Hermitian Metrics,

Einstein Manifolds, &

Conformal Geometry

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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Has same sign as the *scalar curvature*

$$s = r_j^{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

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Much more flexible in higher dimensions.

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On suitable 4-manifolds, Seiberg-Witten theory allows one to mimic Kähler geometry when treating non-Kähler metrics.

Natural Question. *If (M^4, J) is a compact complex surface, when does M^4 admit an Einstein metric g (unrelated to J)?*

Complete answer now available for $\lambda \geq 0$ cases.

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Kähler if the 2-form

$$\omega = g(J\cdot, \cdot)$$

is closed:

$$d\omega = 0.$$

But we do not assume this!

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Only two metrics arise in non-Kähler case!

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Exceptional cases: $\mathbb{CP}_2$ blown up at 1 or 2 points.

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Diffeotypes arising: Del Pezzo surfaces.

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Similarly when M symplectic instead of complex.

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Fano manifolds of complex dimension 2.

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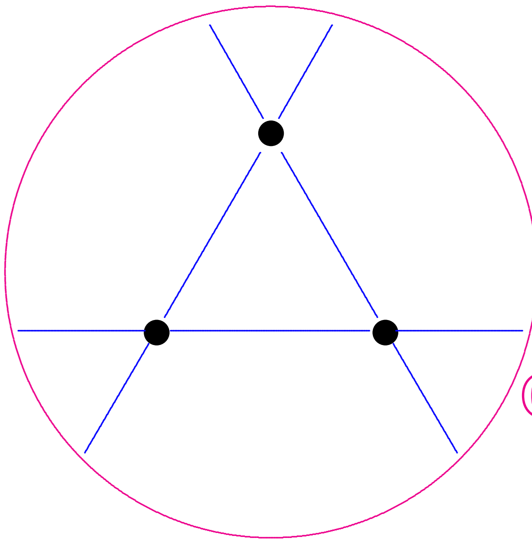

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Will describe a second proof (L '12) which contains much more information.

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Einstein metric is $h = s^{-2}g$.

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$\nabla^{1,0} s$ is a holomorphic vector field.

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X.X. Chen: always minimizers.

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Donaldson/Mabuchi/Chen-Tian:
unique modulo bihomorphisms.

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature *(conformally invariant)*

W_- = anti-self-dual Weyl curvature ″

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1-parameter family of metrics

$$g_t := g + t\dot{g} + O(t^2)$$

First variation

$$\frac{d}{dt}\mathcal{W}_+(g_t)\Big|_{t=0} = -\int \dot{g}^{ab} B_{ab} \, d\mu_g$$

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Conformally Einstein $\Longrightarrow$ $B = 0$

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$$W_{+} = \begin{pmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{pmatrix}$$

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$$|W_{+}|^2 = \frac{s^2}{24}$$

Restriction of $\mathcal{W}_+$ to Kähler metrics?

On Kähler metrics,

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so any critical point of restriction must be extremal in sense of Calabi.

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In fact, for Kähler metrics,

$$B = \frac{1}{12}\Big[2s\mathring{r} + \mathrm{Hess}_0(s) + 3J^*\mathrm{Hess}_0(s)\Big]$$

where Hess_0 denotes trace-free part of $\nabla\nabla$.

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So the critical metrics of restriction of $\mathcal{W}_+$ to $\{$Kähler metrics$\}$ are Bach-flat Kähler metrics.

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Lemma. *For any extremal Kähler g on any Del Pezzo M, scalar curvature $s > 0$ everywhere.*

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among all Kähler metrics on M. *Consequently,* h *is an absolute minimizer of the functional*

$$h \longmapsto \int_M |W|_h^2 \, d\mu_h.$$

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Similarly for $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$, though less interesting...

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Lemma. *For all $[\omega]$ on any Del Pezzo M,*

