Einstein Metrics, Minimizing Sequences, and the Differential Topology of Four-Manifolds

Claude LeBrun
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Ricci curvature measures volume distortion by exponential map:
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\[ d\mu_g = \left[ 1 - \frac{1}{6} r_{jk} x^j x^k + O(|x|^3) \right] d\mu_{\text{Euclidean}}, \]

where \( r \) is the Ricci tensor \( r_{jk} = \mathcal{R}^i_{\ jik} \).
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- When \( n = 4 \): No! (Hitchin)
- When \( n = 5 \): Yes?? (Boyer-Galicki-Kollár)
Question (Yamabe). Does every smooth compact 1-connected $n$-manifold admit an Einstein metric?

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- When $n \geq 6$, wide open. Maybe???
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By contrast, high-dimensional Einstein metrics too common, so have little to do with geometrization.
Variational Problems

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If $M$ smooth compact $n$-manifold, $n \geq 3$,

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then Einstein metrics = critical points of normalized \textit{total scalar curvature} functional

$$\mathcal{G}_M \longrightarrow \mathbb{R}$$

$$g \longmapsto V^{(2-n)/n} \int_M s_g d\mu_g$$

where $V = \text{Vol}(M, g)$ inserted to make scale-invariant.
If \( \not \exists g \in \mathcal{G}_M \) with \( s > 0 \),
\[ \Rightarrow \] any metric minimizing
\[ \mathcal{G}_M \longrightarrow \mathbb{R} \]
\[ g \overset{\longleftrightarrow}{\mapsto} \int_M |s_g|^{n/2}d\mu_g \]
must be Einstein.
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If such Einstein minimizer exists, also minimizes
\[ g \mapsto \int_M |r|_g^{n/2}d\mu_g \]
since
\[ |r|_g^2 = \frac{s^2}{n} + |\dot{r}|_g^2 \geq \frac{s^2}{n} \]
with $\equiv \iff$ Einstein.
Two soft Invariants:

\[ \mathcal{I}_s(M) = \inf_G \int_M |s_g|^{n/2} d\mu_g \]
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Some other goals of this talk:

• compute these invariants for many 4-manifolds;
• describe minimizing sequences for functionals;
• show that above inequality often strict;
• provide context for Anderson’s talk.
What’s so special about dimension 4?

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$$so(4) \cong so(3) \oplus so(3).$$
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On oriented $(M^4, g)$, $\implies$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are $(\pm 1)$-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$
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$\Lambda^+$ self-dual 2-forms.

$\Lambda^-$ anti-self-dual 2-forms.
Riemann curvature of $g$

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splits into 4 irreducible pieces:
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splits into 4 irreducible pieces:

$$\begin{array}{cc}
\Lambda^+ & \Lambda^{+-} \\
W_+ + \frac{s}{12} & \hat{r} \\
\Lambda^- & W_- + \frac{s}{12} \\
\hat{r} & \\
\end{array}$$

where

$s =$ scalar curvature

$\hat{r} =$ trace-free Ricci curvature

$W_+$ = self-dual Weyl curvature

$W_-$ = anti-self-dual Weyl curvature
(\(M, g\)) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

\[
\chi(M) = \frac{1}{8\pi^2} \int_M \left( \frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{\bar{r}^2}{2} \right) d\mu
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for Euler-characteristic \(\chi(M) = \sum_j (-1)^j b_j(M)\).
4-dimensional Hirzebruch signature formula

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Here \( b_{\pm}(M) = \max \text{ dim subspaces } \subset H^2(M, \mathbb{R}) \) on which intersection pairing

\[ H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \longrightarrow \mathbb{R} \]

\( ( [\varphi], [\psi] ) \mapsto \int_M \varphi \wedge \psi \)

is positive (resp. negative) definite.
Associated ‘square-norm’

\[ H^2(M, \mathbb{R}) \longrightarrow \mathbb{R} \]

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- they have the **same Euler characteristic** $\chi$;
- they have the **same signature** $\tau$; and
- both are spin, or both are non-spin.
**Theorem** (Freedman/Donaldson). *Two smooth compact simply connected oriented 4-manifolds are orientedly homeomorphic if and only if*

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Warning: “Exotic differentiable structures!”

