Optimal Metrics,

Curvature Functionals,

and the

Differential Topology of

Four-Manifolds

Claude LeBrun
SUNY Stony Brook

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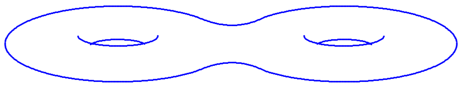

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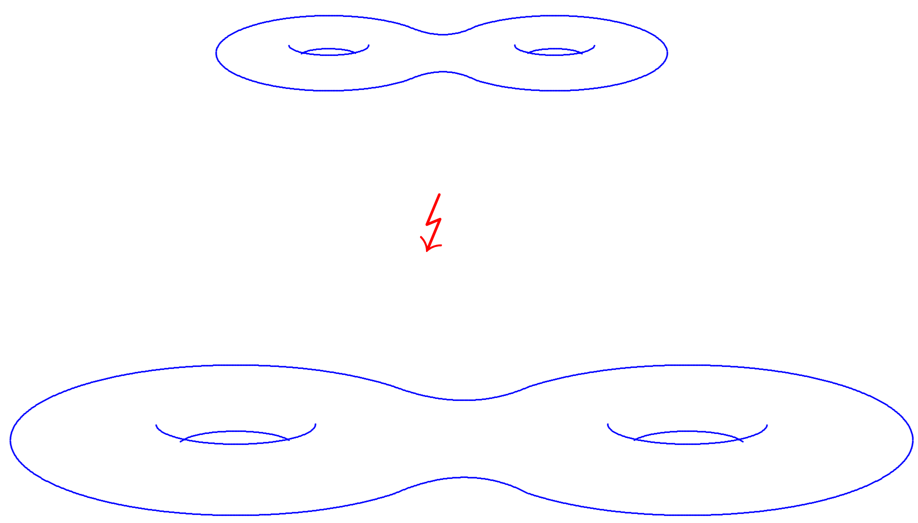

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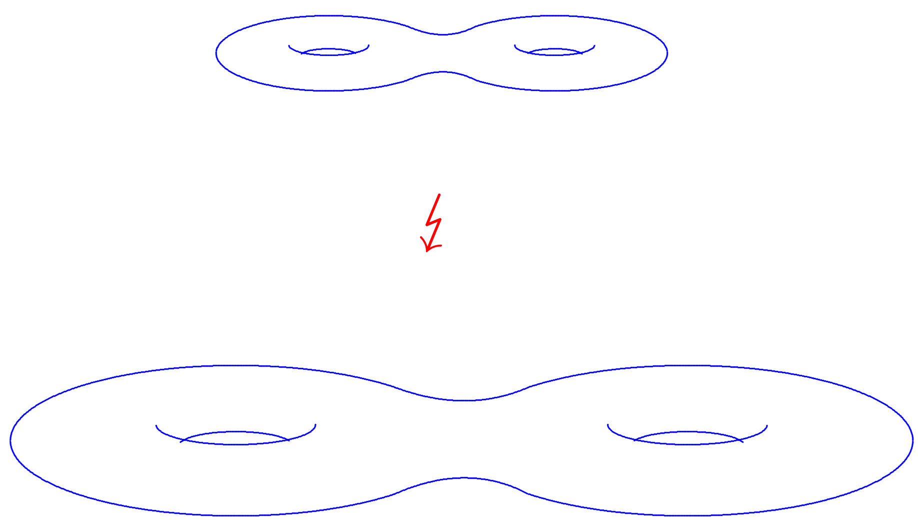

$$g \rightsquigarrow cg \implies |\mathcal{R}| \rightsquigarrow c^{-1}|\mathcal{R}|$$

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Definition (Berger). *Let M^n be a smooth compact n-manifold, $n \geq 3$. A Riemannian metric g on M will be called an* optimal metric *if it is an absolute minimizer of the functional $\mathcal{K}$.*

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- $\mathcal{K}(g) \geq \mathcal{I}_{\mathcal{R}}(M)$ for every metric g on M.
- $\mathcal{K}(g) = \mathcal{I}_{\mathcal{R}}(M) \Longleftrightarrow g$ is optimal.

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Berger's motivation: *Einstein metrics.*

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Berger's motivation: *Einstein metrics.*

Definition. *A Riemannian metric is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$*.*

Proposition (Berger). *Let* (M^4, g) *be a compact Einstein* 4*-manifold. Then* g *is an optimal metric. Moreover, every other optimal metric* $\tilde{g}$ *on* M *is also Einstein.*

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This statement is false in every other dimension!
Standard S^{2k+1}, $S^{2k+1} \times S^3$ not optimal...

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where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W_+ &= \text{self-dual Weyl curvature} \\ W_- &= \text{anti-self-dual Weyl curvature} \end{aligned}$$

(M, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$

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4-dimensional Hirzebruch signature formula

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

for signature $\tau(M) = b_+(M) - b_-(M)$.

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Berger: Einstein $\Longrightarrow$ optimal.

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If also

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then called scalar-flat anti-self-dual *(SFASD).*

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Also get topological obstruction:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$
$$\leq 0.$$

Reverse Hitchin-Thorpe!

Proposition. *Let M^4 be simply connected smooth compact. If M admits a scalar-flat anti-self-dual metric, then*

- *M is homeomorphic to $k\overline{\mathbb{CP}}_2$, $k \geq 5$; or*
- *M is diffeomorphic to $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, $k \geq 10$; or*
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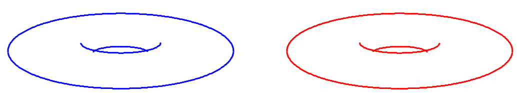
Connected sum:



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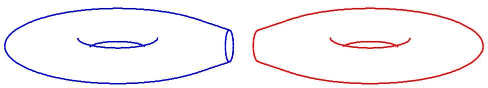

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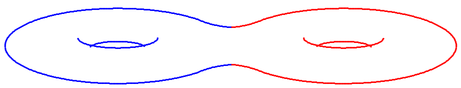

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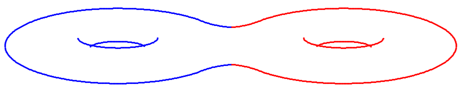

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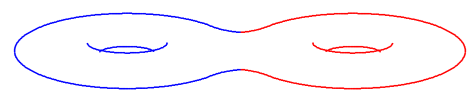

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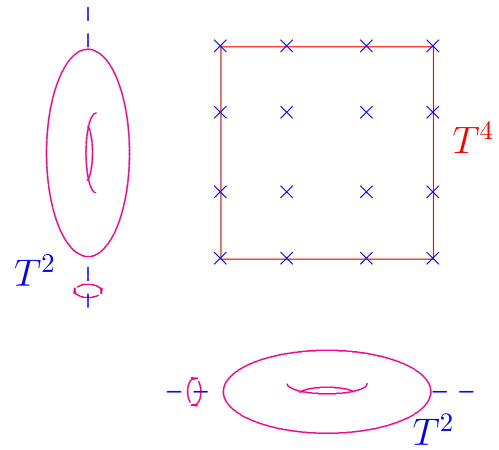

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Weitzenböck formula for $\varphi \in \Gamma(\Lambda^+)$:

$$(d + d^*)^2 \varphi = \nabla^* \nabla \varphi - 2W_+(\varphi, \cdot) + \frac{s}{3}\varphi$$

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For $M^4 \neq K3$, optimal, but not Einstein.

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When $b_+(M) \neq 0$, Weitzenböck formula

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But when $b_+(M) = 0$, key is to find

family g_t of ASD metrics s.t. s changes sign.

