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EINSTEIN METRICS AND MOSTOW RIGIDITY

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Abstract. Using the new diffeomorphism invariants of Seiberg and Witten, a uniqueness theorem is proved for Einstein metrics on compact quotients of irreducible 4-dimensional symmetric spaces of non-compact type. The proof also yields a Riemannian version of the Miyaoka-Yau inequality.

A smooth Riemannian manifold (M, g) is said [1] to be *Einstein* if its Ricci curvature is a constant multiple of g. Any irreducible locally-symmetric space is Einstein, and, in light of Mostow rigidity [5], it is natural to ask whether, up to diffeomorphisms and rescalings, the standard metric is the only Einstein metric on any compact quotient of an irreducible symmetric space of non-compact type and dimension > 2. For example, any Einstein 3-manifold has constant curvature, so the answer is certainly affirmative in dimension 3. In dimension ≥ 4, however, solutions to Einstein's equations can be quite non-trivial. Nonetheless, the following 4-dimensional result was recently proved by means of an entropy comparison theorem [2]:

Theorem 1 (Besson-Courtois-Gallot). *Let M^4 be a smooth compact quotient of hyperbolic 4-space $\mathcal{H}^4 = SO(4,1)/SO(4)$, and let g_0 be its standard metric of constant sectional curvature. Then every Einstein metric g on M is of the form $g = \lambda\varphi^* g_0$, where $\varphi : M \to M$ is a diffeomorphism and $\lambda > 0$ is a constant.*

In this note, we will prove the analogous result for the remaining 4-dimensional cases:

Theorem 2. *Let M^4 be a smooth compact quotient of complex-hyperbolic 2-space $\mathbb{C}\mathcal{H}_2 = SU(2,1)/U(2)$. Let g_0 be its standard complex-hyperbolic metric. Then every Einstein metric g on M is of the form $g = \lambda\varphi^* g_0$, where $\varphi : M \to M$ is a diffeomorphism and $\lambda > 0$ is a constant.*

In contrast to Theorem 1, the proof of this result is based on the new 4-manifold invariants [4] recently introduced by Witten [6].

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1. Seiberg-Witten invariants

While the results in this section are largely due to Edward Witten [6], the crucial sharp form of the scalar-curvature inequality was pointed out to the author by Peter Kronheimer.

Let (M, g) be a smooth compact Riemannian manifold, and suppose that M admits an almost-complex structure. Then the given component of the almost-complex structures on M contains almost-complex structures $J : TM \to TM$, $J^2 = -1$ which are compatible with g in the sense that $J^*g = g$. Fixing such a J, the tangent bundle TM of M may be given the structure of a rank-2 complex vector bundle $T^{1,0}$ by defining scalar multiplication by i to be J. Setting $\wedge^{0,p} := \wedge^p \overline{T^{1,0}}^* \cong \wedge^p T^{1,0}$, we may then define rank-2 complex vector bundles $V_\pm$ on M by

$$V_+ = \wedge^{0,0} \oplus \wedge^{0,2} \tag{1}$$

$$V_- = \wedge^{0,1}, \tag{2}$$

and g induces canonical Hermitian inner products on these bundles.

As described, these bundles depend on the choice of a particular almost-complex structure J, but they have a deeper meaning [3] that depends only on the homotopy class c of J; namely, if we restrict to a contractible open set $U \subset M$, the bundles $V_\pm$ may be canonically identified with $\mathbb{S}_\pm \otimes L^{1/2}$, where $\mathbb{S}_\pm$ are the left- and right-handed spinor bundles of g, and $L^{1/2}$ is a complex line bundle whose square is the 'anti-canonical' line-bundle $L = (\overline{\wedge^{0,2}})^* \cong \wedge^{0,2}$. For each connection A on L compatible with the g-induced inner product, we can thus define a corresponding Dirac operator

$$D_A : C^\infty(V_+) \to C^\infty(V_-).$$

If J is parallel with respect to g, so that (M, g, J) is a Kähler manifold, and if A is the Chern connection on the anti-canonical bundle L, then $D_A = \sqrt{2}(\overline{\partial} \oplus \overline{\partial}^*)$, where $\overline{\partial} : C^\infty(\wedge^{0,0}) \to C^\infty(\wedge^{0,1})$ is the J-antilinear part of the exterior differential d, acting on complex-valued functions, and where $\overline{\partial}^* : C^\infty(\wedge^{0,2}) \to C^\infty(\wedge^{0,1})$ is the formal adjoint of the map induced by the exterior differential d acting on 1-forms; more generally, D_A will differ from $\sqrt{2}(\overline{\partial} \oplus \overline{\partial}^*)$ by only 0^{th} order terms.