No diffeomorphism classification currently known!

Typically, one homeotype $\leftrightarrow \infty$ many diffeotypes.
Hitchin-Thorpe Inequality:

\[(2\chi \pm 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W \pm|^2 - \frac{|r|^2}{2} \right) d\mu_g \]
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Einstein \implies \quad = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W\pm|^2 \right) \, d\mu_g
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Einstein \(\Rightarrow\) \[
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**Theorem (Hitchin-Thorpe Inequality).** If smooth compact oriented \(M^4\) admits Einstein \(g\), then

\[
(2\chi + 3\tau)(M) \geq 0
\]

and

\[
(2\chi - 3\tau)(M) \geq 0.
\]
Example.

Let $\overline{\text{CP}_2} = \text{reverse-oriented CP}_2$.

$$ j\overline{\text{CP}_2} \# k\overline{\text{CP}_2} = \underbrace{\text{CP}_2 \# \cdots \# \text{CP}_2}_{j} \# \underbrace{\overline{\text{CP}_2} \# \cdots \# \overline{\text{CP}_2}}_{k}, $$
Connected sum:
Example.

Let $\overline{\mathbb{CP}}_2 = \text{reverse-oriented } \mathbb{CP}_2$. Then

$$j\overline{\mathbb{CP}}_2 \# k\overline{\mathbb{CP}}_2 = \underbrace{\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2}_j \# \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_k,$$

has

$$2\chi + 3\tau = 4 + 5j - k$$

so $\not\exists$ Einstein metric if $k \geq 4 + 5j$. 

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\[(2\chi \pm 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W\pm|^2 - \frac{|\dot{r}|^2}{2} \right) d\mu_g \]

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Theorem (Hitchin-Thorpe Inequality). If smooth compact oriented \(M^4\) admits Einstein \(g\), then

\[(2\chi + 3\tau)(M) \geq 0\]

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Both inequalities strict unless finitely covered by flat \(T^4\), Calabi-Yau \(K3\), or Calabi-Yau \(\overline{K3}\).
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Spin manifold, $b_+ = 3$, $b_- = 19$. 
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**Theorem (Yau).** $K3$ admits Ricci-flat metrics.
Kummer construction of $K3$: 
Kummer construction of \( \text{K3} \):

Begin with \( T^4/\mathbb{Z}_2 \):

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{kummer_construction.png}
\end{array}
\]
Kummer construction of $K3$:

Begin with $T^4/\mathbb{Z}_2$:

Replace $\mathbb{R}^4/\mathbb{Z}_2$ neighborhood of each singular point with copy of $T^*S^2$. 
Approximate Calabi-Yau metric:

Replace flat metric on $\mathbb{R}^4/\mathbb{Z}_2$

with Eguchi-Hanson metric on $T^*S^2$:

$$g_{EH,\epsilon} = \frac{d\varrho^2}{1 - \epsilon \varrho^{-4}} + \varrho^2 \left( \theta_1^2 + \theta_2^2 + \left[ 1 - \epsilon \varrho^{-4} \right] \theta_3^2 \right)$$