Proposition. *For any integer* $k \geq 6$*, the connected sum*

$$k\overline{\mathbb{CP}}_2 = \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{k}$$

admits 1*-parameter family of* ASD *conformal metrics* $[g_t]$, $t \in [-1, 1]$*, such that*

- $\exists g_{-1} \in [g_{-1}]$ *with* $s < 0$*; and*
- $\exists g_1 \in [g_1]$ *with* $s > 0$*.*

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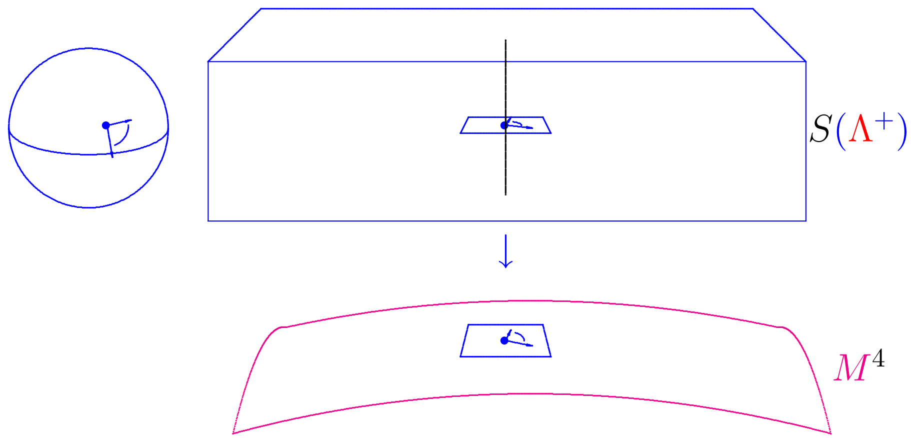

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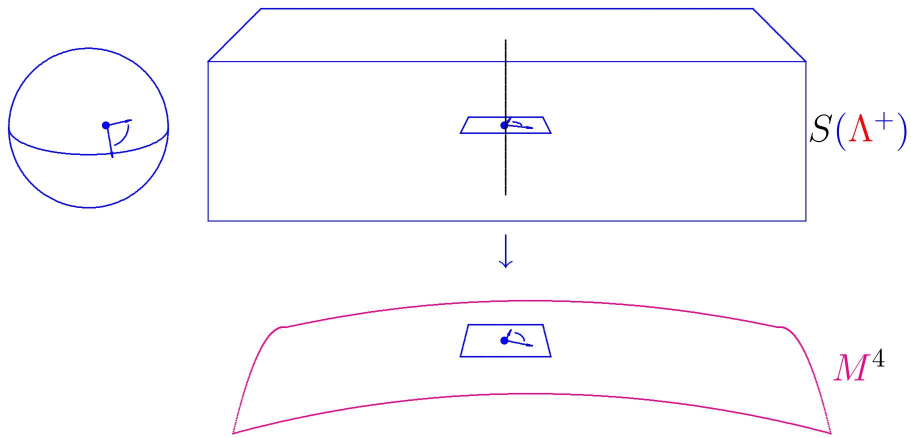

Theorem (Atiyah-Hitchin-Singer). (Z, J) *is a complex 3-manifold iff* $W_+ = 0$.

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Warning: M could be empty; or disconnected!

Scalar-flat Kähler case:

$\exists$ complex surface $\Sigma \subset Z$ such that

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These auxiliary structures detect $s \equiv 0$ metric.

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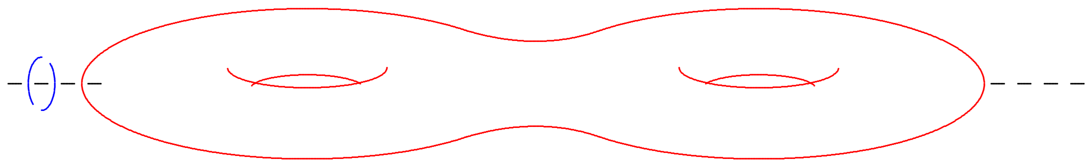
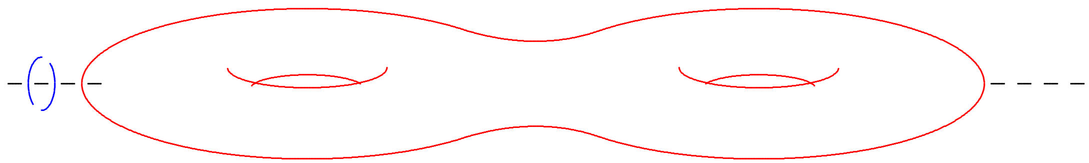
Strategy:

- Find such metrics on related orbifold.
- Then smooth singularities.



$$X = [(S^1 \times S^3)\#(S^1 \times S^3)]/\mathbb{Z}_2$$



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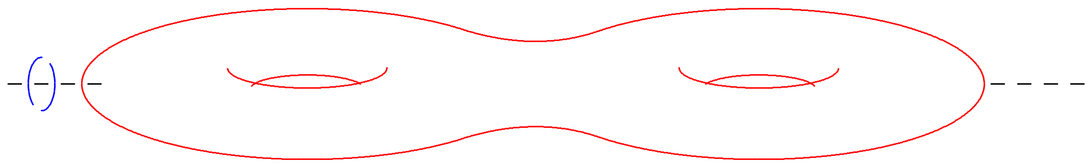

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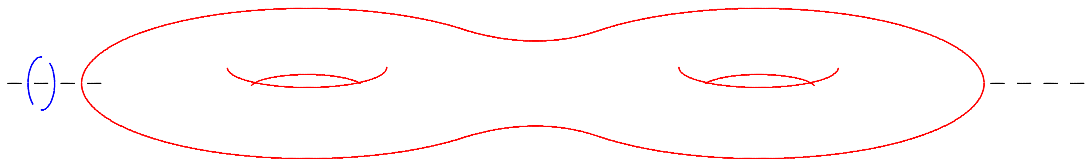

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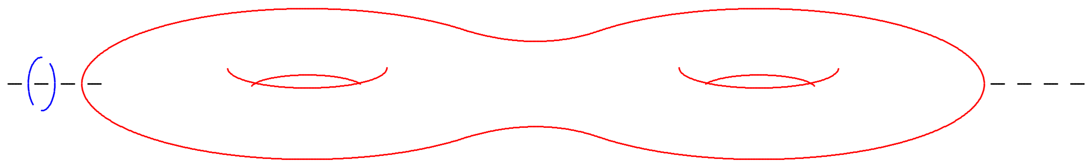

