Edges, Orbifolds,

and

Seiberg-Witten Theory

Claude LeBrun
Stony Brook University

Mathematical Congress of the Americas
Guanajuato, Mexico
August 9, 2013

Main reference:

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arXiv:1305.1960 [math.DG].

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arXiv:1203.6389 [math.DG],
Math. Proc. Cambr. Phil. Soc. (2013)

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Has same sign as the *scalar curvature*

$$s = r_j^j = \mathcal{R}^{ij}{}_{ij}.$$

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Proposition. *If* $n \geq 3$, *A Riemannian* n*-manifold* (M^n, g) *is Einstein iff the trace-free part of its Ricci tensor vanishes:*

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Proof. Bianchi identity $\Longrightarrow \nabla \cdot \mathring{r} = (\frac{1}{2} - \frac{1}{n})ds.$

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Anderson, Bando-Kasue-Nakajima, . . .

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But today we will focus on codimension 2 case ...

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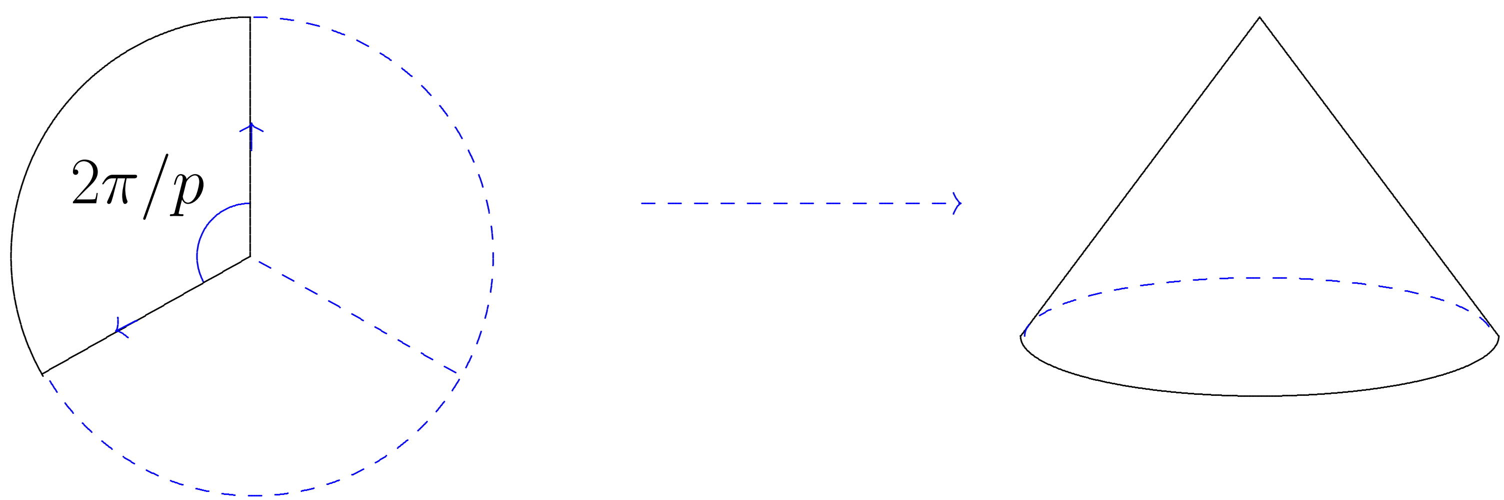

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Strategy: Deform cone angle $2\pi\beta$ with $\beta \in (0, 1/p]$.

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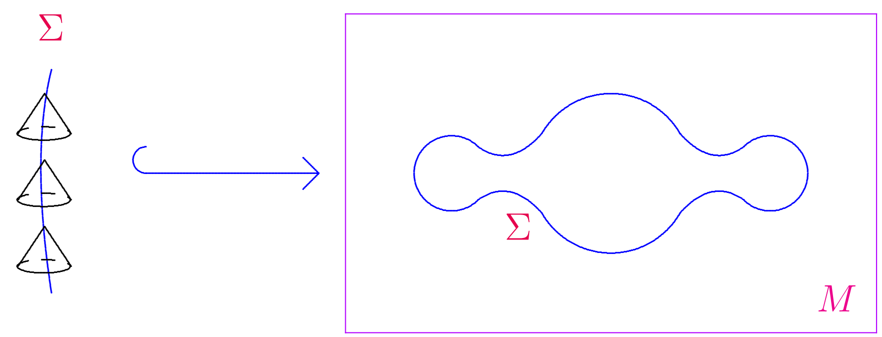

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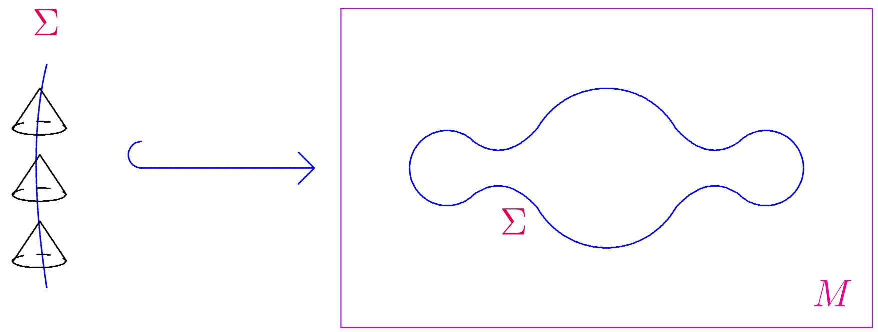

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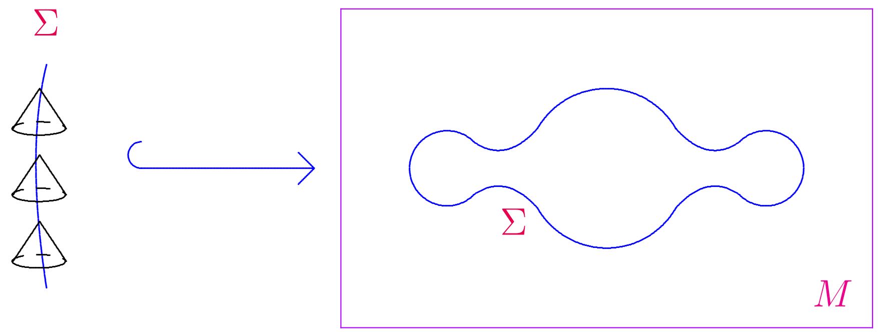

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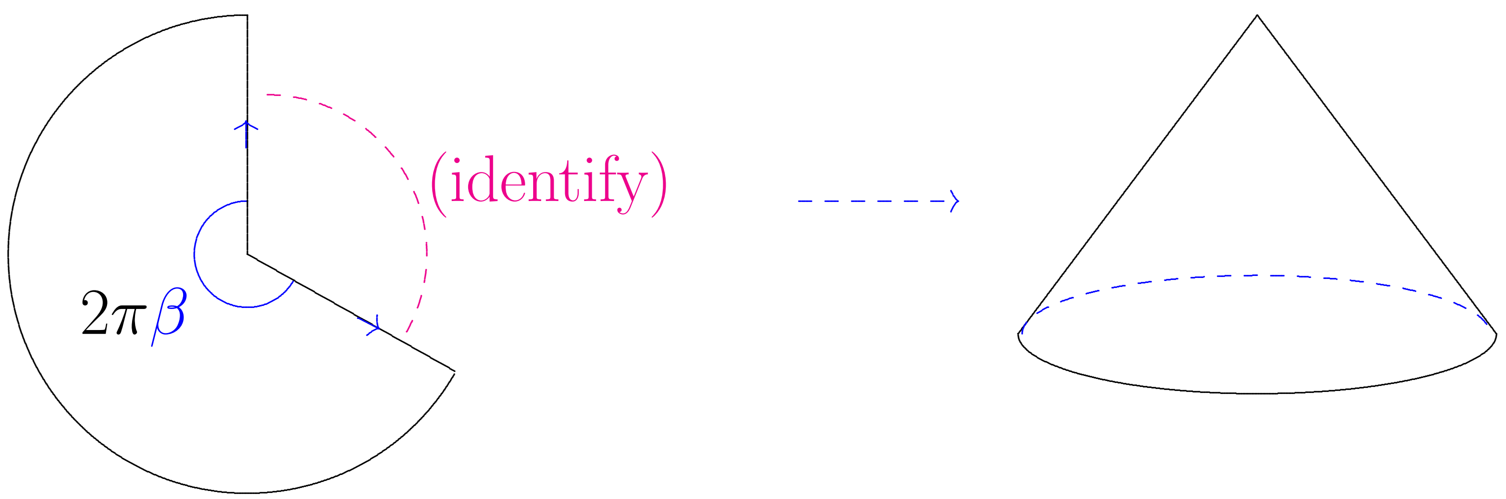

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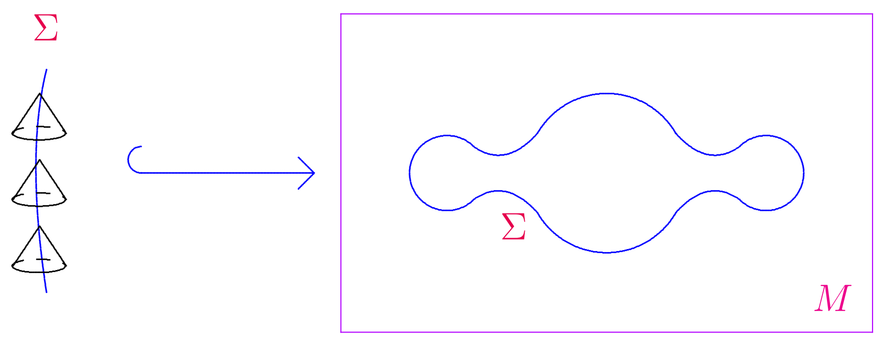