(Page, Kobayashi-Todorov, LeBrun-Singer)
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**Theorem** (Aubin/Yau). *Compact complex manifold* \((M^{2m}, J)\) *admits compatible Kähler-Einstein metric with* \(s < 0\) \iff \(\exists\) holomorphic embedding.*
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\[ j : M \hookrightarrow \mathbb{CP}_k \]
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**Theorem (Aubin/Yau).** Compact complex manifold \((M^{2m}, J)\) admits compatible Kähler-Einstein metric with \(s < 0 \iff \exists \text{ holomorphic embedding } j : M \hookrightarrow \mathbb{C}P_k\) such that \(c_1(M)\) is negative multiple of \(j^*c_1(\mathbb{C}P_k)\).
Corollary. For any $\ell \geq 5$, the degree $\ell$ surface

$$t^\ell + u^\ell + v^\ell + w^\ell = 0$$

in $\mathbb{CP}_3$ admits $s < 0$ Kähler-Einstein metric.
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**Remark.** This happens \(\iff -c_1(M)\) is a Kähler class. Short-hand: \(c_1(M) < 0\).
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**Remark.** This happens $\iff -c_1(M)$ is a Kähler class. Short-hand: $c_1(M) < 0$.

**Remark.** When $m = 2$, such $M$ are necessarily minimal complex surfaces of general type.
Blowing up:

If $N$ is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}_2}$$

in which new $\mathbb{CP}_1$ has self-intersection $-1$. 
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A complex surface $X$ is called minimal if it is not the blow-up of another complex surface.

Any complex surface $M$ can be obtained from a minimal surface $X$ by blowing up a finite number of times:

$$M \approx X \# k\mathbb{C}P_2$$

One says that $X$ is minimal model of $M$. 
Compact complex surface \((M^4, J)\) general type if
\[\dim \Gamma(M, \mathcal{O}(K^\otimes \ell)) \sim a\ell^2, \quad \ell \gg 0,\]
where \(K = \Lambda^{2,0}\) is canonical line bundle.
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\] where $K = \Lambda^{2,0}$ is canonical line bundle.

If $\ell \geq 5$, then $\Gamma(M, \mathcal{O}(K^\otimes \ell))$ gives holomorphic map

$$f_\ell : M \to \mathbb{CP}_N$$

which just collapses each $\mathbb{CP}_1$ with self-intersection $-1$ or $-2$ to a point. Image $X = f_\ell(M)$ called pluricanonical model of $M$. 
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\[ f_\ell : M \to \mathbb{CP}_N \]

which just collapses each \(\mathbb{CP}_1\) with self-intersection \(-1\) or \(-2\) to a point. Image \(X = f_\ell(M)\) called pluricanonical model of \(M\).

Pluricanonical model \(X\) is a complex orbifold with \(c_1 < 0\) and singularities \(\mathbb{C}^2/G, \ G \subset SU(2)\).
Aubin-Yau proof \implies

**Corollary** (R. Kobayashi). The *pluricanonical model* $X$ of any compact complex surface $M$ of general type admits and *orbifold* Kähler-Einstein metric with $s < 0$. 
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Seiberg-Witten theory:

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spin$^c$ Dirac operator, preferred connection on $L$. 
Spin$^c$ structures:
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\[ w_2(TM) \in H^2(M, \mathbb{Z}_2) \]

in image of

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\[ L \to M \]

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where \( S_\pm \) are the (locally defined) left- and right-handed spinor bundles of \((M, g)\).
Every unitary connection $A$ on $L$
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Weitzenböck formula: $\forall \Phi \in \Gamma(\mathbb{V}_+)$,

$$\langle \Phi, D_A^* D_A \Phi \rangle = \frac{1}{2} \Delta |\Phi|^2 + |\nabla_A \Phi|^2 + \frac{s}{4} |\Phi|^2$$
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where $F_A^+$ = self-dual part curvature of $A$, and $\sigma : \mathcal{V}_+ \rightarrow \Lambda^+$ is a natural real-quadratic map,

$$|\sigma(\Phi)| = \frac{1}{2\sqrt{2}} |\Phi|^2.$$
Witten:

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Non-linear, but elliptic once ‘gauge-fixing’

\[
d^*(A - A_0) = 0
\]

imposed to eliminate automorphisms of \( L \to M \).
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When invariant is non-zero, solutions guaranteed.
Definition. Let $M$ be a smooth compact oriented 4-manifold with $b_+ \geq 2$. 
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have a solution $(\Phi, A)$ for every metric $g$ on $M$. 
Proposition. Let $M$ be a smooth compact oriented 4-manifold with $b_+ \geq 2$. 
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If \( \mathcal{C} \neq \emptyset \), let

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If \( \mathcal{C} = \emptyset \), set \( \beta^2(M) = 0 \).
Example  If $X$ is a minimal complex surface with $b_+ > 1$, and if

$$M = X \# \ell \overline{\mathbb{CP}}_2$$

then ‘classical’ Seiberg-Witten invariant allows one to show that

$$\beta^2(M) = c_1^2(X).$$
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**Example** If $X$, $Y$, $Z$ are minimal complex surfaces with $b_1 = 0$ and $b_+ \equiv 3 \mod 4$, and if

$$M = X \# Y \# Z \# \ell \overline{\mathbb{CP}}_2$$

Bauer-Furuta invariant allows one to show that

$$\beta^2(M) = c_1^2(X) + c_1^2(Y) + c_1^2(Z)$$

Similarly for 2 or 4...
Theorem (Curvature Estimates). For any $C^2$ Riemannian metric $g$
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Theorem (Curvature Estimates). For any $C^2$ Riemannian metric $g$ on any smooth compact oriented 4-manifold $M$ with $b_+ \geq 2$, the following curvature bounds are satisfied:

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Moreover, if $\beta^2(M) \neq 0$, equality holds in either case iff $(M, g)$ is a Kähler-Einstein manifold with $s < 0$. 
\[ \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W - |^2 \right) d\mu_g \geq \frac{1}{4\pi^2} \int_M \frac{s^2}{24} d\mu_g \]
First curvature estimate implies

\[
\frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W - |^2 \right) d\mu_g \geq \frac{1}{4\pi^2} \int_M \frac{s^2}{24} d\mu_g \\
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Hence:

**Theorem A.** Let \( M \) be a smooth compact oriented 4-manifold with \( b_+(M) \geq 2 \).
First curvature estimate implies

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\frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g \geq \frac{1}{4\pi^2} \int_M \frac{s^2}{24} d\mu_g \\
\geq \frac{1}{3} \beta_2^2(M)
\]

Hence:

**Theorem A.** Let \( M \) be a smooth compact oriented 4-manifold with \( b_+(M) \geq 2 \). If \( M \) admits an Einstein metric \( g \),
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Hence:

**Theorem A.** Let \( M \) be a smooth compact oriented 4-manifold with \( b_+ (M) \geq 2 \). If \( M \) admits an Einstein metric \( g \), then

\[
(2\chi - 3\tau)(M) \geq \frac{1}{3} \beta^2(M)
\]
First curvature estimate implies

\[ \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_1|^2 \right) d\mu_g \geq \frac{1}{4\pi^2} \int_M \frac{s^2}{24} d\mu_g \geq \frac{1}{3} \beta^2(M) \]

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\[ (2\chi - 3\tau)(M) \geq \frac{1}{3} \beta^2(M) \]

with equality only if \((M, g)\) is flat \( T^4 \) or complex hyperbolic \( \mathbb{C}H_2/\Gamma \).
First curvature estimate implies

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**Theorem A.** Let $M$ be a smooth compact oriented 4-manifold with $b_+(M) \geq 2$. If $M$ admits an Einstein metric $g$, then

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(2\chi - 3\tau)(M) \geq \frac{1}{3} \beta^2(M)
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with equality only if $(M, g)$ is flat $T^4$ or complex hyperbolic $\mathbb{C}H_2/\Gamma$.