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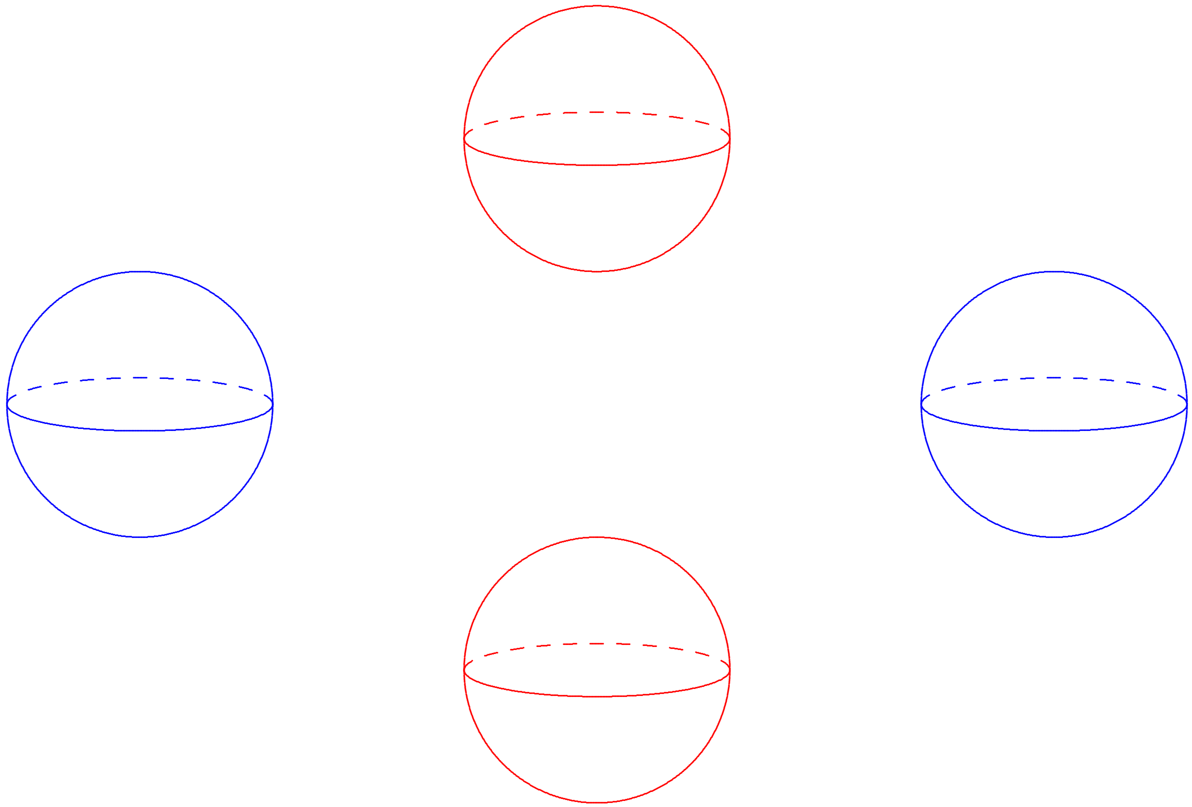
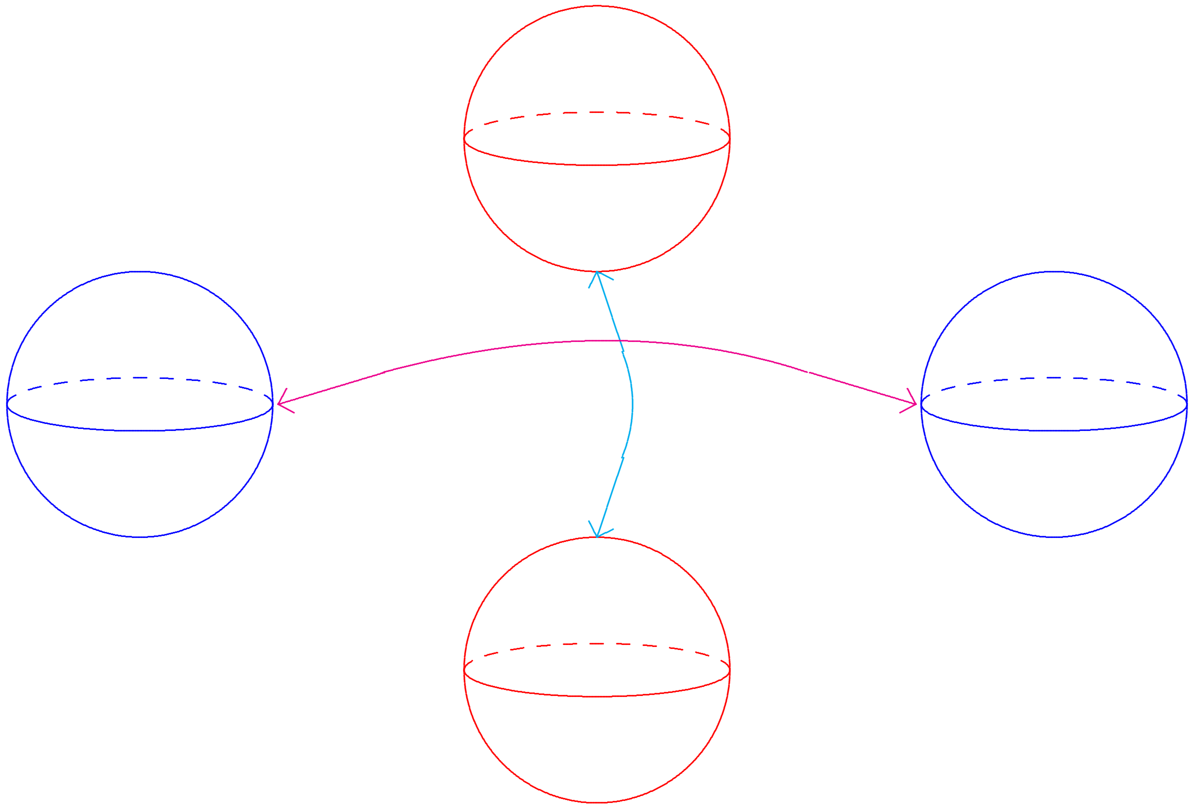
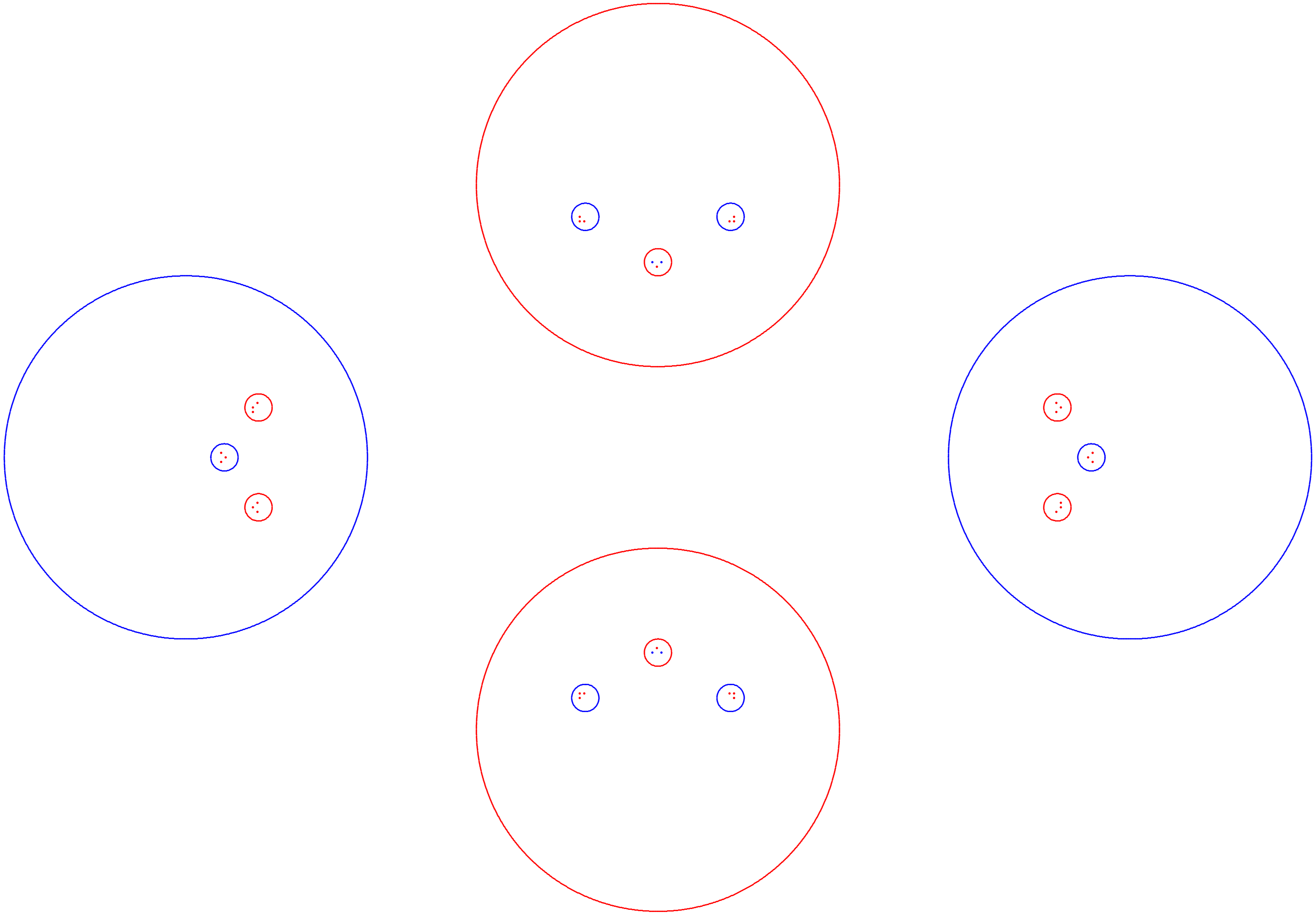
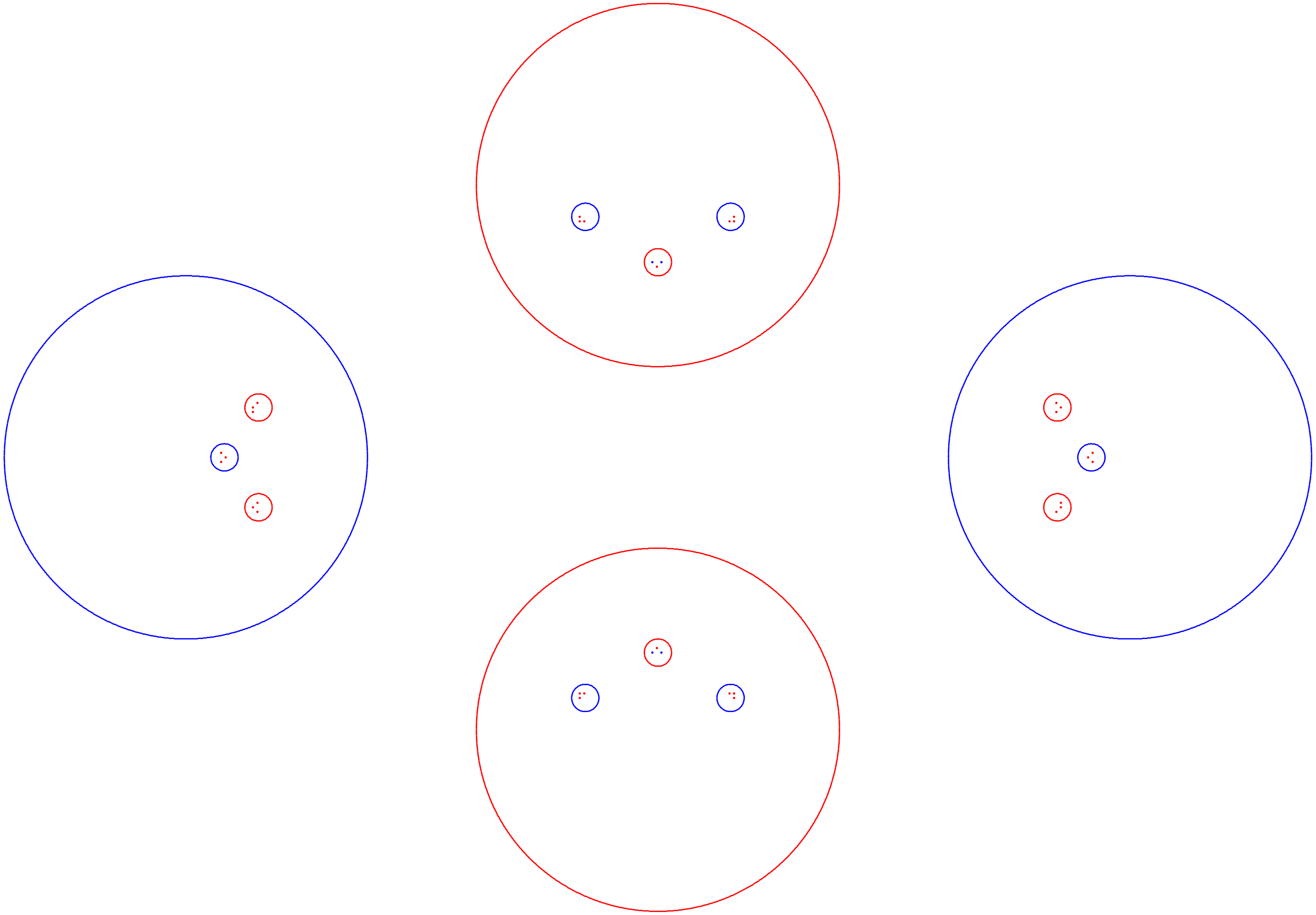