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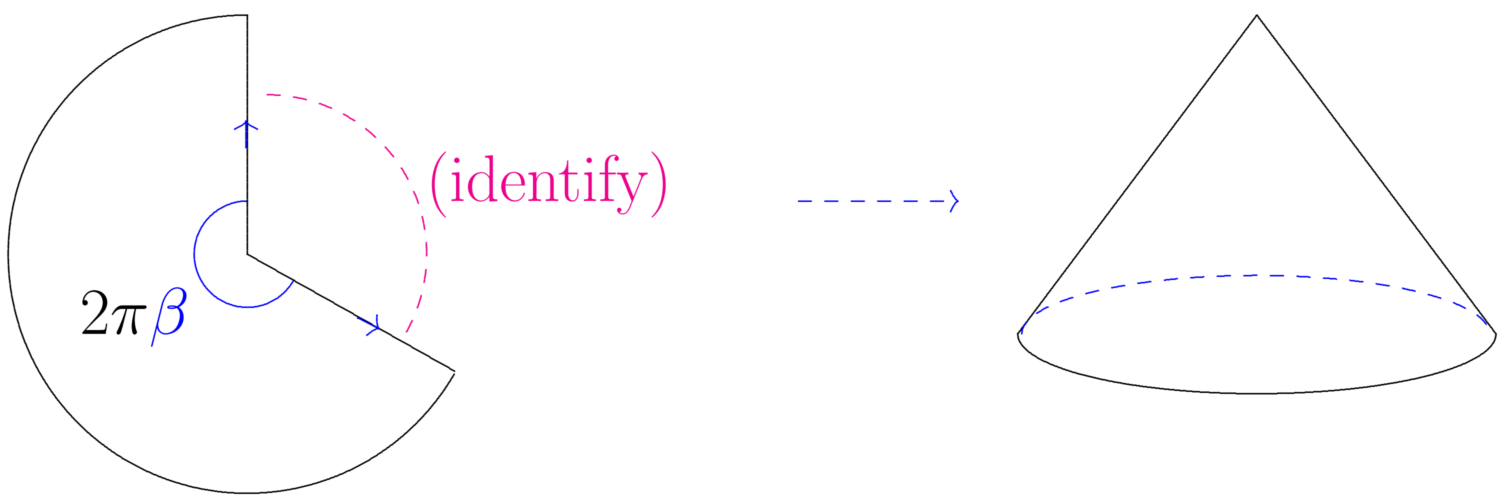

Program for $c_1 > 0$ assumes $[\Sigma] \propto c_1$.

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Theorem (Chen-Donaldson-Sun 2012-13)**.** *Let* (M, J) *be a compact complex manifold with* $c_1 > 0$*.*

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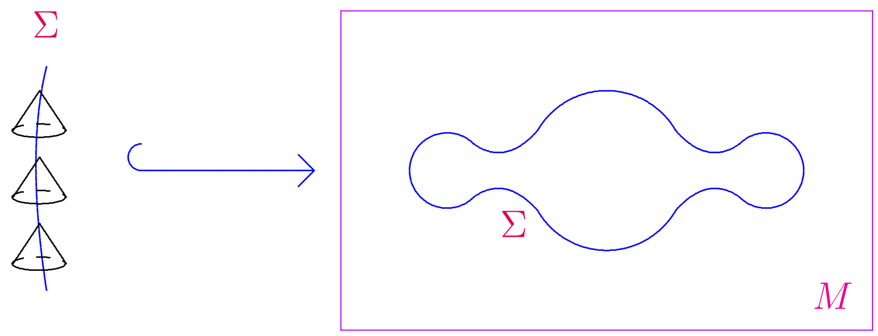
K-stability is a criterion which is formulated purely in terms of algebraic geometry. It concerns singular limits of embeddings $(M, J) \hookrightarrow \mathbb{CP}_N$.

Edge-Cone Metrics:



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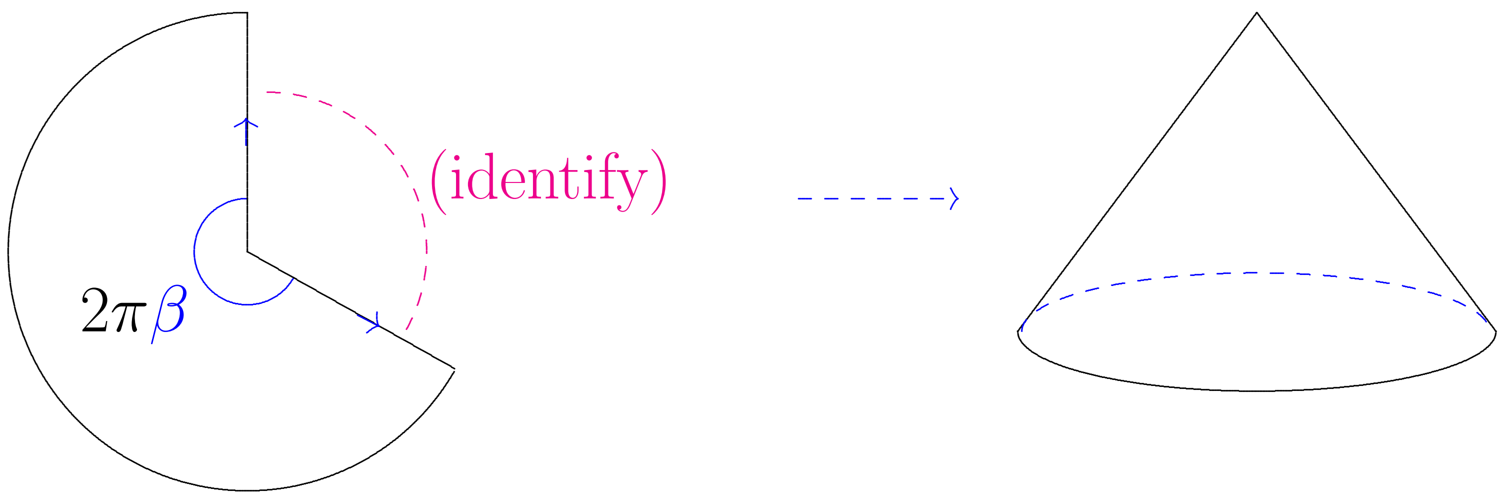

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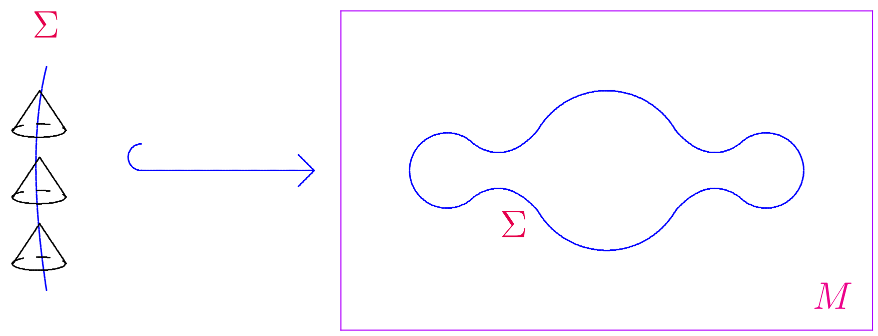

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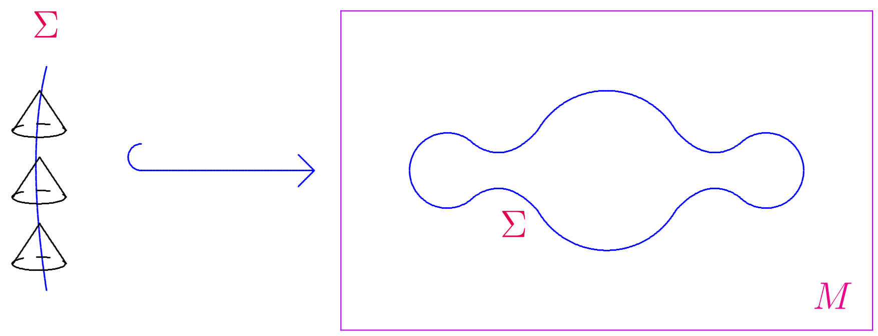

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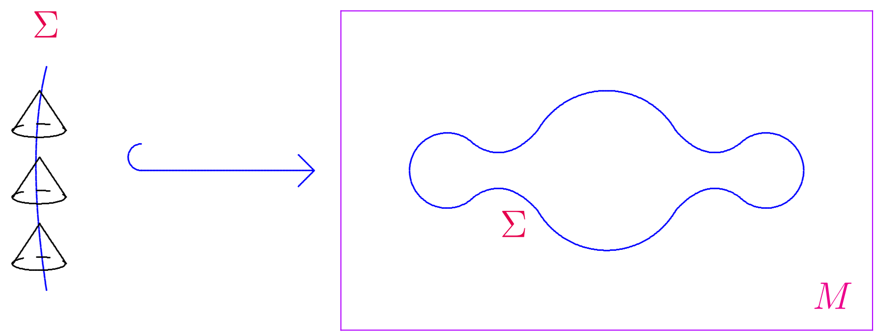

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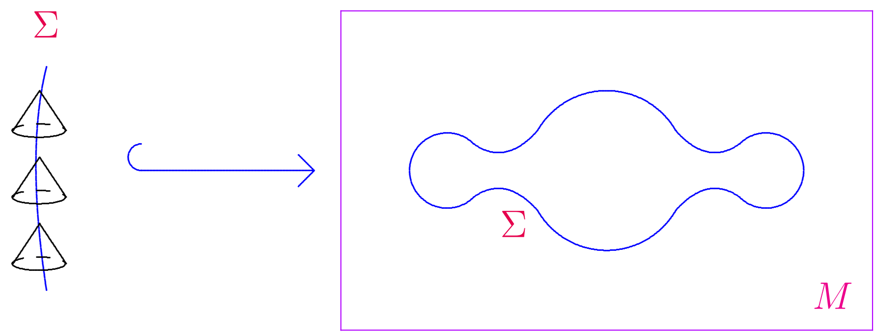