$\implies$ Einstein metric on $\mathbb{C}H_2/\Gamma$ unique, . . .
\[ \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W + |^2 \right) d\mu_g \]
Second curvature estimate implies

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\frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W|^2 \right) d\mu_g \geq \frac{2}{3} \beta^2(M)
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Hence:

**Theorem B.** Let $M$ be a smooth compact oriented 4-manifold with $b_+(M) \geq 2$. 

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**Theorem B.** Let \( M \) be a smooth compact oriented 4-manifold with \( b_+(M) \geq 2 \). If \( M \) admits an Einstein metric \( g \), then

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with equality only if both sides vanish, in which case \( g \) must be hyper-Kähler, and \( M \) must be diffeomorphic to either \( K3 \) or \( T^4 \).
**Example**  Let $N$ be double branched cover $\mathbb{CP}^2$, ramified at a smooth octic:

\[ \begin{array}{c}
\begin{array}{c}
N \\
\downarrow \\
B'
\end{array} \\
\rightarrow \\
\begin{array}{c}
\mathbb{CP}^2 \\
B'
\end{array}
\end{array} \]

Aubin/Yau $\implies N$ carries Einstein metric.
Now let $X$ be a triple cyclic cover $\mathbb{CP}_2$, ramified at a smooth sextic.
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\[ M = X \# \overline{\mathbb{CP}^2}. \]
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and set

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Then

$$\beta^2(M) = c_1^2(X) = 3$$
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\[ (2\chi + 3\tau)(M) = c_1^2(X) - 1 = 2 \]
Theorem B. Let $M$ be a smooth compact oriented 4-manifold with $b_+(M) \geq 2$. If $M$ admits an Einstein metric $g$, then

$$\begin{align*}
(2\chi + 3\tau)(M) &\geq \frac{2}{3}\beta^2(M) \\
\text{& equality only if } M \text{ diffeomorphic to } K3 \text{ or } T^4.
\end{align*}$$
Theorem B. Let $M$ be a smooth compact oriented 4-manifold with $b_+(M) \geq 2$. If $M$ admits an Einstein metric $g$, then

$$\left(2\chi + 3\tau\right)(M) \geq \frac{2}{3}\beta^2(M)$$

& equality only if $M$ diffeomorphic to $K3$ or $T^4$.

In example:

$$\beta^2(M) = 3$$
$$\left(2\chi + 3\tau\right)(M) = 2$$
\( X \) is triple cover \( \mathbb{CP}_2 \) ramified at sextic

\[
M = X \# \overline{\mathbb{CP}_2}.
\]

Theorem B \( \implies \) no Einstein metric on \( M \).
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Hence Freedman $\implies M$ homeomorphic to $N$!
But $M$ and $N$ are both simply connected & non-spin, and both have $c_1^2 = 2$, $h^{2,0} = 3$, so

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Hence Freedman $\Rightarrow M$ homeomorphic to $N$! ♦

Moral: *Existence depends on diffeotype!*

Same ideas lead to infinitely many other examples.

Typically get **non-existence** for infinitely many smooth structures on fixed topological manifold.

Until now, discussed arbitrary Einstein metrics.

Instead, focus on Einstein metrics which minimize

\[ g \mapsto \int_M s_g^2 d\mu_g \]

Related to soft invariants

\[ \mathcal{I}_s(M) = \inf_g \int_M s_g^2 d\mu_g \]

\[ \mathcal{I}_r(M) = \inf_g \int_M |r|^2_g d\mu_g \]

which satisfy

\[ \mathcal{I}_r(M) \geq \frac{1}{4} \mathcal{I}_s(M) \]

with \( \iff \exists \) Einstein minimizer.
**Theorem (Curvature Estimates).** For any $C^2$ Riemannian metric $g$ on any smooth compact oriented 4-manifold $M$ with $b_+ \geq 2$, the following curvature bounds are satisfied:

\[
\int_M s^2 d\mu_g \geq 32\pi^2 \beta^2(M)
\]