$S^4 - 4$ balls

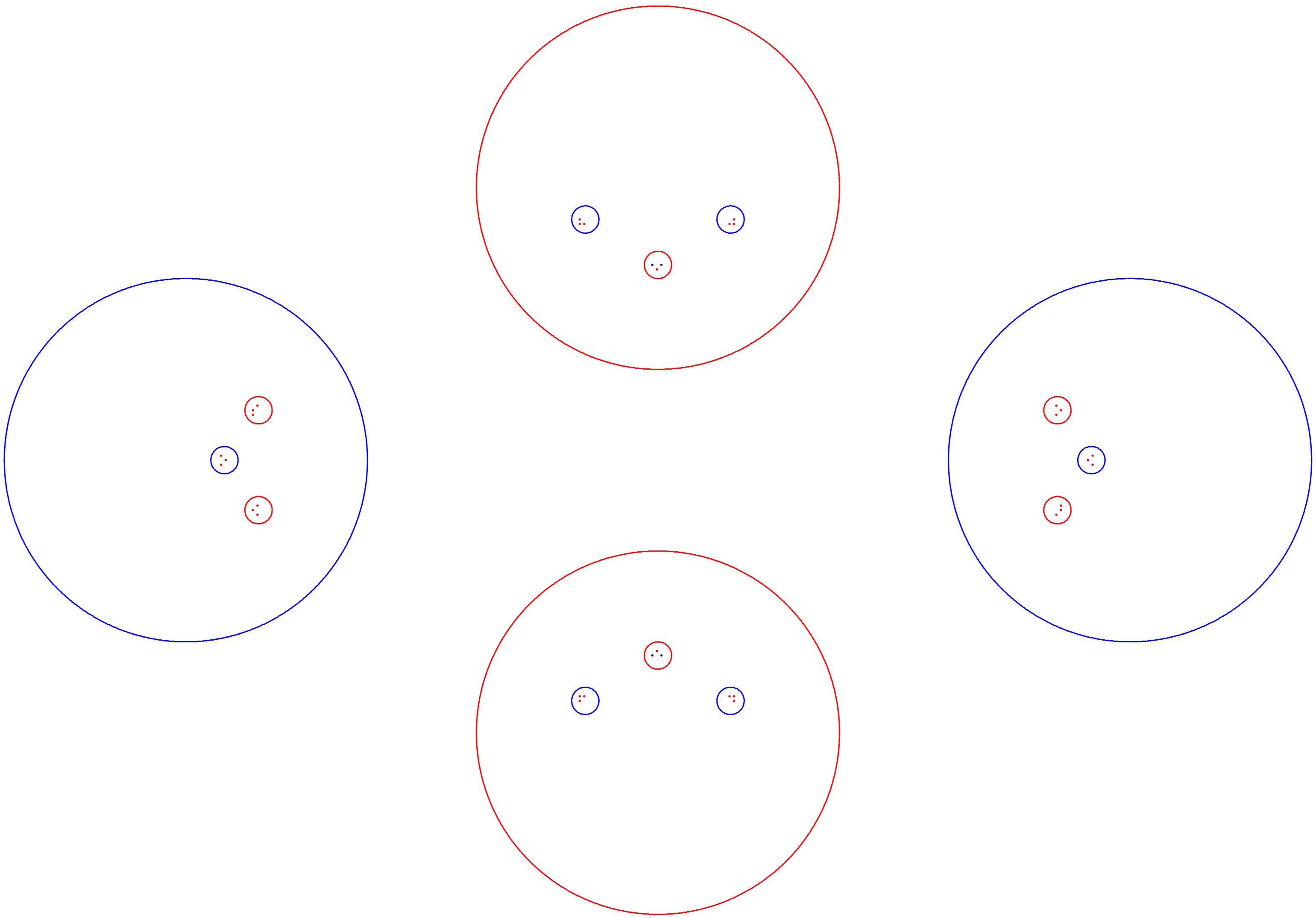


Identify boundary 3-spheres.