The Seiberg-Witten equations

$$D_A \Phi = 0 \tag{3}$$

$$F_A^+ = i\sigma(\Phi). \tag{4}$$

are equations for an unknown smooth connection A on L and an unknown smooth section Φ of V_+. Here the purely imaginary 2-form F_A^+ is the self-dual part of the curvature of A, and, in terms of (1), the real-quadratic map $\sigma : V_+ \to \wedge^2_+$ is given by

$$\sigma(f,\phi) = (|f|^2 - |\phi|^2)\frac{\omega}{4} + \Im m(\bar{f}\phi),$$

where $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ is the 'Kähler' form. Notice that $|F^+| = 2^{-3/2}|\Phi|^2$.

For each solution (A,Φ) of (3) and (4), one can generate a new solution $(A + 2d\log f, f\Phi)$ for any $f : M \to S^1 \subset \mathbb{C}$; two solutions which are related in this way are called *gauge equivalent*, and may be considered to be geometrically identical. A solution is called *reducible* if $\Phi \equiv 0$; otherwise, it is called *irreducible*.

A useful generalization of the Seiberg-Witten equations is obtained by replacing (4) with the equation

$$iF^+ + \sigma(\Phi) = \varepsilon \tag{5}$$

for an arbitrary $\varepsilon \in C^\infty(\wedge^+)$. We can then consider the map which sends solutions of (3) and (5) to the corresponding $\varepsilon \in C^\infty(\wedge^+)$, and define a solution to be *transverse* if it is a regular point of this map, i.e. if the linearization $C^\infty(V_+ \oplus \wedge^1) \to C^\infty(\wedge^2_+)$ of the left-hand-side of (5), constrained by the linearization of (3), is surjective.

Example. Let (M,g,J) be a Kähler surface of constant scalar curvature $s<0$. Let $\Phi = (\sqrt{-s},0) \in \wedge^{0,0} \oplus \wedge^{0,2}$, and let A be the Chern connection on the anti-canonical bundle. Since $F_A^+ = -is\omega/4$, (Φ, A) is an irreducible solution of the Seiberg-Witten equations (3) and (4).

The linearization of (3) at this solution is just

$$(\overline{\partial} \oplus \overline{\partial}^*)(u+\psi) = -\frac{\sqrt{-s}}{2}\alpha, \tag{6}$$

where $(u,\psi) \in C^\infty(V_+)$ is the linearization of $\Phi = (f,\phi)$ and $\alpha \in \wedge^{0,1}$ is the $(0,1)$-part of the purely imaginary 1-form which is the linearization of A. Linearizing (5) at our solution yields the operator

$$(u,\psi,\alpha) \mapsto id^+(\alpha - \bar{\alpha}) + \frac{\sqrt{-s}}{2}(\Re e u)\omega + \sqrt{-s}\Im m\psi.$$

Since the right-hand-side is a real self-dual form, it is completely characterized by its component in the ω direction and its $(0,2)$-part. The ω-component of this operator is just

$$(u,\psi,\alpha) \mapsto \Re e\left[-\bar{\partial}^*\alpha + \frac{\sqrt{-s}}{2}u\right],$$

while the $(0,2)$-component is

$$(u,\psi,\alpha)\mapsto i\bar{\partial}\alpha - i\frac{\sqrt{-s}}{2}\psi.$$

Substituting (6) into these expressions, we obtain the operator

$$\begin{array}{ccc} C^{\infty}(\mathbb{C}\oplus\wedge^{0,2}) & \longrightarrow & C^{\infty}(\mathbb{R}\oplus\wedge^{0,2}) \\ (u,\psi) & \mapsto & (\frac{1}{\sqrt{-s}}\Re e\left[\Delta-\frac{s}{2}\right]u, -\frac{i}{\sqrt{-s}}\left[\Delta-\frac{s}{2}\right]\psi), \end{array}$$

which is surjective because $s/2<0$ is not in the spectrum of the Laplacian. The constructed solution is therefore transverse.

Relative to $c=[J]$, a metric g will be called *excellent* if it admits only irreducible transverse solutions of (3) and (4). Relative to any excellent metric, the set of solutions of (3) and (4), modulo gauge equivalence, is finite [4, 6]. Notice that a metric g is automatically excellent if the corresponding equations (3) and (4) admit no solutions at all.