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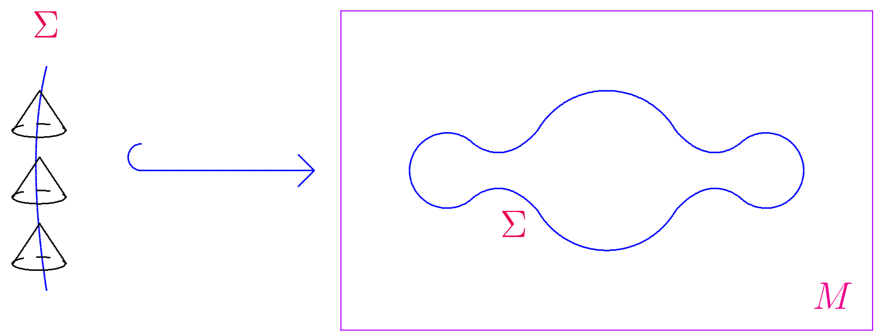

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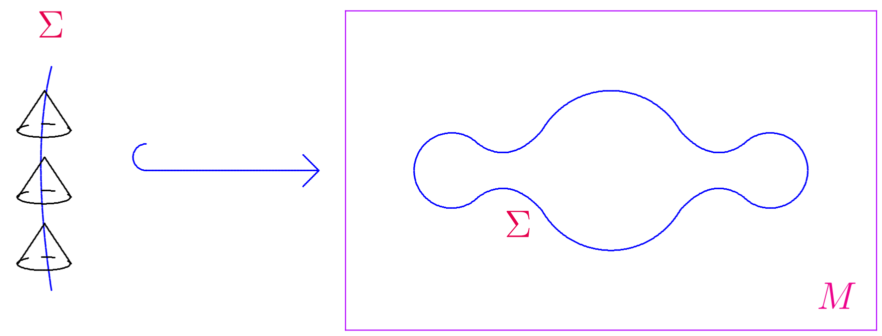

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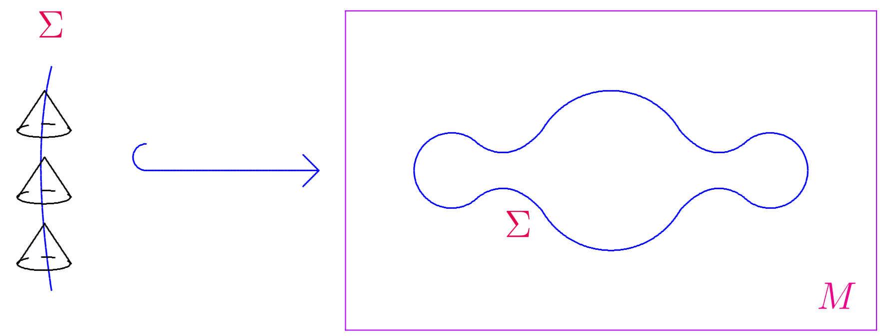

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But converse true in Einstein case...

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Begin with review of smooth case...

Two homotopy invariants

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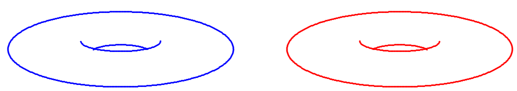

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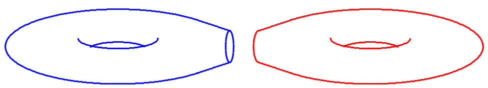

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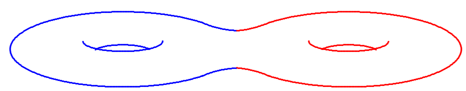

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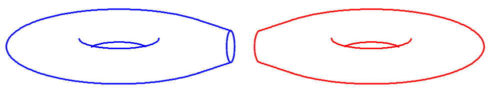

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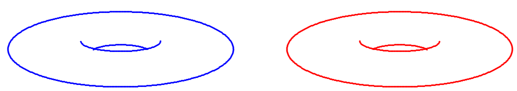

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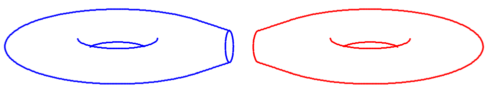

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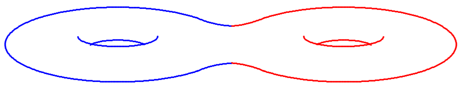

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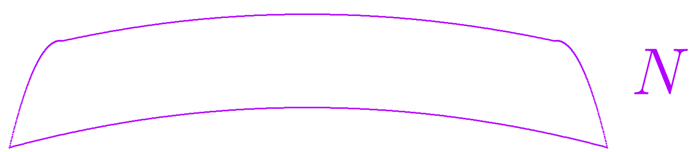

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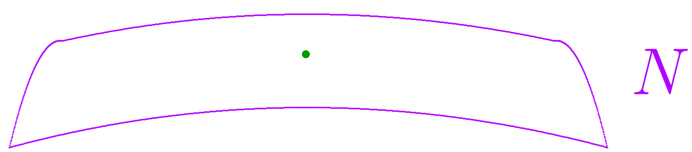

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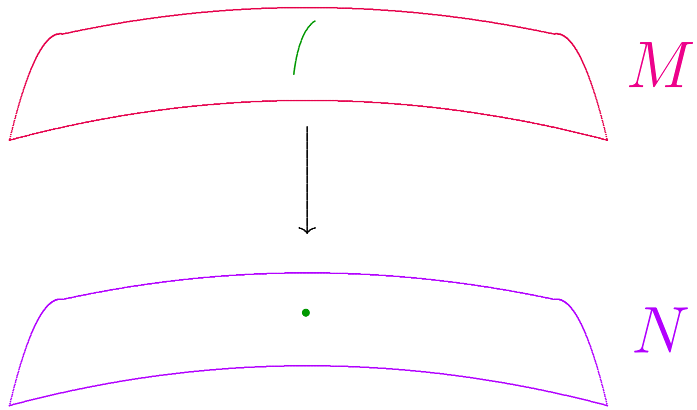

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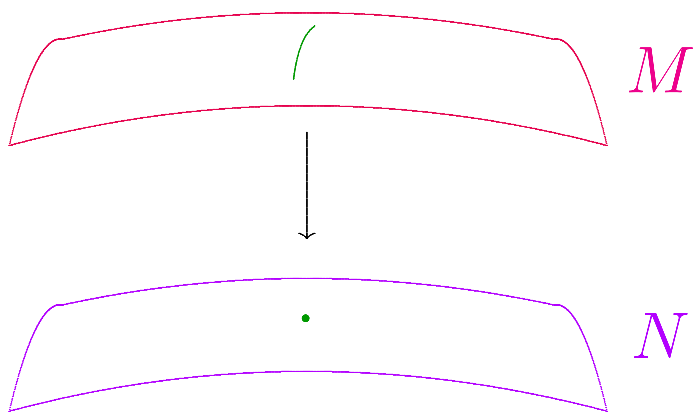