$$\dim(\Lambda) < 1 = \frac{4}{2} - 1$$



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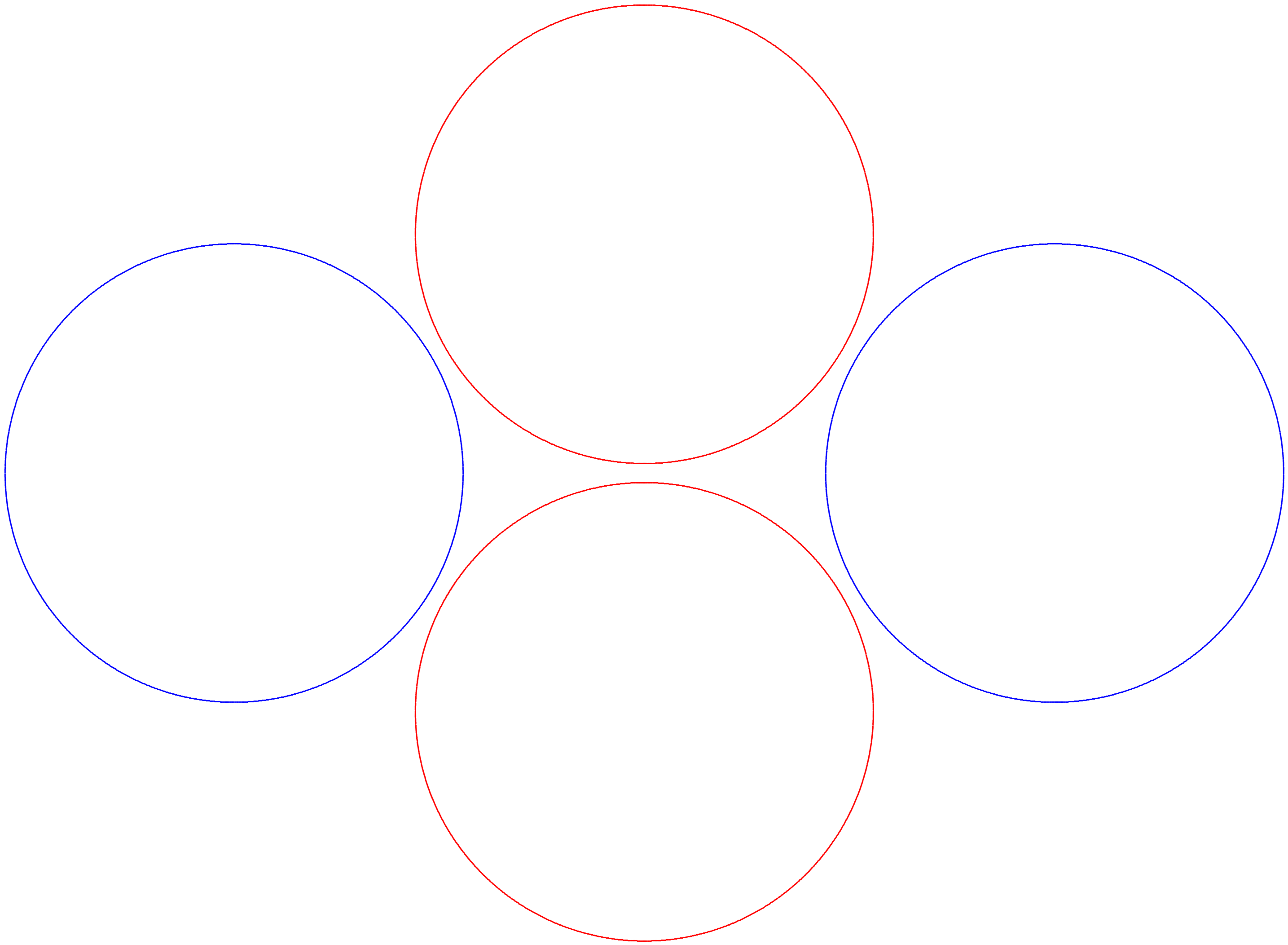
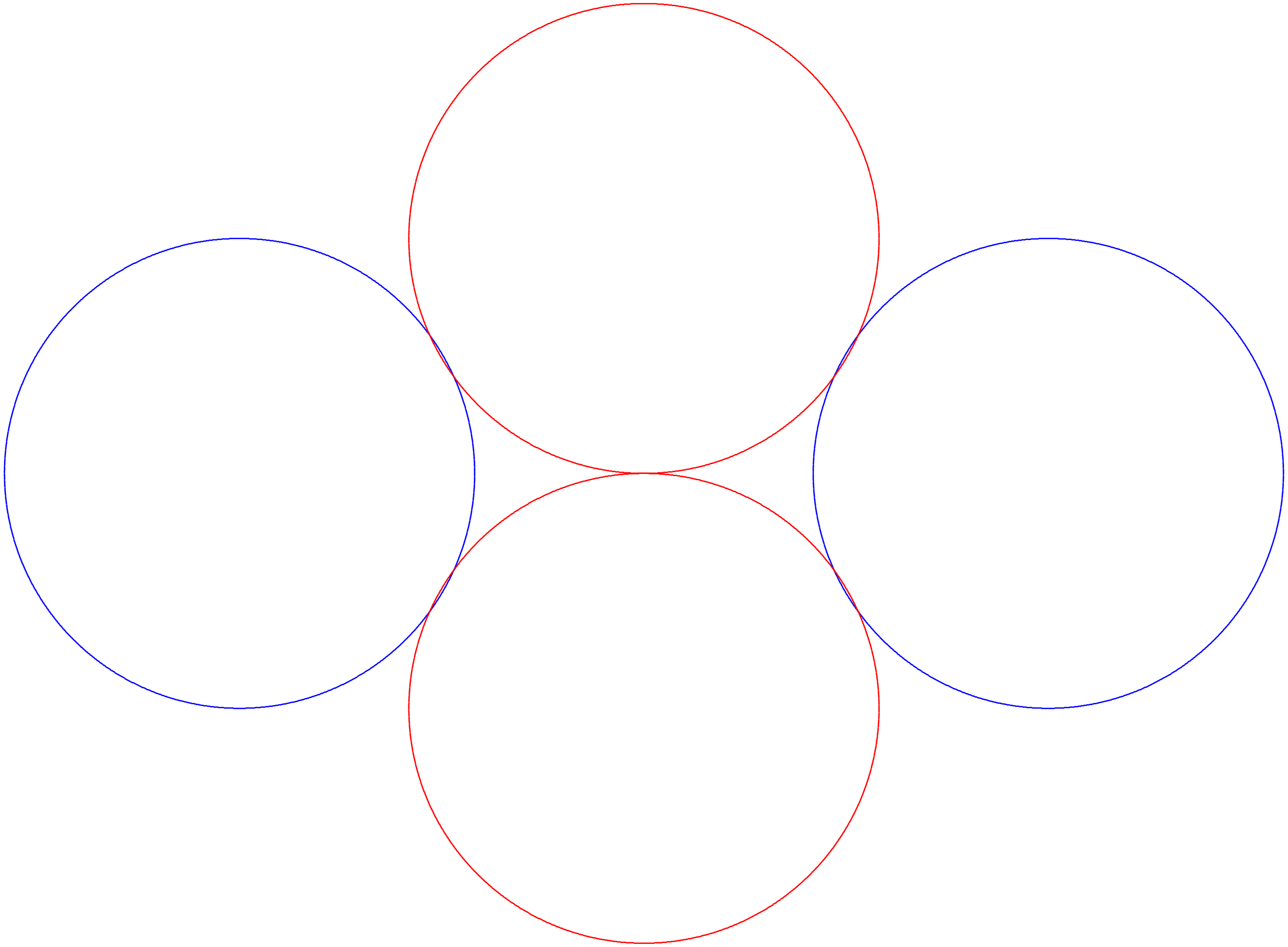
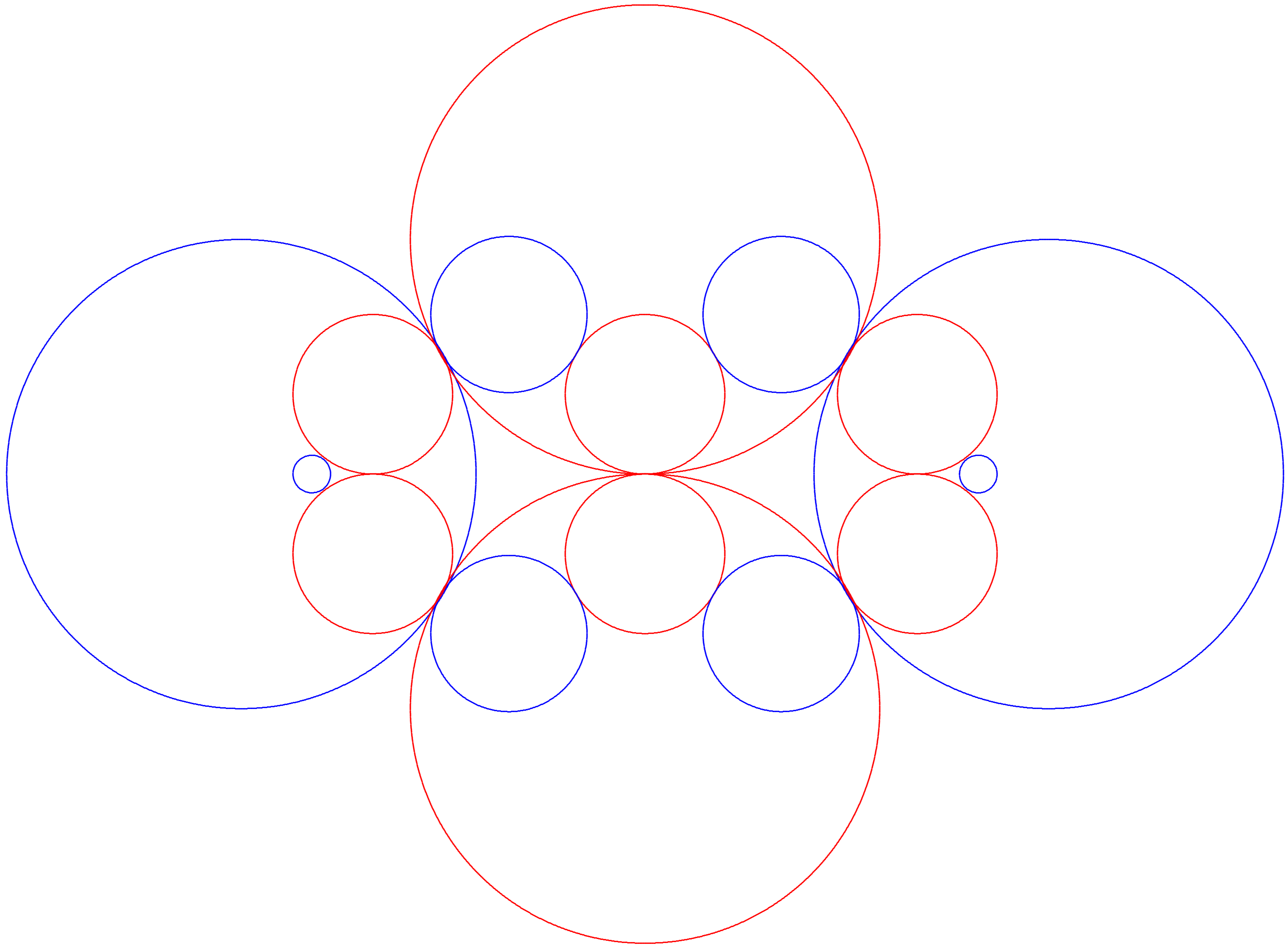
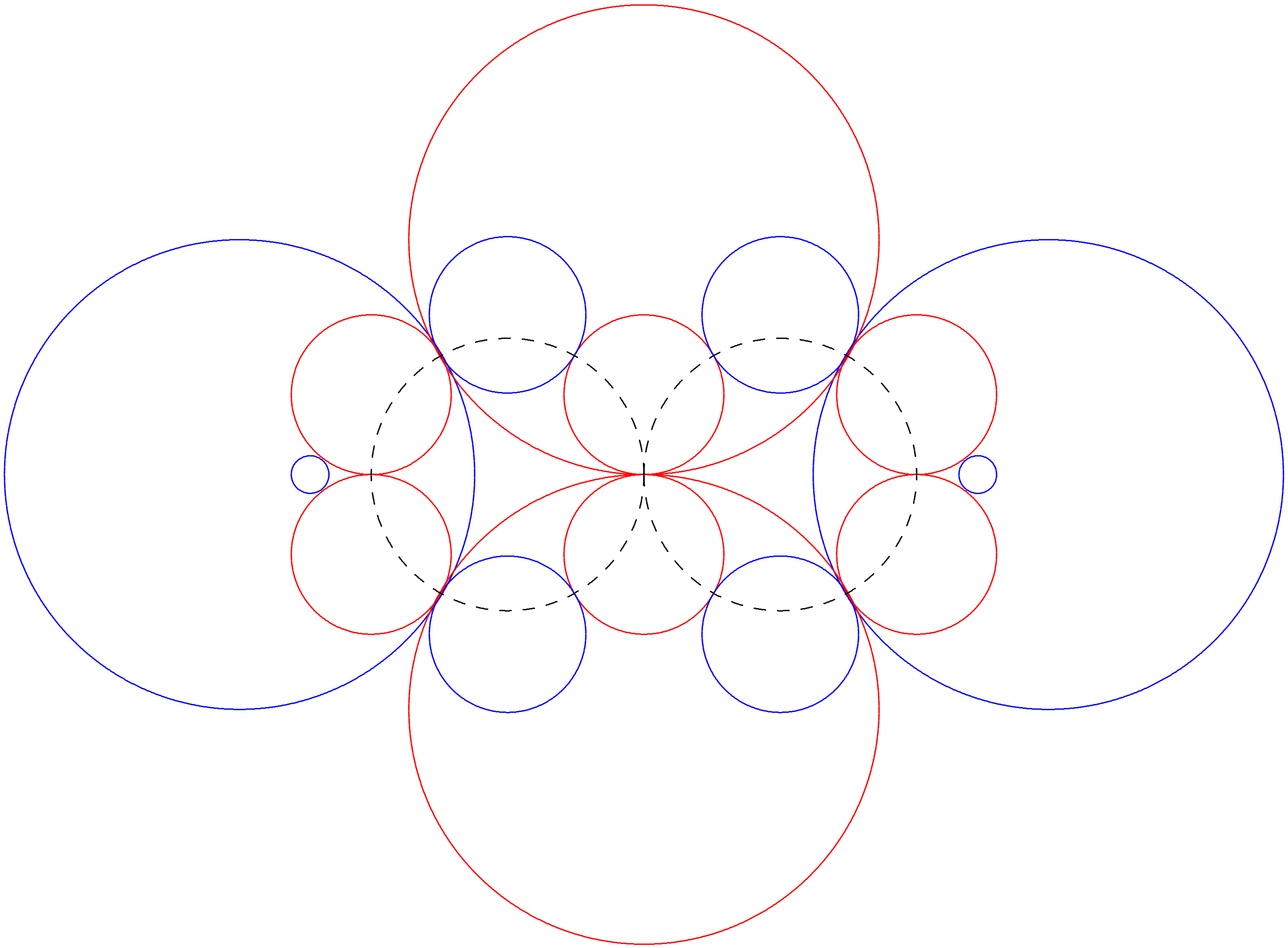