Definition 1. *Let (M,c) be a compact 4-manifold equipped with a a homotopy class $c=[J]$ of almost-complex structures. Assume either*

$$b_+(M)>1$$

$$\textit{or}$$

$$b_+=1 \quad \textit{and} \quad (2\chi+3\tau)(M)>0.$$

If g is an excellent metric on M, define the (mod 2) Seiberg-Witten invariant *$n_c(M)\in\mathbb{Z}_2$ to be*

$$n_c(M)=\#\{\textit{gauge classes of solutions of } (3) \textit{ and } (4)\} \bmod 2$$

calculated with respect to g.

It turns out [4] that $n_c(M)$ is actually metric-independent; when $b_+=1$, this fact depends on the assumption that $c_1(L)^2=2\chi+3\tau>0$, which guarantees that (3) and (4) cannot admit reducible solutions for any metric.

Theorem 3. *Let (M,J) be a compact complex surface, where the underlying oriented 4-manifold M is as in Definition 1. Suppose that (M,J) admits a Kähler metric g of constant scalar curvature $s<0$, and let $c=[J]$. Then $n_c(M)=1\in\mathbb{Z}_2$.*

Proof. With respect to g we shall show that, up to gauge equivalence, there is exactly one solution of the Seiberg-Witten equations, namely the one described in the above example. Indeed, the Weitzenböck formula for the twisted Dirac operator and equation (4) tell us that

$$0 = D_A^* D_A \Phi = \nabla^* \nabla \Phi + \frac{s}{4}\Phi + \frac{1}{4}|\Phi|^2\Phi,$$

which implies [4] the C^0 estimate $|\Phi|^2 \leq -s$, with equality only at points where $\nabla\Phi = 0$. Since

$$|F_A^+|^2 = \frac{1}{8}|\Phi|^4 \leq \frac{s^2}{8},$$

it follows that

$$\int_M |F_A^+|^2 d\mu \leq \int_M \left(\frac{s}{4}|\omega|\right)^2 d\mu = \int_M |\rho^+|^2 d\mu$$

where the Ricci form ρ is in the same cohomology class as the closed form F_A, namely $2\pi c_1(L) = 2\pi c_1(M, J)$. But since s is constant, ρ is harmonic, and we must therefore have that

$$\begin{aligned}\int_M |\rho^+|^2 d\mu &= 2\pi^2 c_1(L)^2 + \frac{1}{2}\int_M |\rho|^2 d\mu \\ &\leq 2\pi^2 c_1(L)^2 + \frac{1}{2}\int_M |F_A|^2 d\mu \\ &= \int_M |F_A^+|^2 d\mu\end{aligned}$$

because a harmonic form minimizes the L^2 norm among closed forms in its deRham class. Hence $F_A = \rho$, and A differs from the Chern connection on L by twisting with a flat connection. But also $|\Phi|^2 \equiv -s$, which forces $\nabla\Phi \equiv 0$. Since $c_1(L) \neq 0$, the induced connection on $\wedge^{0,2} \subset V_+$ has non-trivial curvature, and Φ must therefore be a section of $\wedge^{0,0}$. Since Φ is parallel, the induced connection on $\wedge^{0,0}$ must not only be flat, but also have trivial holonomy. Thus A must exactly be the Chern connection on L, and our solution coincides, up to gauge transformation, with that of the example. In particular, every solution with respect to g is irreducible and transverse, so g is excellent. But since there is only one gauge class of solutions with respect to g, we conclude that $n_c(M) = 1 \bmod 2$. □

The following refinement of an observation of Witten [6, §3] is the real key to the proof of Theorem 2.

Theorem 4. *Let M be a smooth compact oriented 4-manifold with*

$$2\chi(M) + 3\tau(M) > 0.$$

Suppose that there is a an orientation-compatible class $c = [J]$ of almost-complex structures for which the Seiberg-Witten invariant $n_c(M) \in \mathbb{Z}_2$ is non-zero. Let g be a metric of constant scalar curvature s and volume V on M. Then

$$s\sqrt{V} \leq -2^{5/2}\pi\sqrt{2\chi + 3\tau},$$

with equality iff g is Kähler-Einstein with respect to some integrable complex structure J in the homotopy class c.