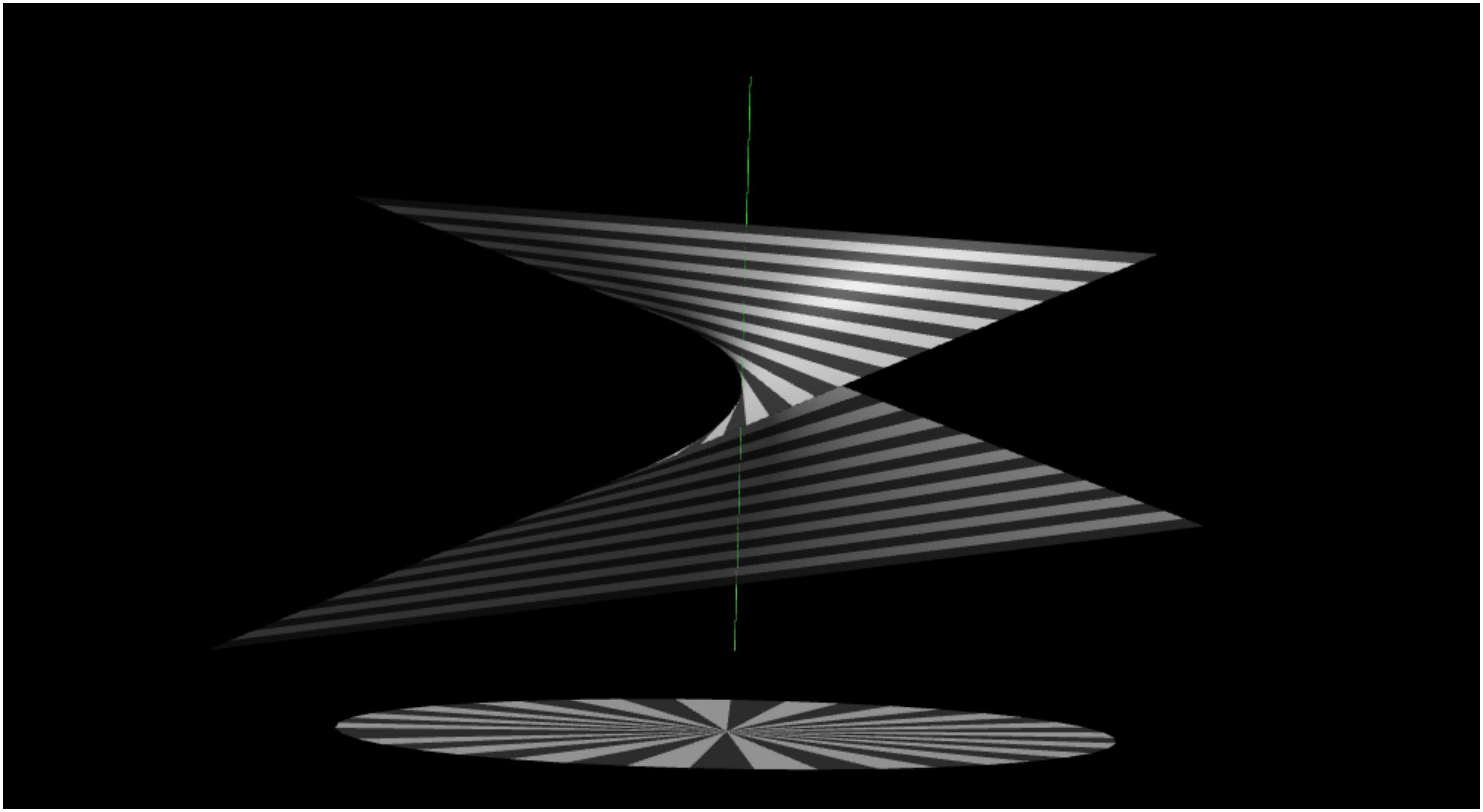

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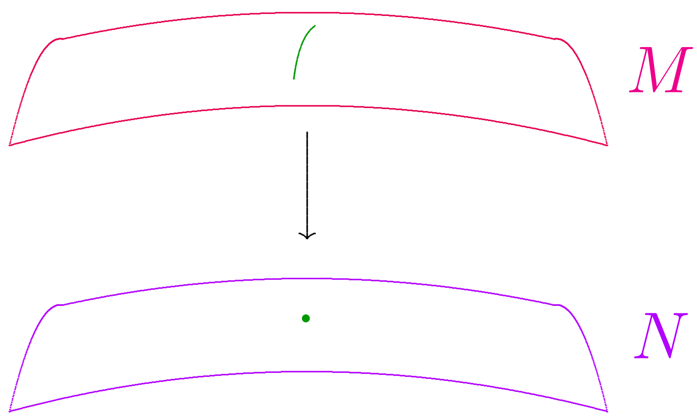

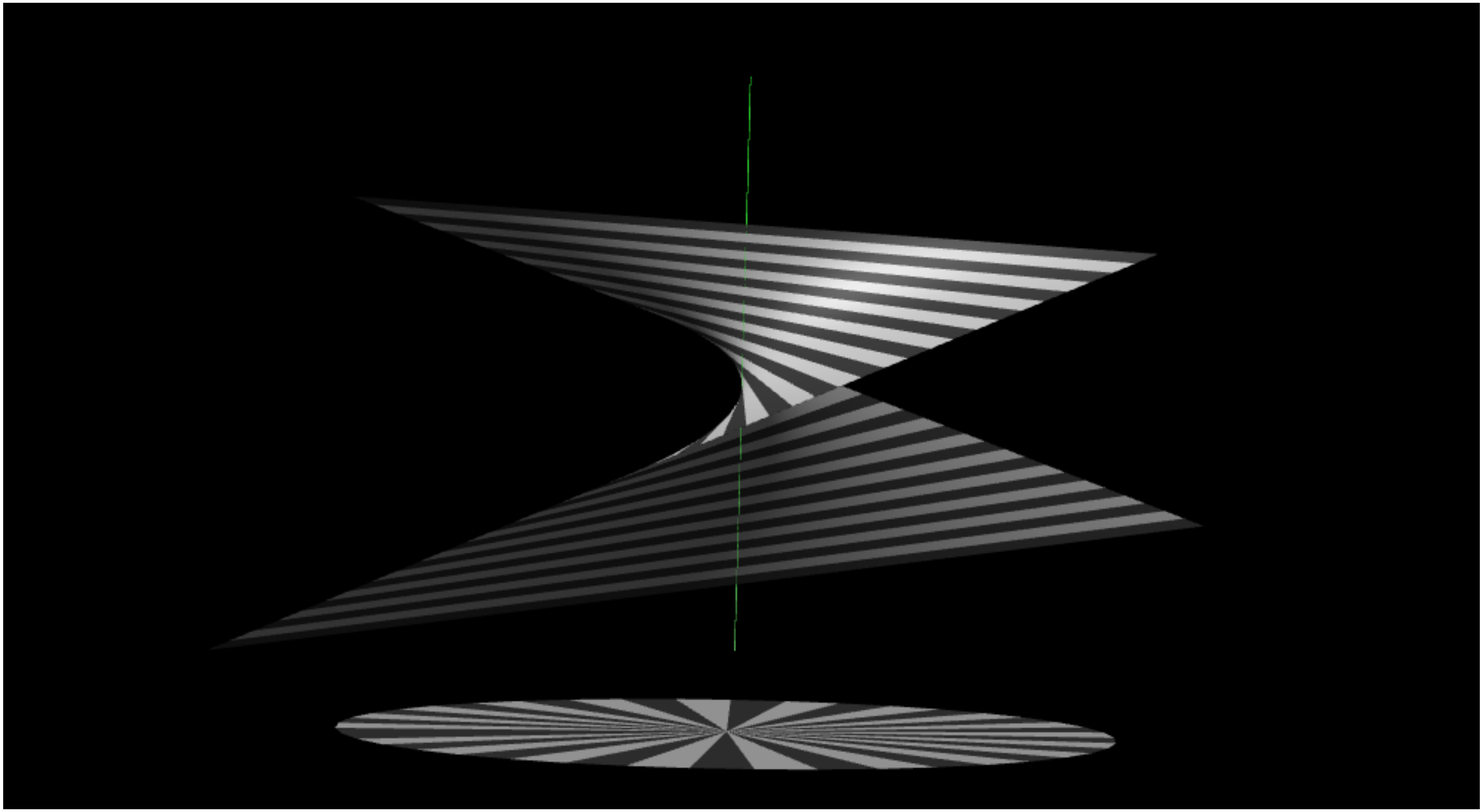

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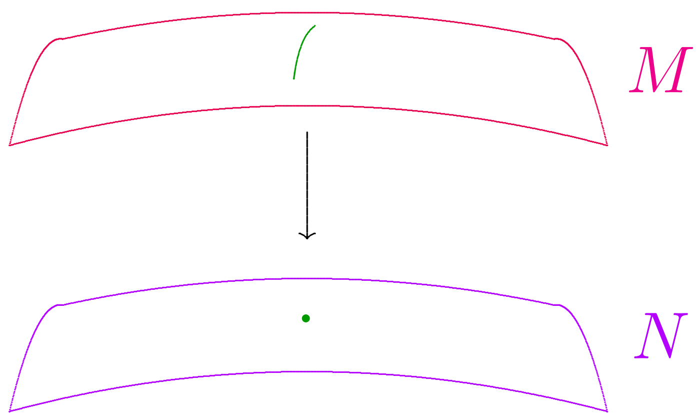