$$\dim(\Lambda) = ?$$







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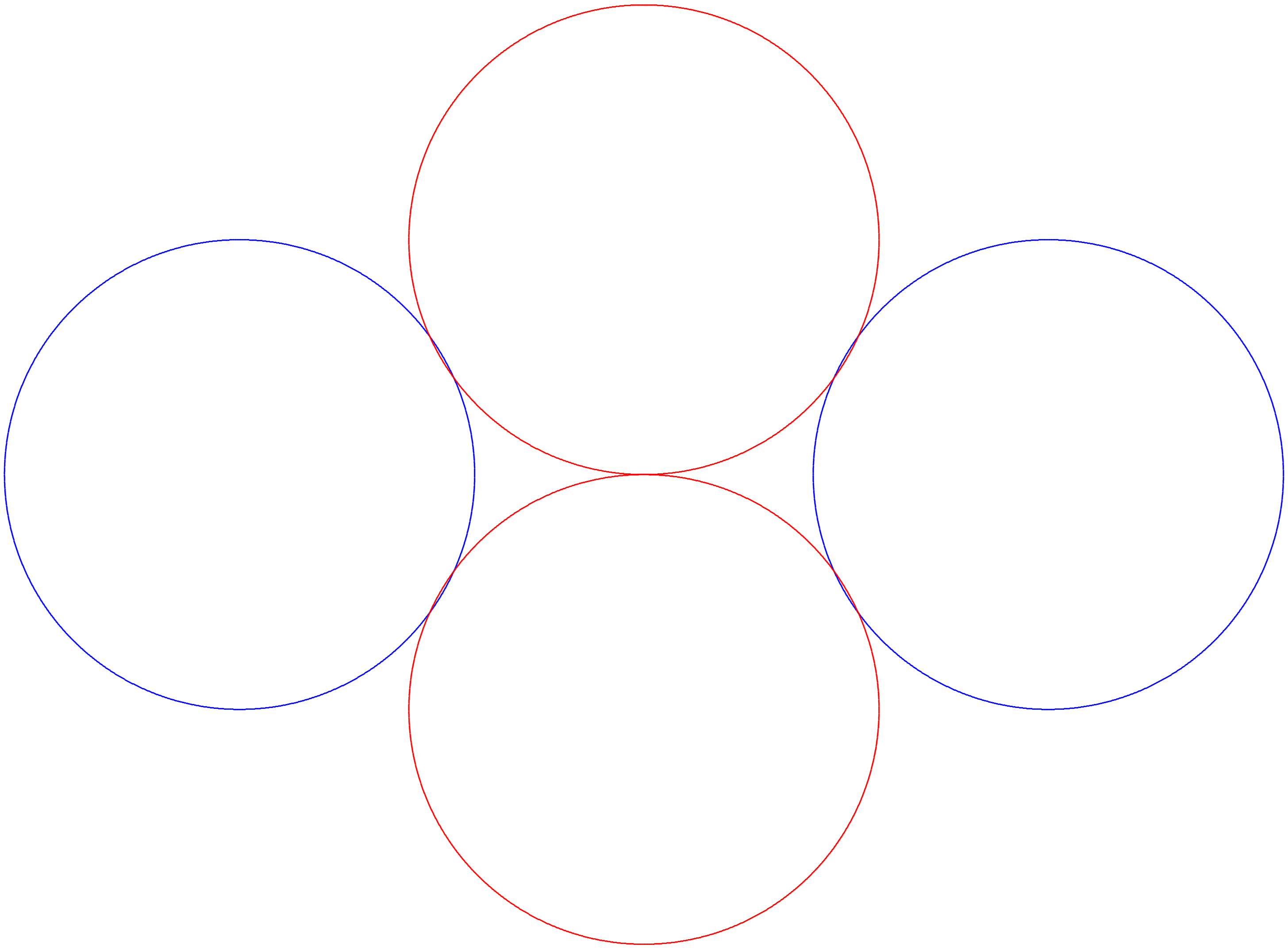