Proof. For any given metric g on M, there must exist a solution of (3) and (4), since otherwise we would have $n_c(M) = 0$. But since $|F_A^+|^2 = |\Phi|^4/8 \leq s^2/8$, with equality iff $\nabla\Phi = 0$, it follows that

$$2\chi + 3\tau = c_1(L)^2 = \frac{1}{4\pi^2}\int_M \left(|F_A^+|^2 - |F_A^-|^2\right) d\mu \leq \frac{1}{32\pi^2}\int_M s^2 d\mu,$$

with equality only if

$$\nabla F_A^+ \equiv 0 \quad \text{and} \quad F_A^- = 0.$$

If equality holds, the parallel self-dual form

$$\sqrt{2}F_A/|F_A|$$

corresponds via g to a parallel almost-complex structure J, and the manifold is thus Kähler, with Kähler class $8\pi/s$ times $c_1(M, J) = c_1(L)$. But since s is constant, the Ricci form is harmonic, and the manifold is Kähler-Einstein.

On the other hand, any Kähler-Einstein metric will saturate the bound in question, since the first Chern class of a Kähler-Einstein surface is $[s\omega/8\pi]$, and the metric volume form is $d\mu = \omega^2/2$. □

2. The Miyaoka-Yau inequality

For any compact oriented Riemannian 4-manifold (M, g), the Euler characteristic and signature can be expressed as

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(|W_+|^2 + |W_-|^2 + \frac{s^2}{24} - \frac{|\mathrm{ric}_0|^2}{2}\right) d\mu$$

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

where s, ric_0, W_+ and W_- are respectively the scalar, trace-free Ricci, self-dual Weyl, and anti-self-dual Weyl parts of the curvature tensor; pointwise norms are calculated with respect to g, and $d\mu$ is the metric volume form. If g is Einstein, $\text{ric}_0 = 0$, and M therefore satisfies

$$(2\chi \pm 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(2|W_\pm|^2 + \frac{s^2}{24}\right) d\mu,$$

so the *Hitchin-Thorpe inequality* $2\chi + 3\tau \geq 0$ holds, with strict inequality unless M is finitely covered by a 4-torus or $K3$ surface.

Now assume that M admits a homotopy class of almost-complex structures for which the Seiberg-Witten invariant is non-zero. If g is an Einstein metric on M, Theorem 4 then tells us that

$$\begin{aligned} 2\chi + 3\tau &\leq \frac{1}{32\pi^2}\int_M s^2 d\mu \\ &\leq 3\left[\frac{1}{4\pi^2}\int_M \left(|2W_-|^2 + \frac{s^2}{24}\right) d\mu\right] \\ &= 3(2\chi - 3\tau) \end{aligned}$$

with equality iff the metric is Kähler and $W_- = 0$. But the curvature operator of any Kähler manifold is an element of $\wedge^{1,1} \otimes \wedge^{1,1}$, and in real dimension 4 one also has $\wedge^{1,1} = \wedge^- \oplus \mathbb{C}\omega$, where ω is the Kähler form; when

$$W_- : \wedge_- \to \wedge_- \quad \text{and} \quad \text{ric}_0 : \wedge_- \to \wedge_+$$

both vanish, the curvature operator must therefore be of the form

$$\mathcal{R} = \frac{s}{8}\omega \otimes \omega + \frac{s}{12}1_{\wedge_-}$$

and so satisfy

$$\nabla\mathcal{R} = 0,$$

which is to say that (M, g) must be locally symmetric. Unless g is flat, the non-triviality of the Seiberg-Witten invariant now forces s to be negative, and the point-wise form of the curvature tensor then implies that the exponential map induces an isometry between the universal cover of (M, g) and a complex-hyperbolic space which has been rescaled so as to have the same scalar curvature. This proves the following generalization of the Miyaoka-Yau inequality [7]:

Theorem 5. *Let (M, g) be a compact Einstein 4-manifold, and suppose that M admits an almost-complex structure J for which the Seiberg-Witten invariant is non-zero. Also assume that M is not finitely covered by the*

4-torus T^4*. Then, with respect to the orientation of* M *determined by* J*, the Euler characteristic and signature of* M *satisfy*

$$\chi \geq 3\tau,$$

with equality iff the universal cover of (M, g) *is complex-hyperbolic 2-space* $\mathbb{C}\mathcal{H}_2 := SU(2,1)/U(2)$ *with a constant multiple of its standard metric.*

On the other hand, Theorem 3 tells us the Seiberg-Witten invariant of any complex hyperbolic 4-manifold $M = \mathbb{C}\mathcal{H}_2/\Gamma$ is actually non-zero. Theorem 5 and Mostow rigidity thus imply Theorem 2.

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