$$\mathcal{B}([\omega]) < \frac{1}{4}$$

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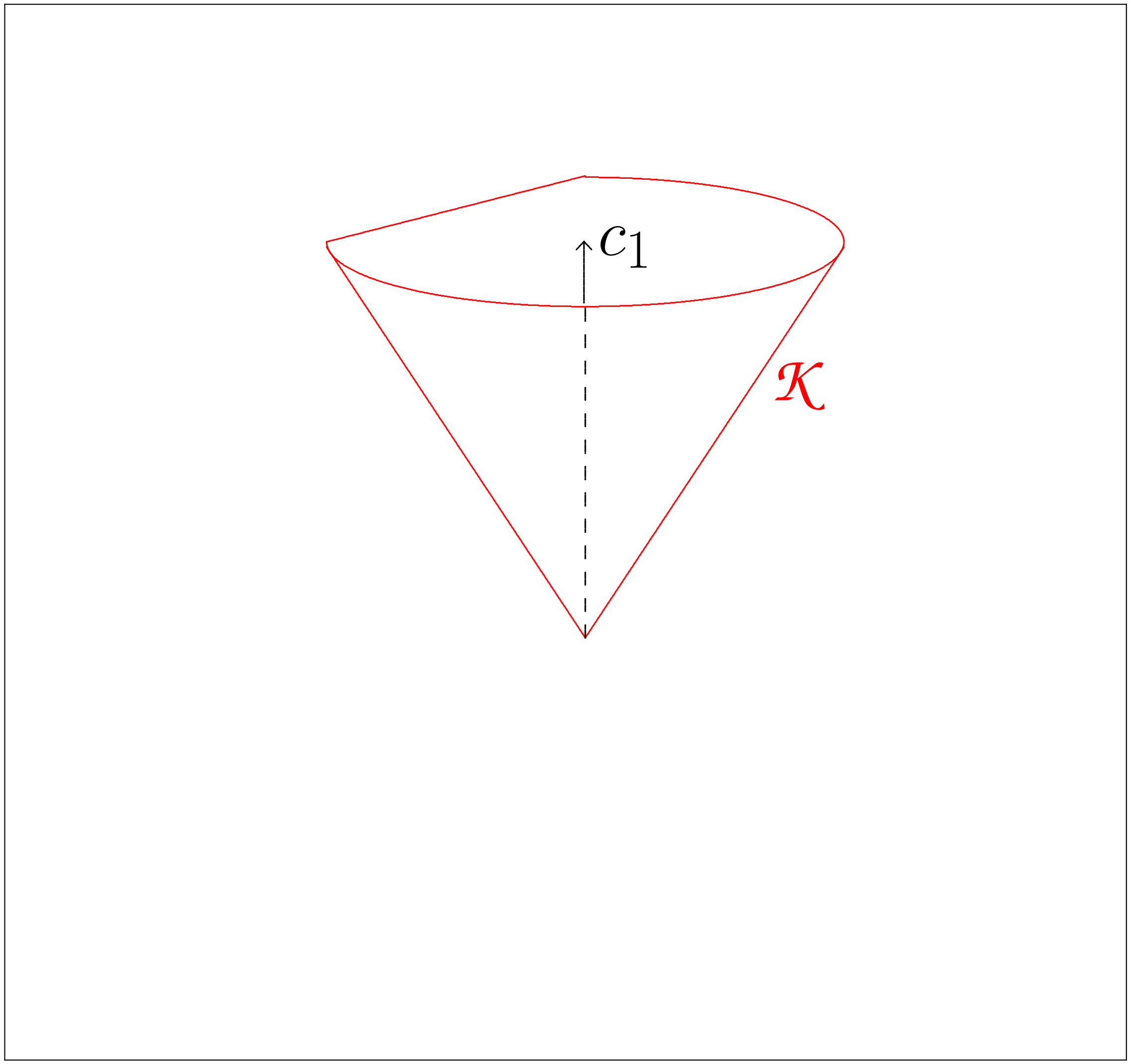
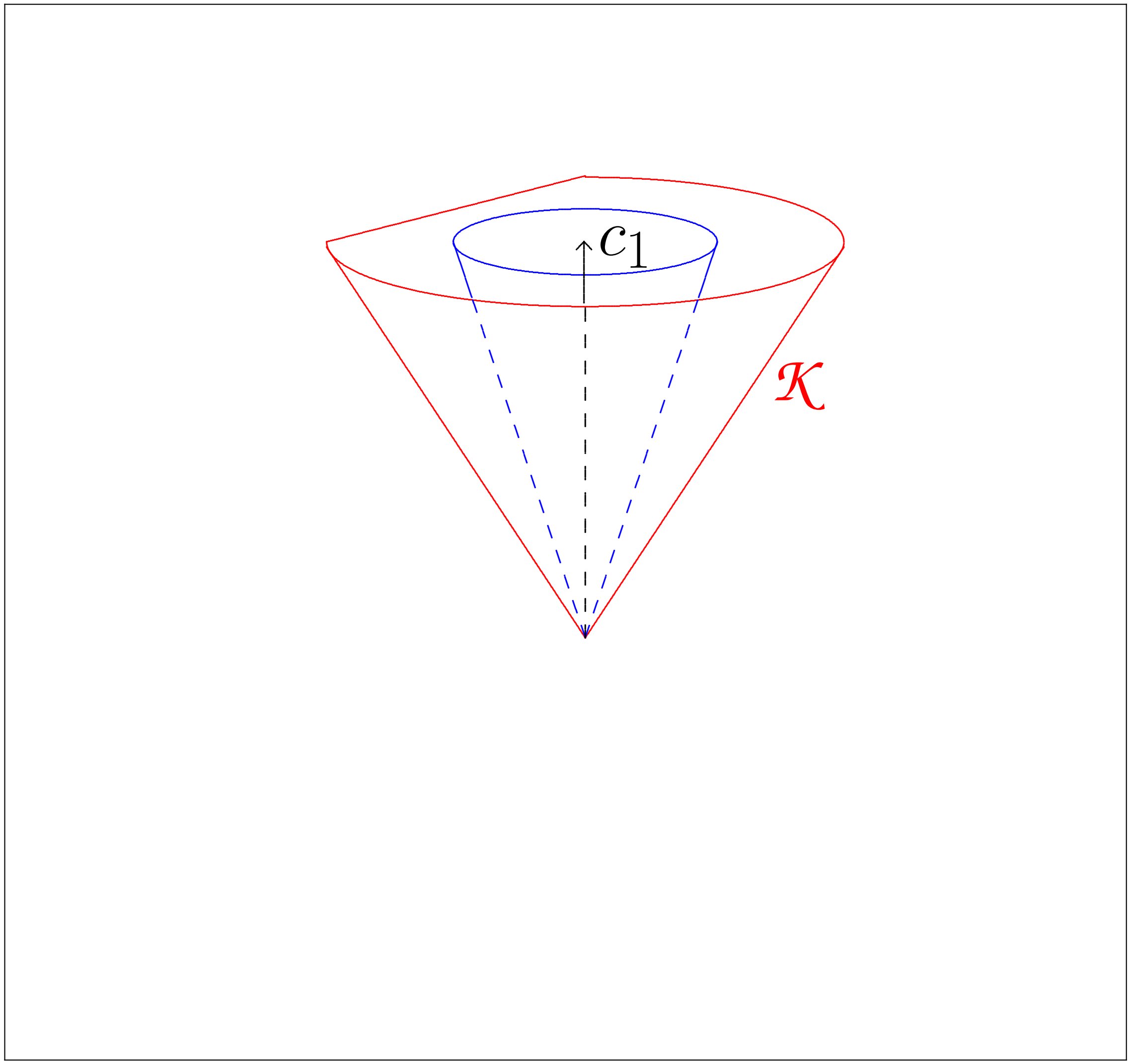
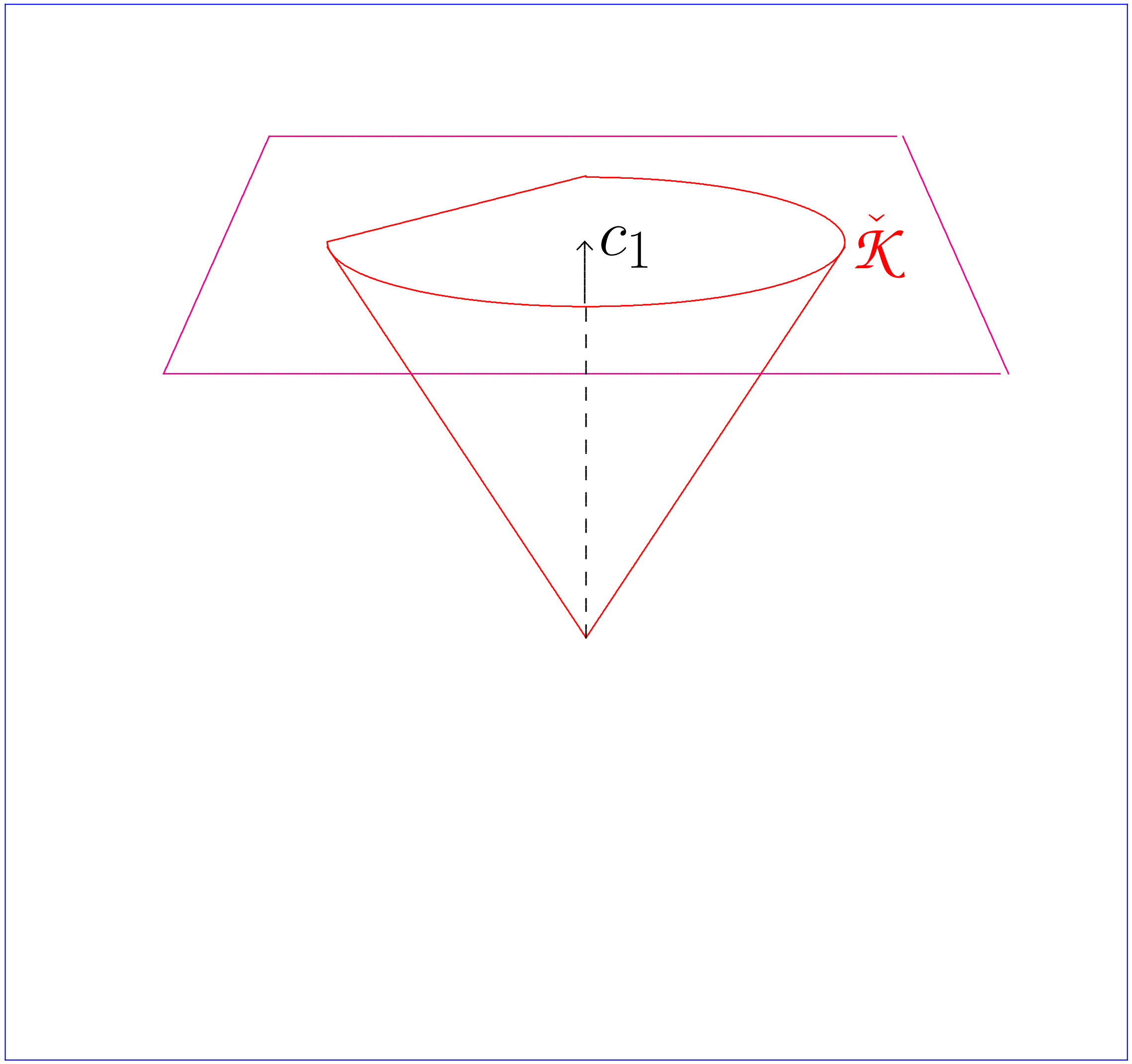