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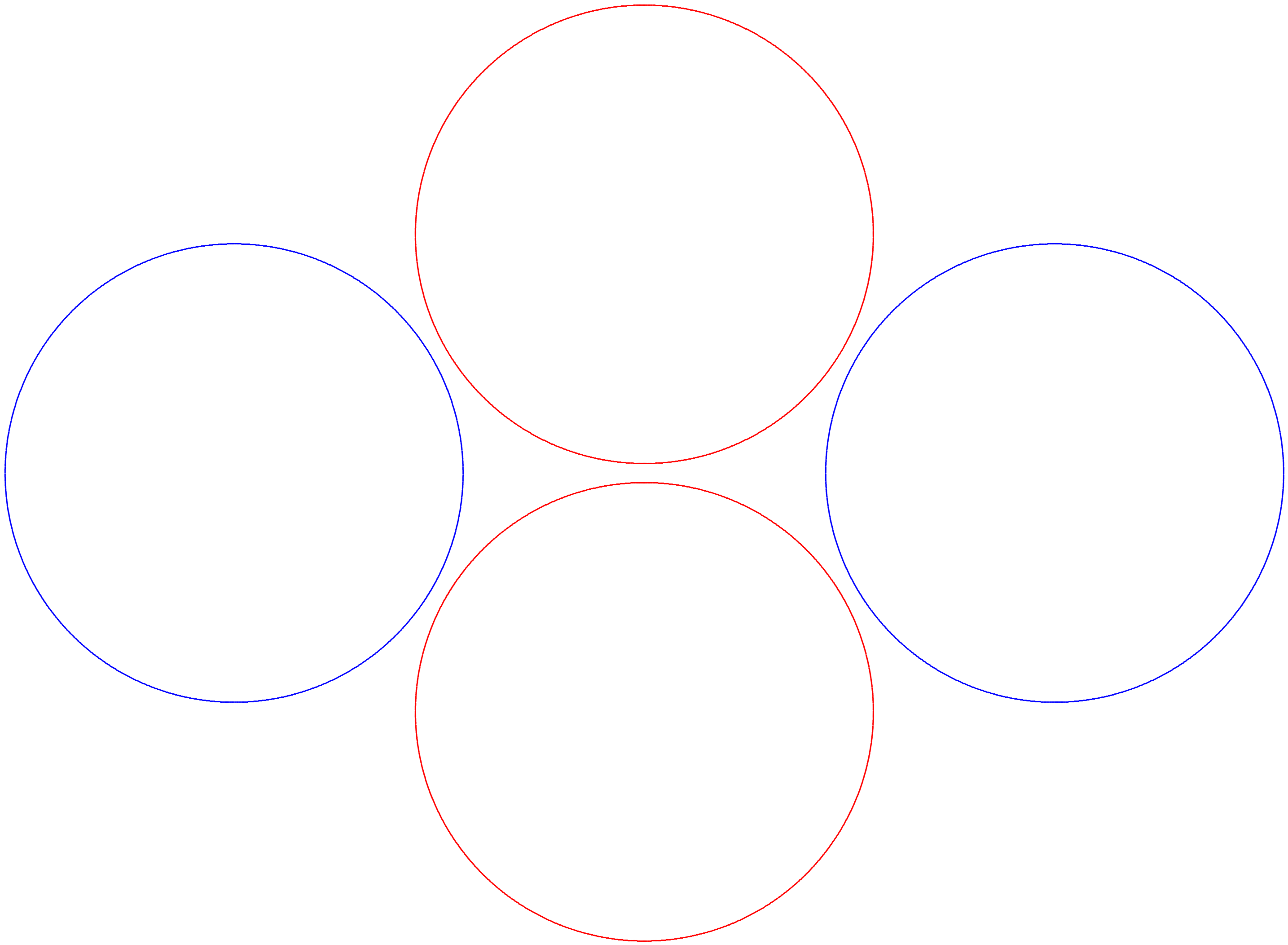

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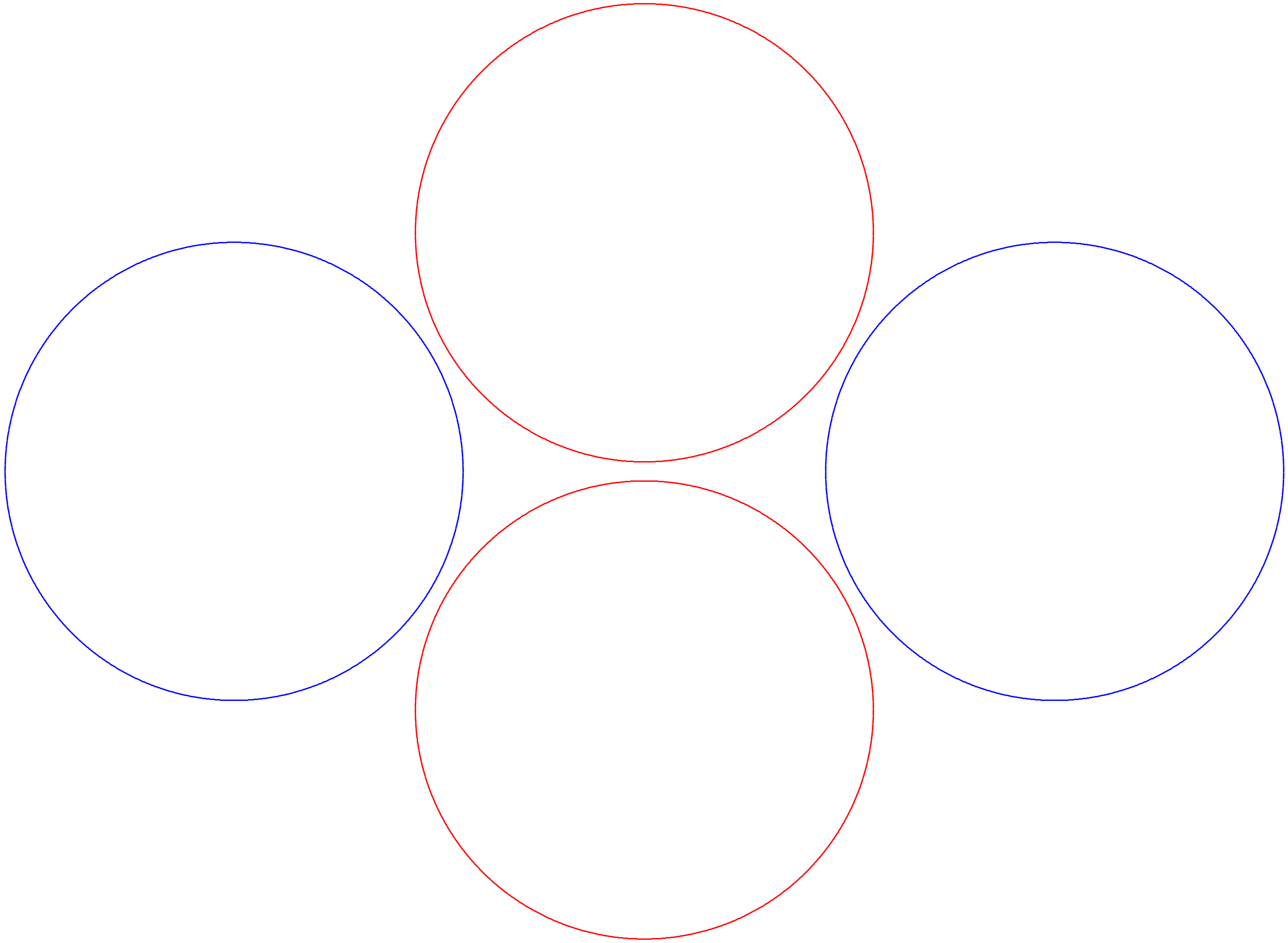
(Bishop-Jones)



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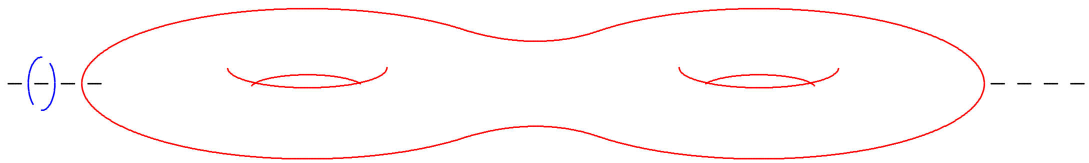

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Lemma. *There is a real-analytic 1-parameter family g_t, $t \in [-1, 1]$, of conformally flat orbifold metrics on X such that*

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Twistor space of $(S^1 \times S^3)\#(S^1 \times S^3)$:

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Lemma (Eastwood-Singer). *Twistor space Z any conformally flat g on*

$$k(S^1 \times S^3) = \underbrace{(S^1 \times S^3)\# \cdots \#(S^1 \times S^3)}_{k}$$

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But blowing up twistor lines

$$\mathbb{CP}_1 \rightsquigarrow Q = \mathbb{CP}_1 \times \mathbb{CP}_1$$

of six fixed points gives complex manifold $\tilde{Z}_X$.

Other building blocks:

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Eguchi-Hanson metric on T^*S^2:

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Burns metric on $\overline{\mathbb{CP}}_2 - \{\infty\}$:

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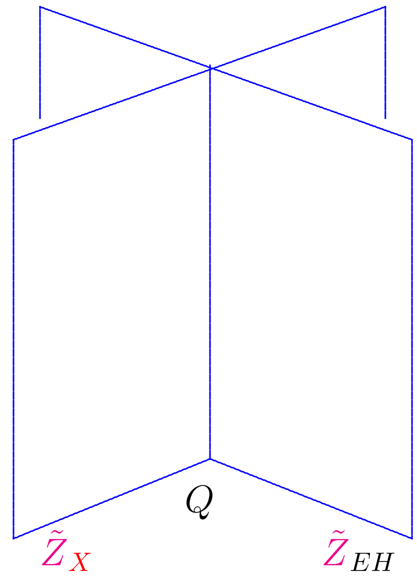
Fubini-Study metric on $\overline{\mathbb{CP}}_2$:

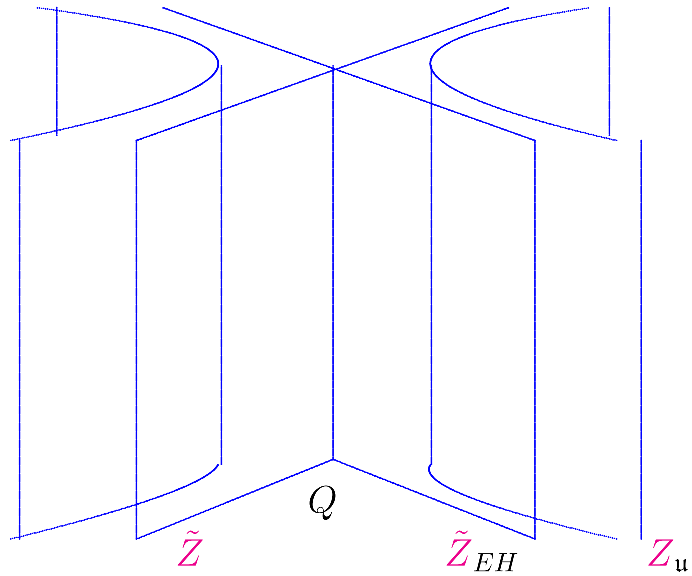
$$\left\{ ([\vec{z}],[\vec{w}]) \in \mathbb{CP}_2 \times \mathbb{CP}_2 \;\middle|\; z_1 w_1 + z_2 w_2 + z_3 w_3 = 0 \right\}$$



Complex space with normal-crossing singularities:

$$Z_0 = \tilde{Z}_X \cup 6\tilde{Z}_{EH} \cup \ell\tilde{Z}_{FS}$$

$\tilde{Z}_{EH} = Z_{EH} \cup Q$
$\tilde{Z}_{FS} =$ blow up of Z_{FS} at twistor line.



Donaldson-Friedman, LeBrun-Singer:

Obtain twistor spaces of ASD metrics on

$$M = (6+\ell)\overline{\mathbb{CP}}_2$$

as smoothing $Z_{\mathfrak{u}}$ of normal crossings.

Carry out uniformly in additional parameter t.

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Understand Green's functions:
Atiyah's construction, uniform in $\mathfrak{u}$.

Carry out uniformly in additional parameter t.

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Hence

$$M = (6+\ell)\overline{\mathbb{CP}}_2$$

admits metrics with $W_+ \equiv 0$, $s = 0$.

Theorem A. *Simply connected smooth compact M^4 actually admits a scalar-flat anti-self-dual metric if*

- *M is diffeomorphic to $k\overline{\mathbb{CP}}_2$, $k > 5$; or*
- *M is diffeomorphic to $\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$, $k \geq 10$; or*
- *M is diffeomorphic to $K3$.*

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However. . .

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Theorem B. *For each* $k \geq 9$*, the topological manifold* $\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$ *admits infinitely many distinct exotic smooth structures for which no compatible optimal metric exists.*

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Similar conclusion also holds for $K3$.

Definition. *An* anorexic sequence *is a sequence of metrics* g_j *on smooth compact oriented* M^4 *for which* $\int s^2 d\mu \to 0$ *and* $\int |W_+|^2 d\mu \to 0$.

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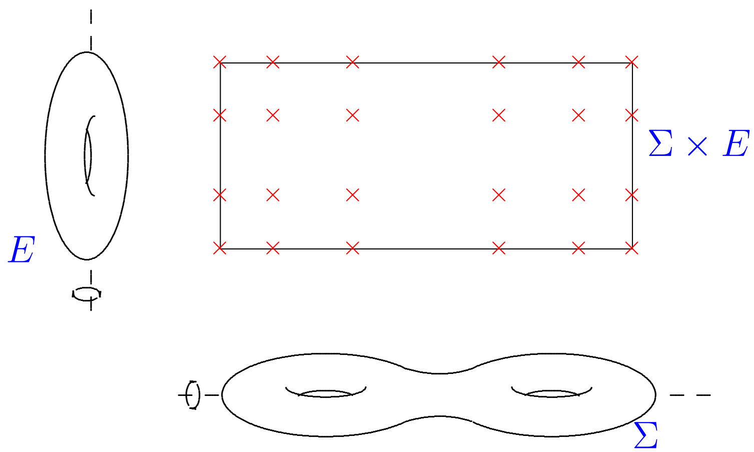
Typical example:



Lemma. *If* M^4 *admits an anorexic sequence, then*

$$\mathcal{I}_{\mathcal{R}}(M) = -8\pi^2(\chi + 3\tau)(M),$$

and any optimal metric on M *is SFASD.*

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and any optimal metric on M is SFASD.

$$\mathcal{K}(g) = -8\pi^2(\chi+3\tau)(M)+2\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

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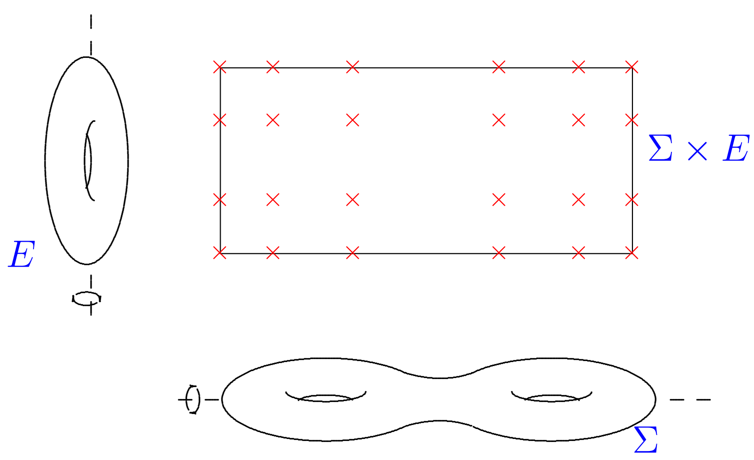

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Donaldson, Friedman-Morgan, et al.

Other homeotypes:

Theorem C. *If $j \geq 2$ and $k \geq 9j$, the smooth simply connected 4-manifold $j\mathbb{CP}_2\#k\overline{\mathbb{CP}}_2$ does not admit optimal metrics.*

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Theorem C. *If* $j \geq 2$ *and* $k \geq 9j$*, the smooth simply connected 4-manifold* $j\mathbb{CP}_2\#k\overline{\mathbb{CP}_2}$ *does not admit optimal metrics.*

Moreover, if $j \geq 5$ *and* $j \not\equiv 0 \bmod 8$*, the underlying topological manifold of this space admits infinitely many distinct differentiable structures for which no optimal metric exists.*

Proposition. *Suppose that* $Y_1, \ldots, Y_k$ *are the underlying 4-manifolds of elliptic complex surfaces. Then the connected sum*

$$Y_1 \# Y_2 \# \cdots \# Y_k$$

admits an anorexic sequence of Riemannian metrics.

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If $k \leq 4$, and Y's have $q = 0$, p_g odd,

Bauer-Furuta invariant distinguishes diffeotypes.

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4-manifolds need not carry optimal metrics.

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Geometrization of 3-manifolds:

Wrong question!

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Can 4-manifolds be decomposed into, say,

- Einstein and
- collapsed pieces?

Thank you, Nigel,

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Happy Birthday!