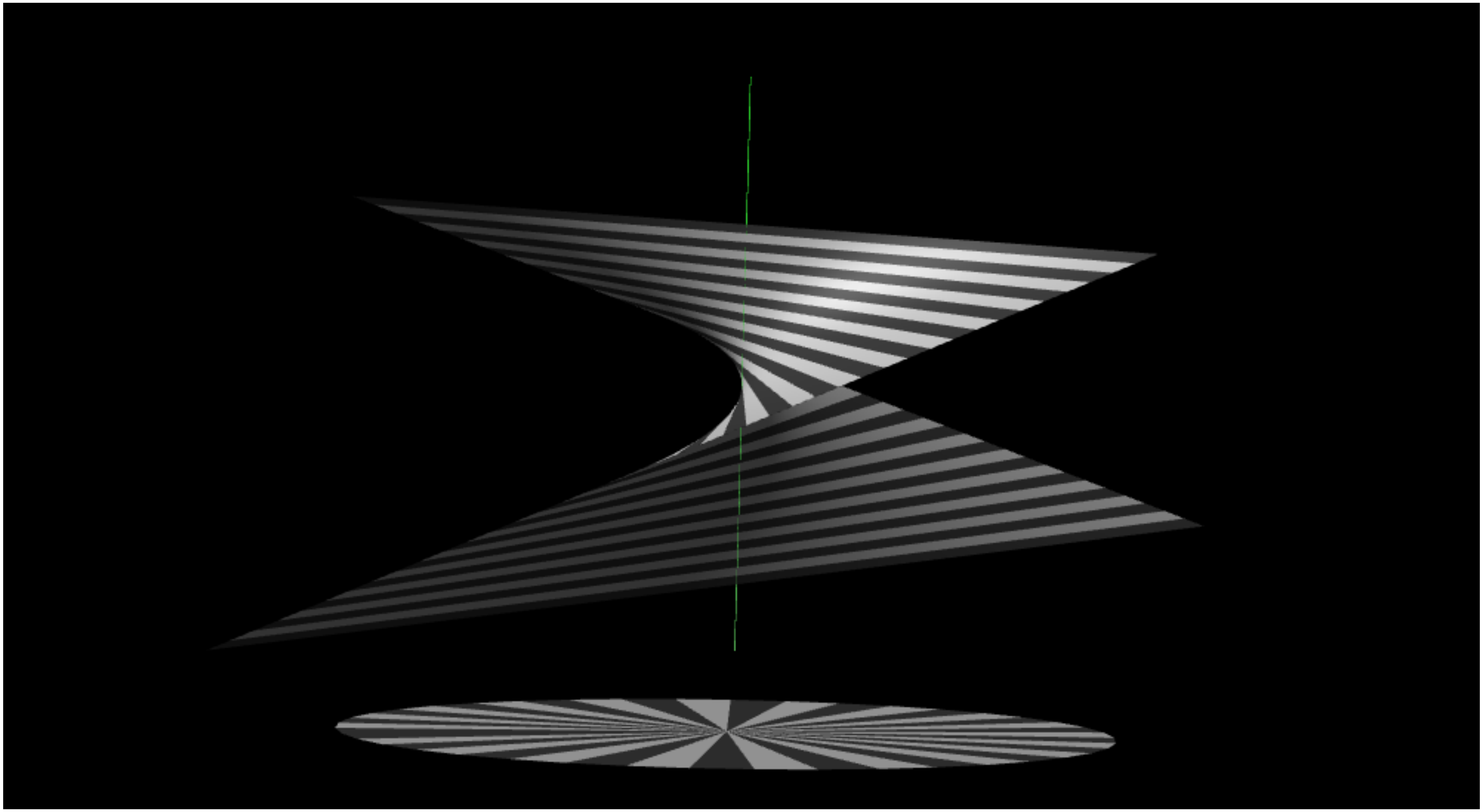

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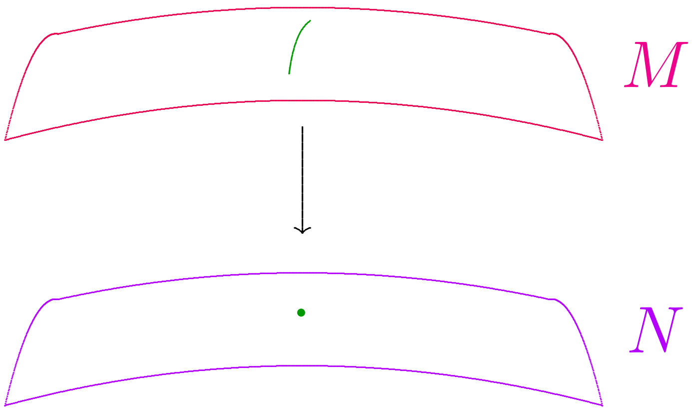

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Yau, Tian-Yau, Chen-LeBrun-Weber…

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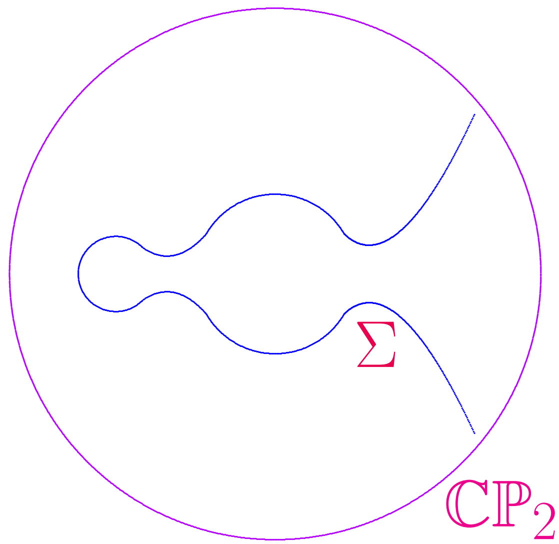

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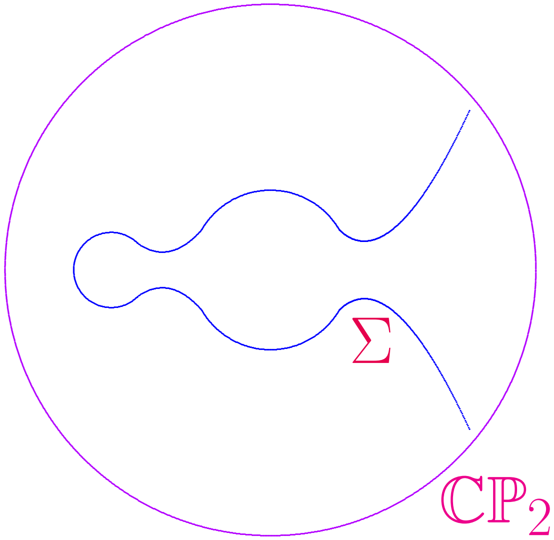

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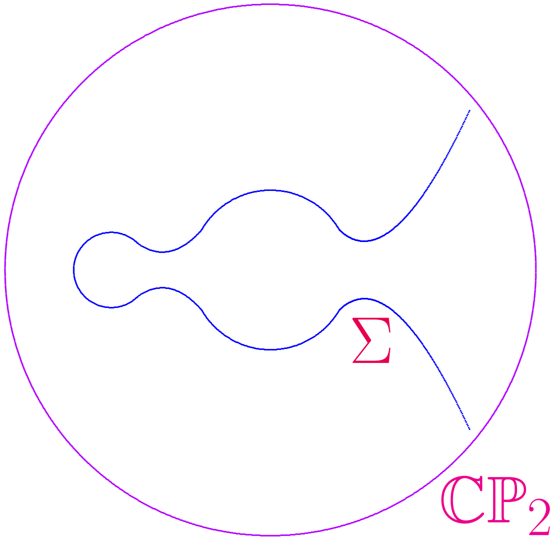

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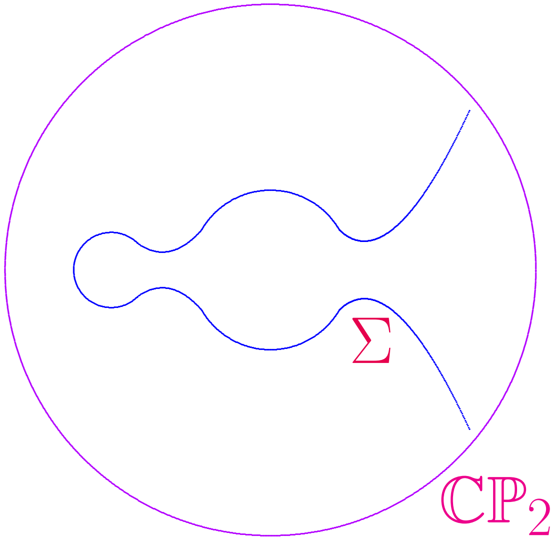

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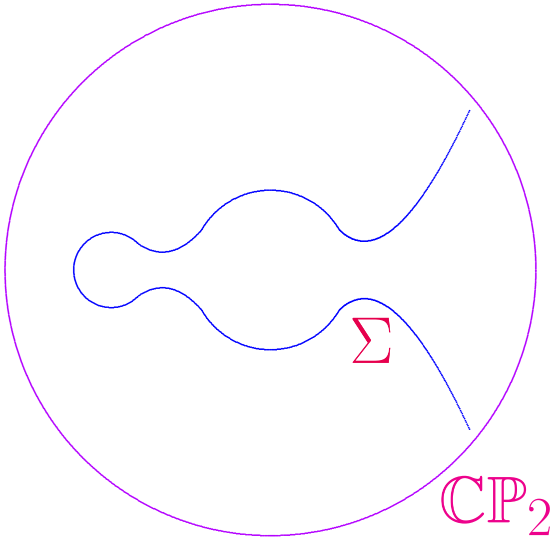

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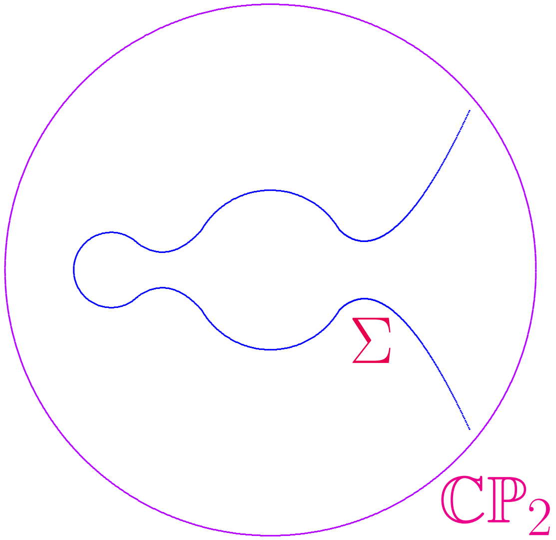

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Theorem $\Longrightarrow$ unique Einstein metric, up to scale.

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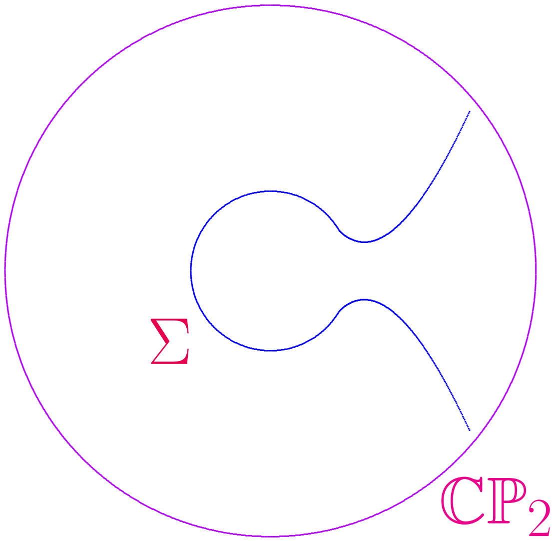

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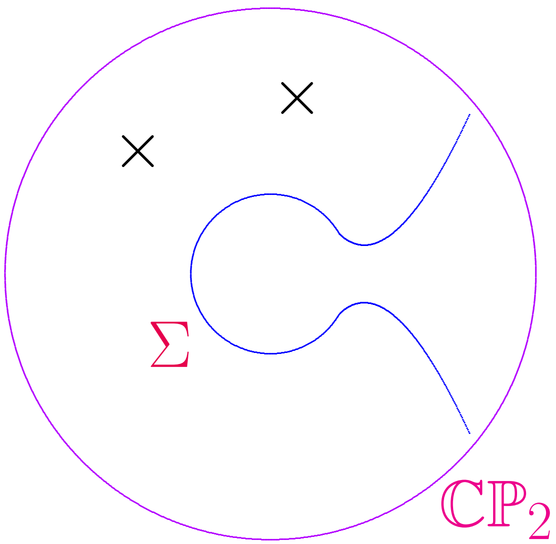

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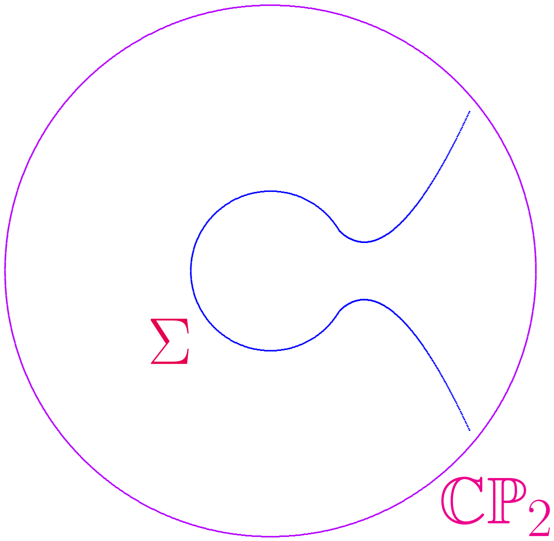

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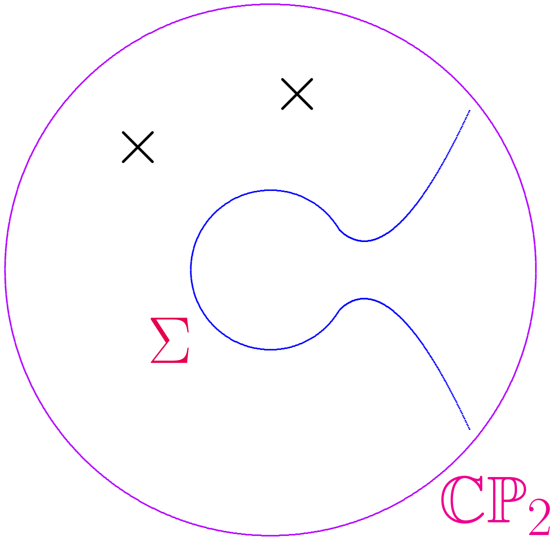

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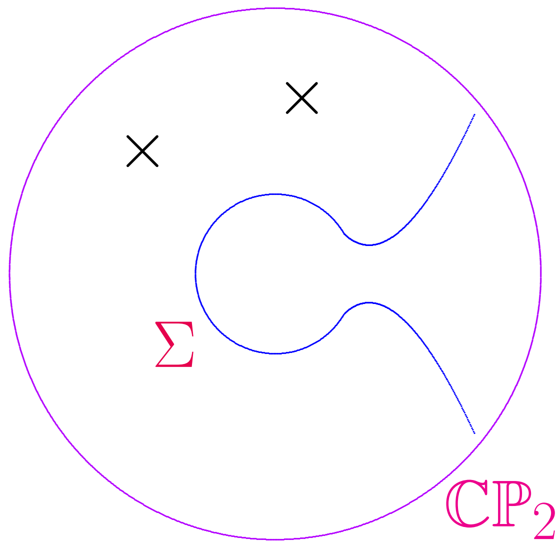

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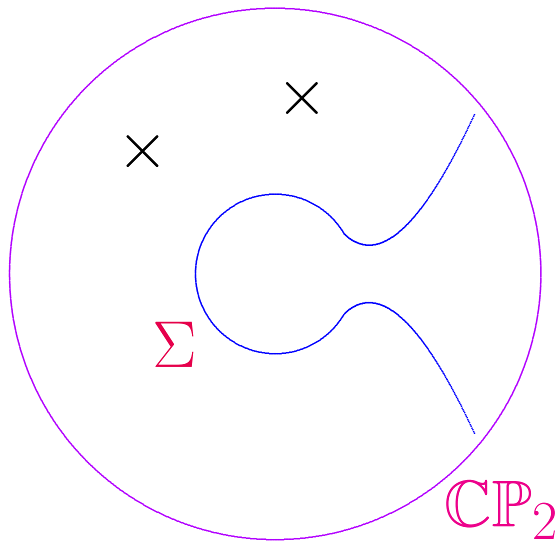

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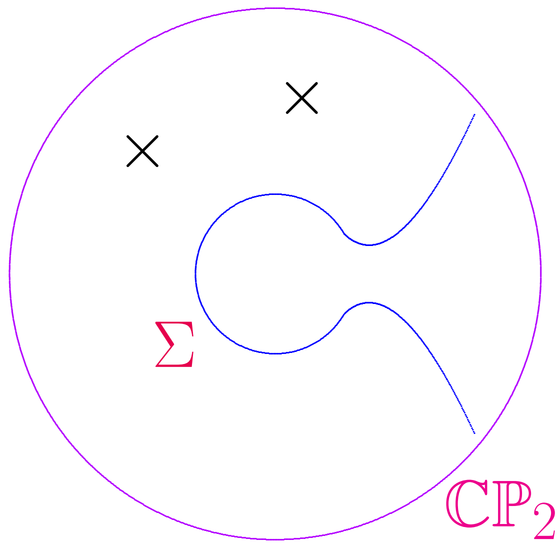

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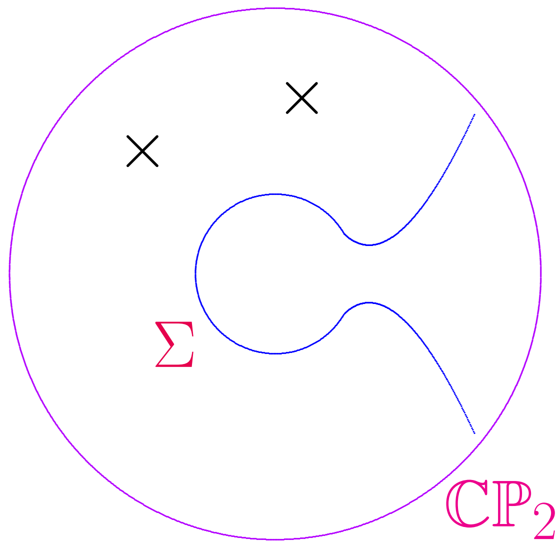

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For small β, Theorem $\Longrightarrow$ (M, Σ) never Einstein.

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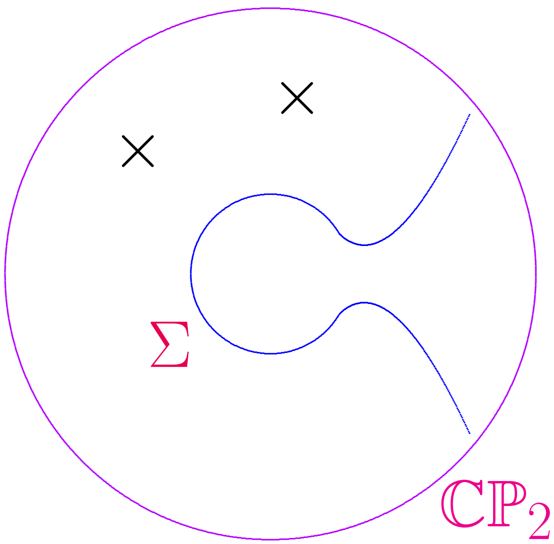

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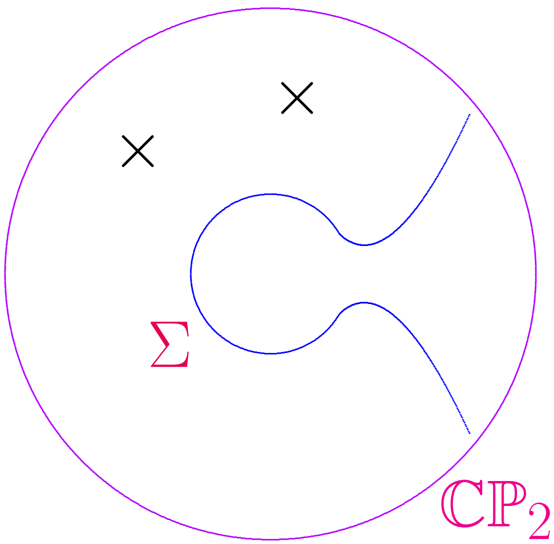

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Berman, Li-Sun, ...

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Nothing analogous known in other dimensions.

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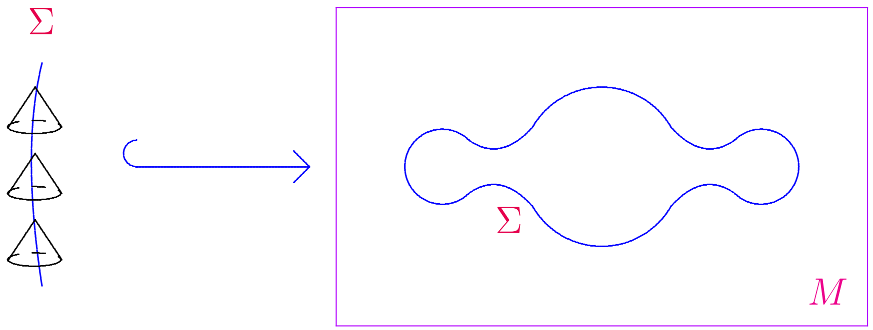

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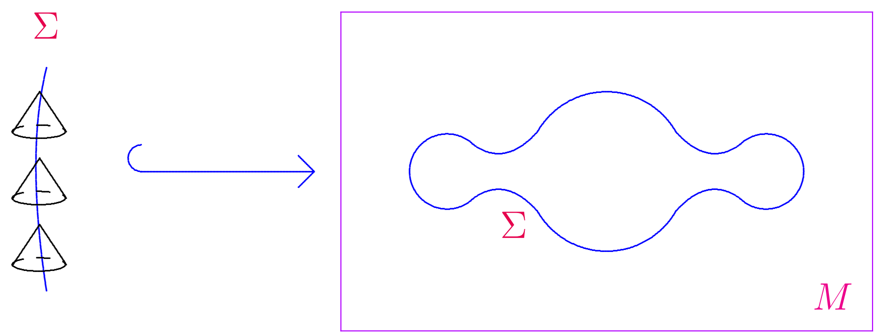
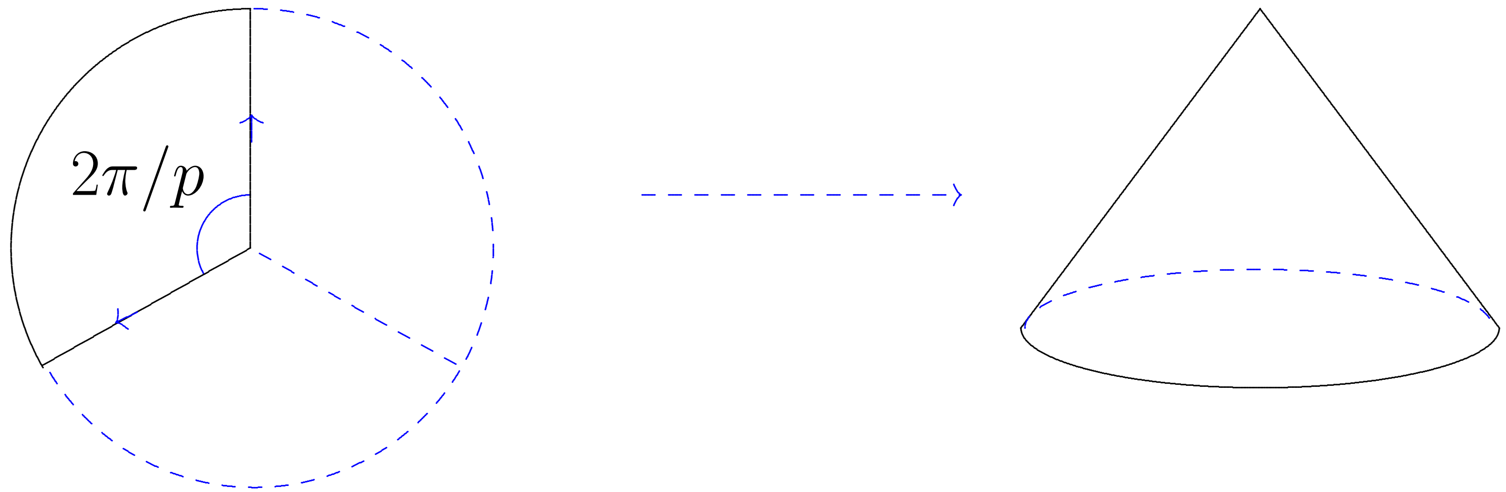


Transverse Picture:



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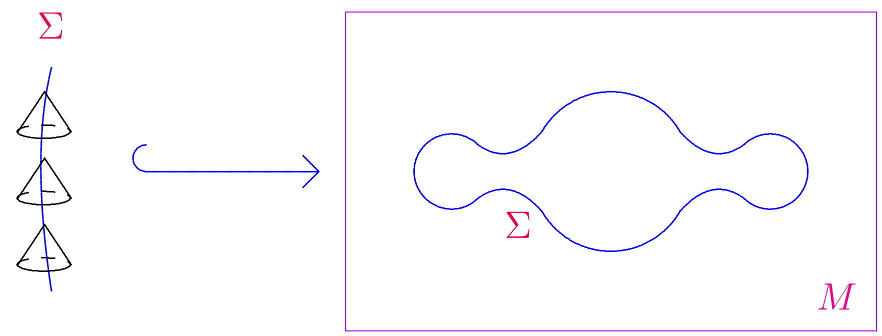

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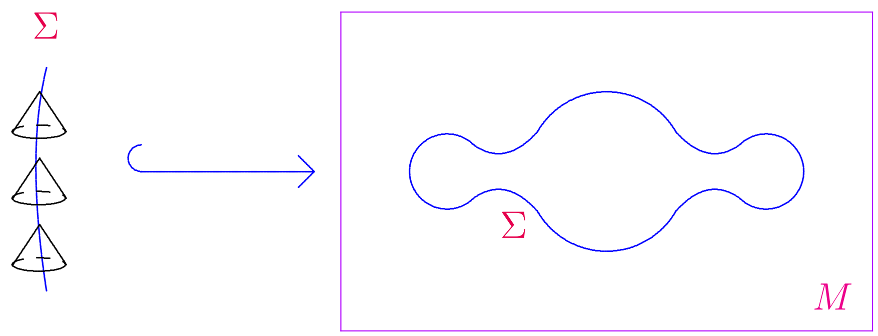

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Once again, 4-dimensional phenomenon.

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3 times better than Hitchin-Thorpe obstruction!

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Symplectic form ω has nothing to do with metric! Inequality guarantees that SW invariant is non-zero. Automatic if $b_+(M) > 1$.

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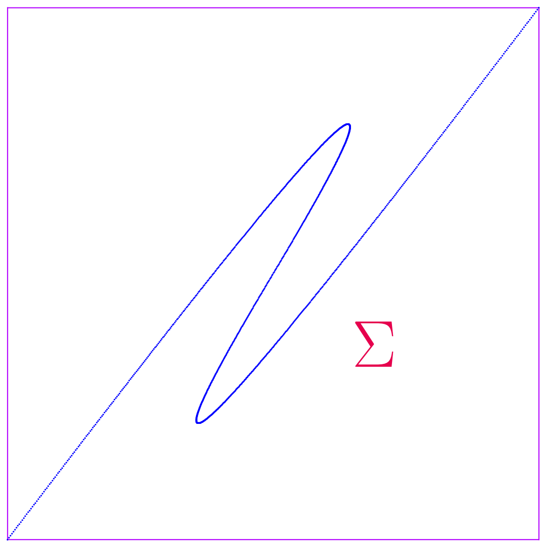

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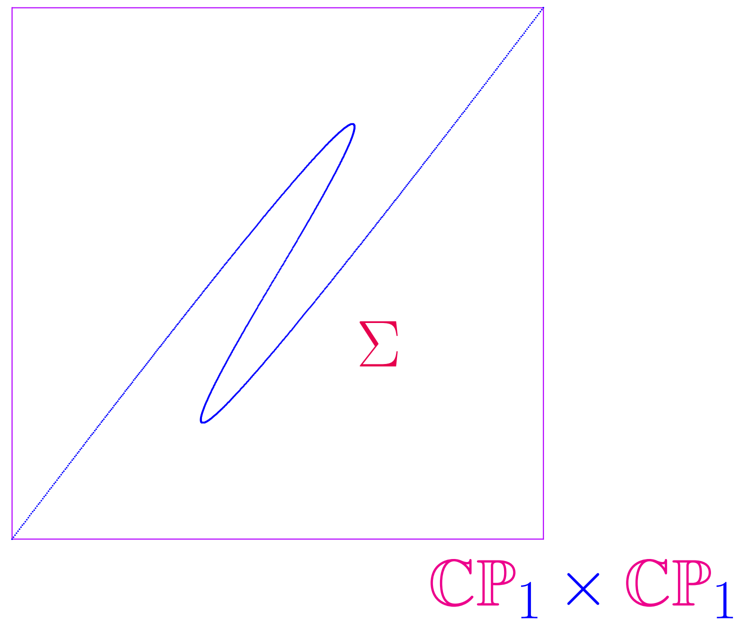

Chen-Donaldson-Sun:

$(\mathbb{CP}_1 \times \mathbb{CP}_1, \Sigma)$ admits Kähler-Einstein metrics of all cone angles $2\pi\beta$, $\beta \in (0, 1]$.

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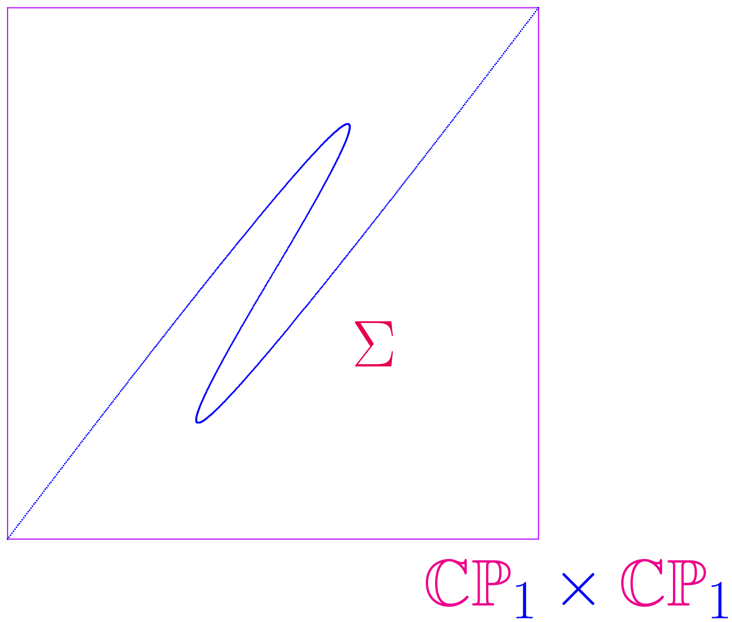

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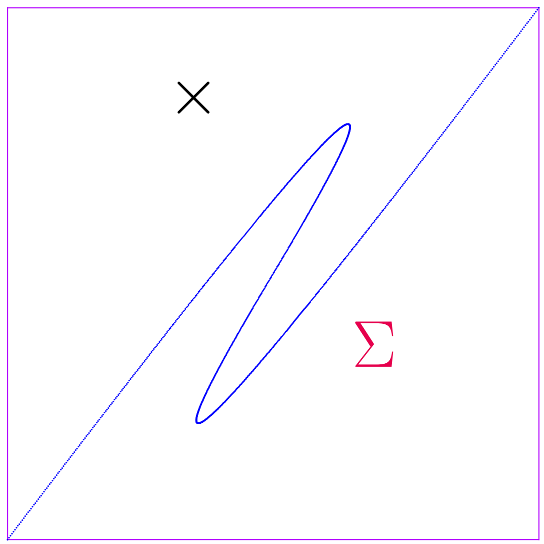

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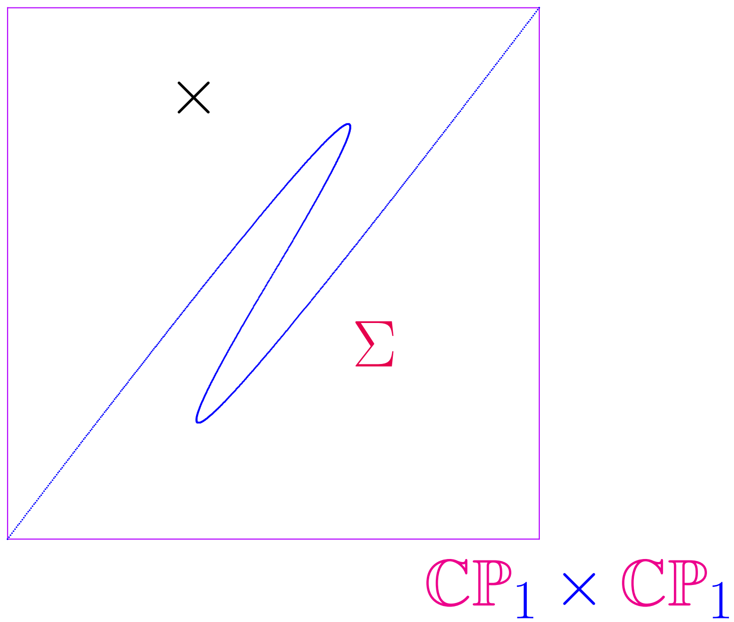

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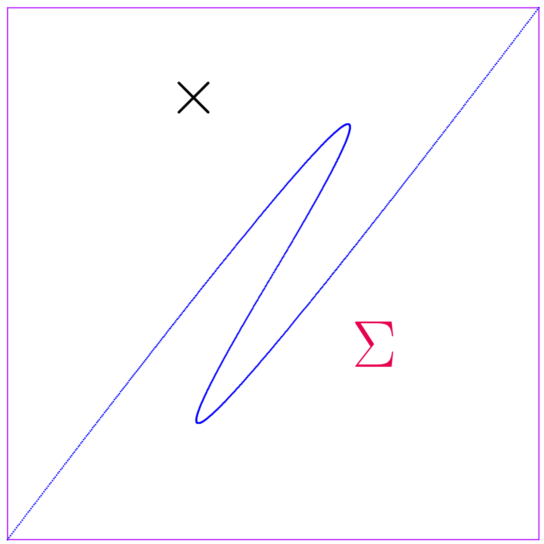

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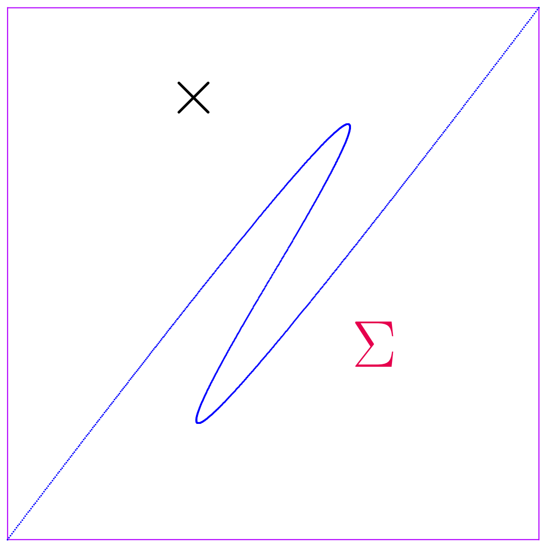

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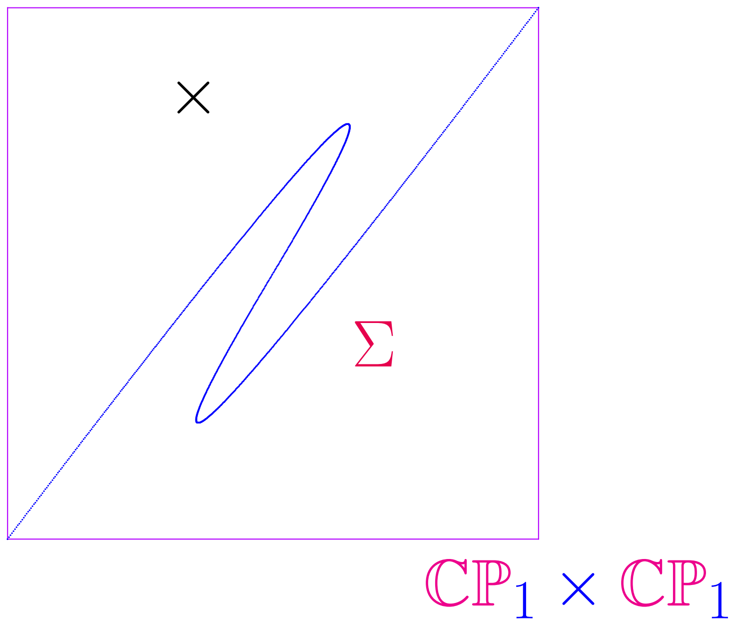

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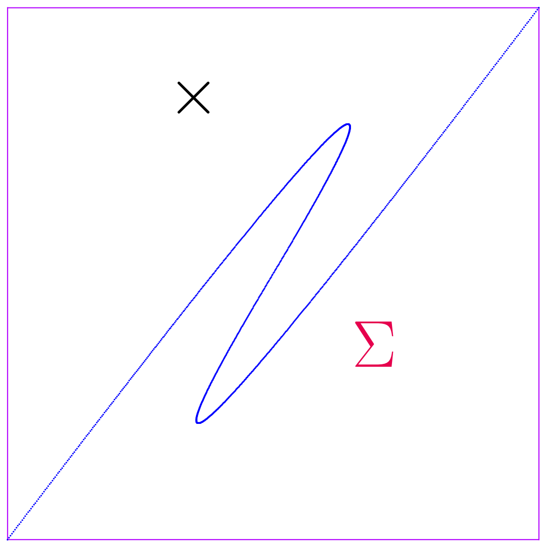

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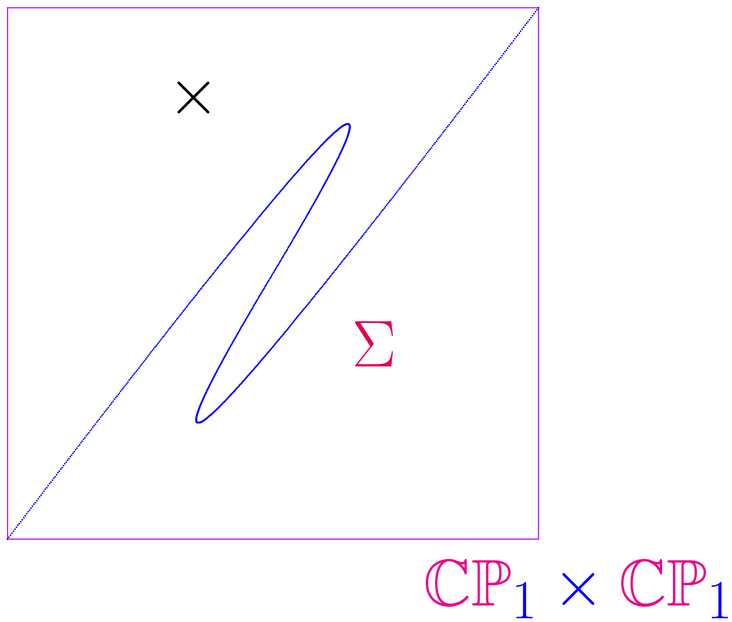

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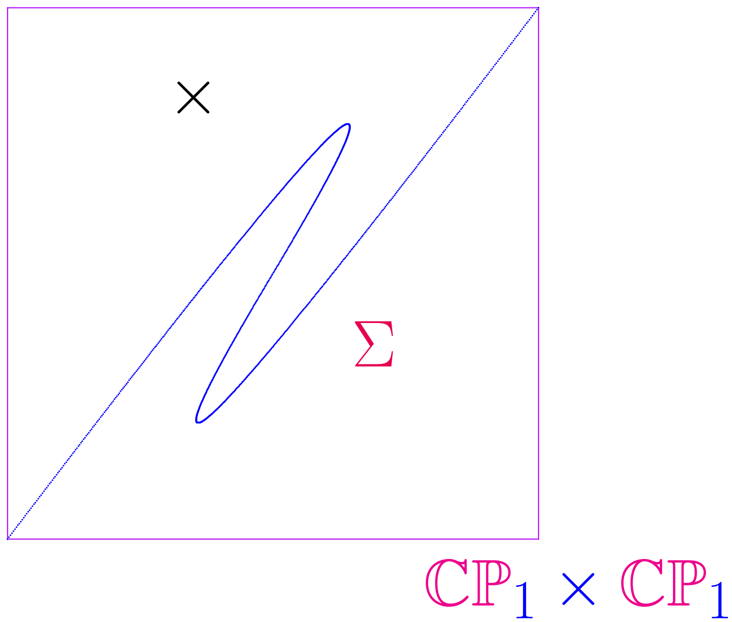

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What about edge-cone metrics?

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Theorem A then follows by integral formulæ.

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Conjecture: Same estimates hold for general β.

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Work in progress: Seiberg-Witten for general β.