\[
\int_M |r|_g^2 d\mu_g \geq 8\pi^2 \left[ 2\beta^2 - (2\chi + 3\tau) \right](M)
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**Theorem (Curvature Estimates).** For any $C^2$ Riemannian metric $g$ on any smooth compact oriented 4-manifold $M$ with $b_+ \geq 2$, the following curvature bounds are satisfied:

\[ \int_M s^2 d\mu_g \geq 32\pi^2 \beta^2(M) \]
\[ \int_M |r|^2_g d\mu_g \geq 8\pi^2 \left[ 2\beta^2 - (2\chi + 3\tau) \right](M) \]
\[ \int_M |r|^2_g d\mu_g = -8\pi^2(2\chi + 3\tau)(M) \]
\[ + 8 \int_M \left( \frac{s^2}{24} + \frac{1}{2}|W + |^2 \right) d\mu_g \]
Theorem. Suppose $M^4$ diffeo to non-minimal compact complex surface with $b_+ > 1$. Then $M$ does not admit a metric which minimizes either

\[ g \mapsto \int_M s_g^2 d\mu_g \quad \text{or} \quad \int_M |r|^2_g d\mu_g \]
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$$ g \mapsto \int_M s_g^2 d\mu_g \quad \text{or} \quad \int_M |r|^2_d d\mu_g $$

By hypothesis

$$ M = X \# k\overline{\mathbb{C}P}_2 $$

where $X$ minimal and $k > 0$.

One shows

$$ \mathcal{I}_s(M) = 32\pi^2 c_1^2(X) $$

$$ \mathcal{I}_r(M) = 8\pi^2 [c_1^2(X) + k] $$

so that

$$ \mathcal{I}_r(M) > \frac{1}{4} \mathcal{I}_s(M) $$
Theorem. Let $X$, $Y$ and $Z$ be simply connected minimal complex surfaces with $b_+ \equiv 3 \mod 4$. Then

$$M = X \# Y \# Z \# k\overline{\mathbb{CP}_2}$$

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In fact,

$$\mathcal{I}_s(M) = 32\pi^2 [c_1^2(X) + c_1^2(Y) + c_1^2(Z)]$$
$$\mathcal{I}_r(M) = 8\pi^2 [c_1^2(X) + c_1^2(Y) + c_1^2(Z) + 8 + k]$$
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Similarly for $\#$ of 2 or 4 complex surfaces.
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Similarly for $\#$ of 2 or 4 complex surfaces.

Mystery: More summands? $b_+ \equiv 1 \text{ mod } 4$?
When $X$, $Y$ and $Z$ general type, however,

$\exists$ minimizing $\{g_j\}$ with Gromov-Hausdorff limit

3 Kähler-Einstein orbifolds touching at points.
\exists \text{ points where curvature has accumulated.}
Predictable amount of $\hat{r}$ accumulates on necks.
Rescaled limit of neck carries $AE$ metric with

$$s = 0$$

$$W_+ = 0$$

Example:

$$g = \left(1 + \frac{1}{\varrho^2}\right) g_{\text{Euclidean}}$$
Orbifold singularities:
rescaled metric tends to gravitational instanton:
Asymptotically Locally Euclidean metric with

\[ r = 0 \]
\[ W_+ = 0 \]
Bubbling off $\overline{\mathbb{CP}_2}$’s:

Asymptotically Euclidean metric with

\[
\begin{align*}
  s &= 0 \\
  W_+ &= 0
\end{align*}
\]
Basic example:

Burns metric on $\mathbb{CP}_2 - \{\infty\}$:

$$g_{B,\epsilon} = \frac{d\rho^2}{1 - \epsilon \rho^{-2}} + \rho^2 \left( \theta_1^2 + \theta_2^2 + \left[ 1 - \epsilon \rho^{-2} \right] \theta_3^2 \right)$$

Conformal Greens rescaling of Fubini-Study.
If one of $X$, $Y$ and $Z$ is elliptic, collapses in limit to orbifold Riemann surface.
Typical example: