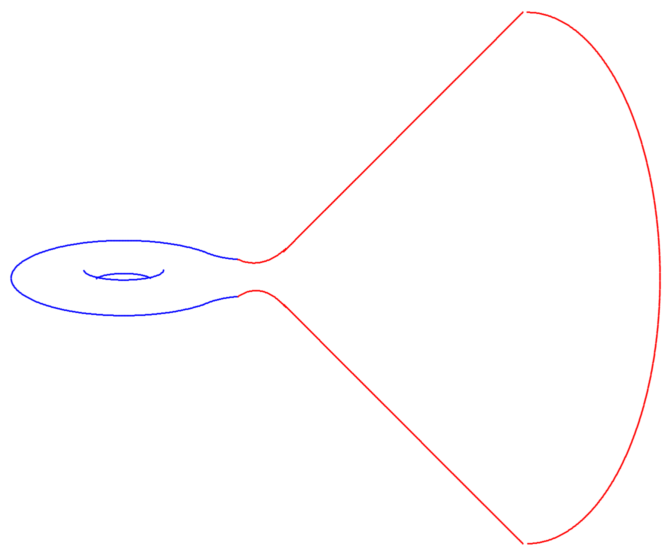
with $= \iff (M, g, J)$ *is scalar-flat Kähler.*

$$\mathfrak{m}(M,g) = \frac{\langle \clubsuit(-c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$



$m = 2:$

$$\mathfrak{m}(M, g) = \frac{\langle \clubsuit(-c_1), [\omega]\rangle}{3\pi} + \frac{1}{12\pi}\int_M s_g d\mu_g$$



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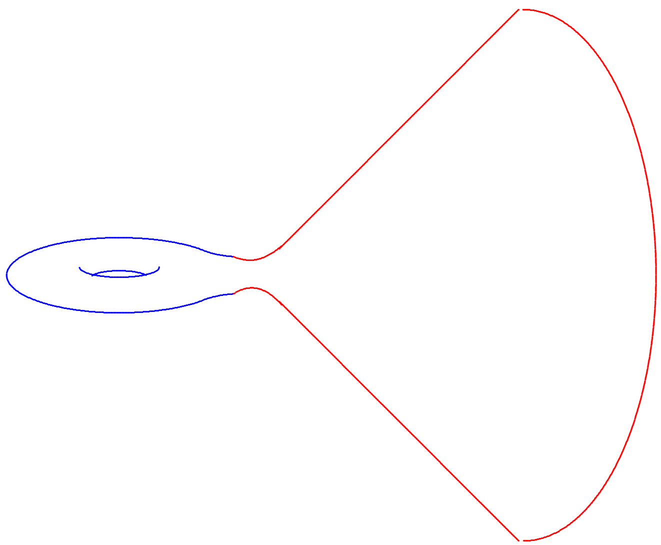
Technical challenge: Loss of control of derivatives!

Distance-decreasing map:



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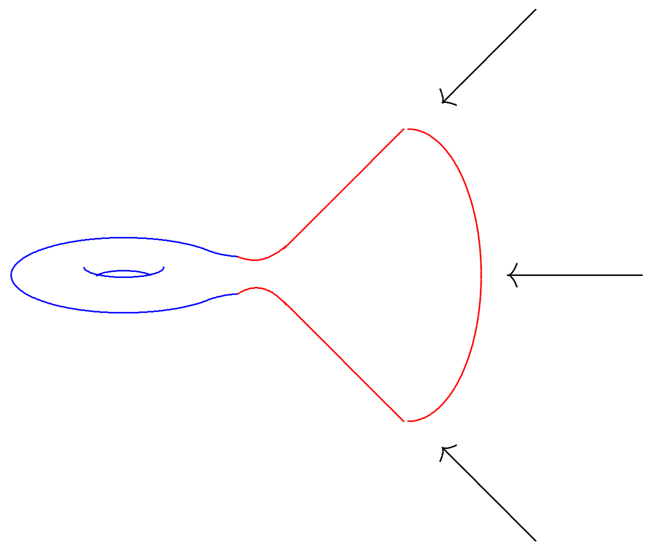

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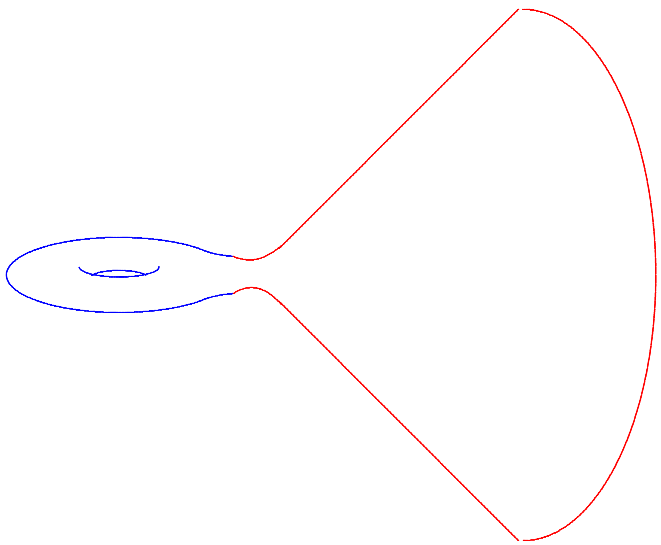

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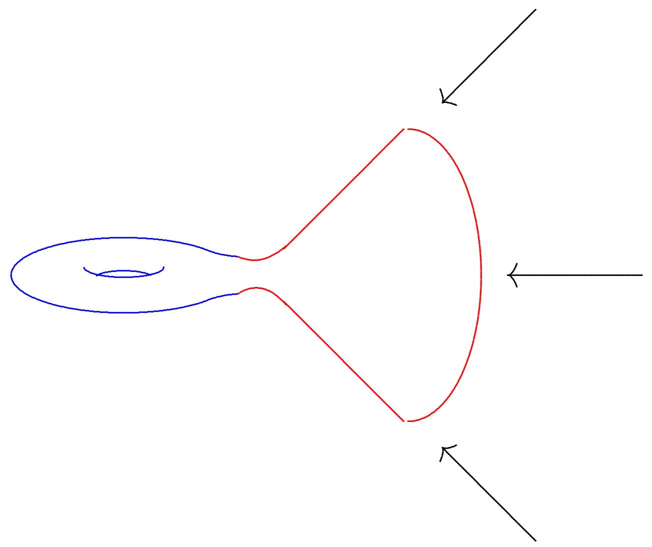

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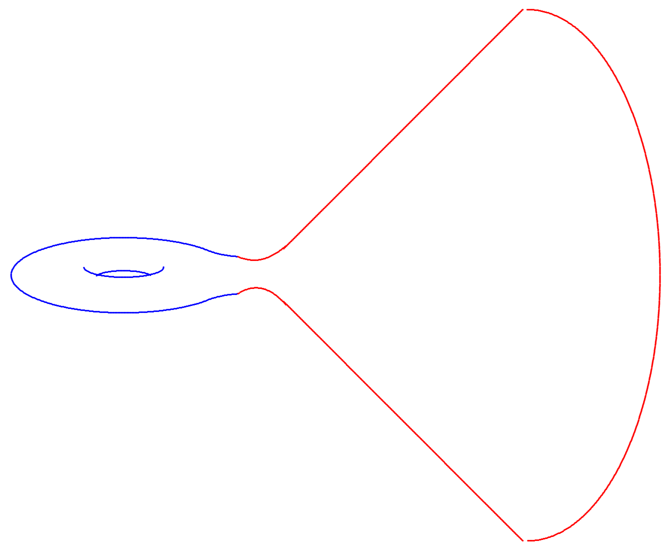

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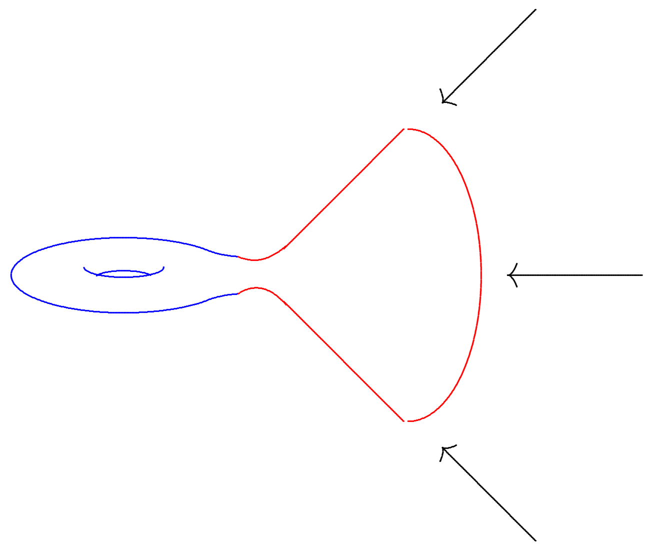

Robust under distortion of metric in outer region.

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In (M, J), this gives desired Poincaré dual of $-c_1$.

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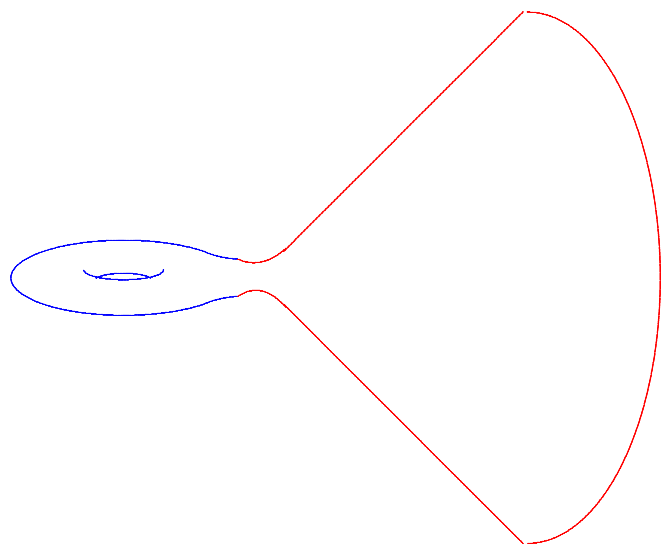

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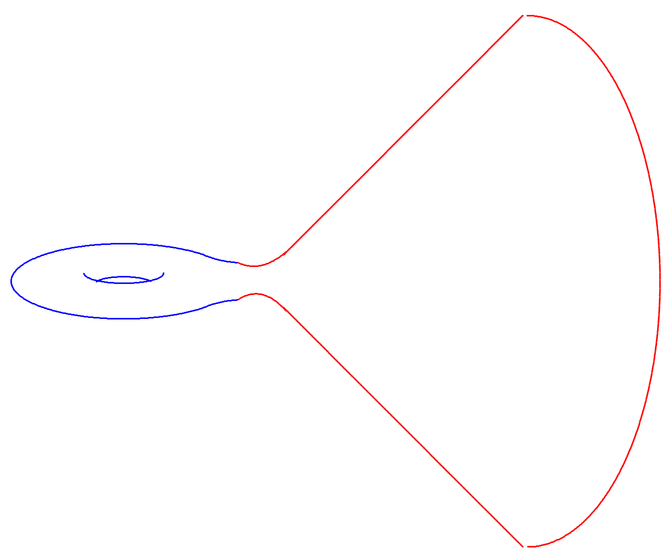

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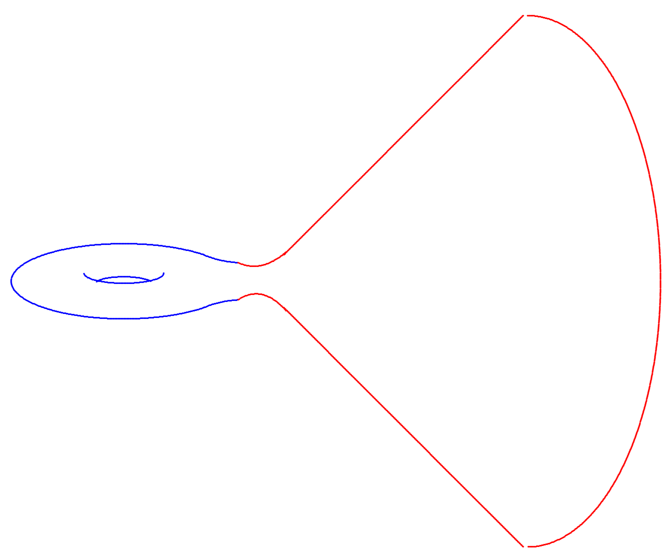

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