$$H^{1,1}(M, \mathbb{R}) = H^2(M, \mathbb{R})$$



$$\mathcal{T}([\omega]) = \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} \leq const$$



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Generalizes work of

Anderson, Bando-Kasue-Nakajima, Tian-Viaclovsky...

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Recall:

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have Sobolev bound on convex cone

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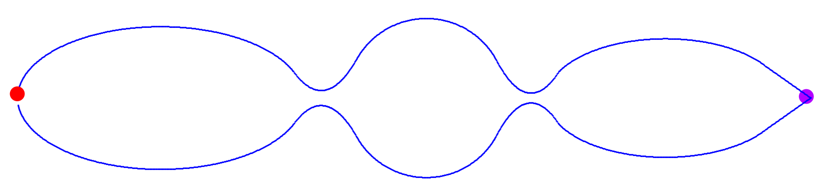

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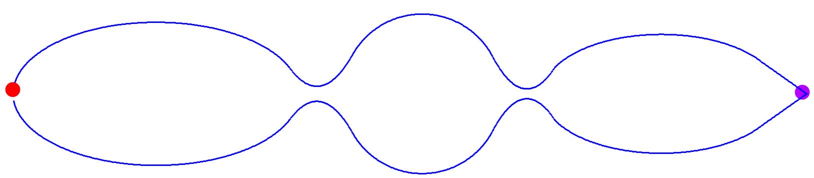

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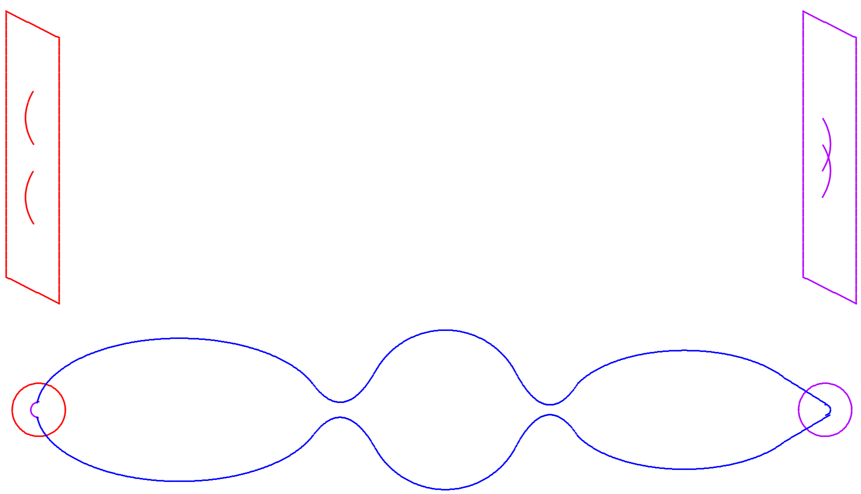

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Suggests continuity method...

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Difficulty: rule out deepest bubbles.

Lemma. *If* M *is toric, any deepest bubble* (X, g_∞) *must be toric, too, with* $H_2(X, \mathbb{R}) \neq 0$ *generated by holomorphic* $\mathbb{CP}_1$*'s.*

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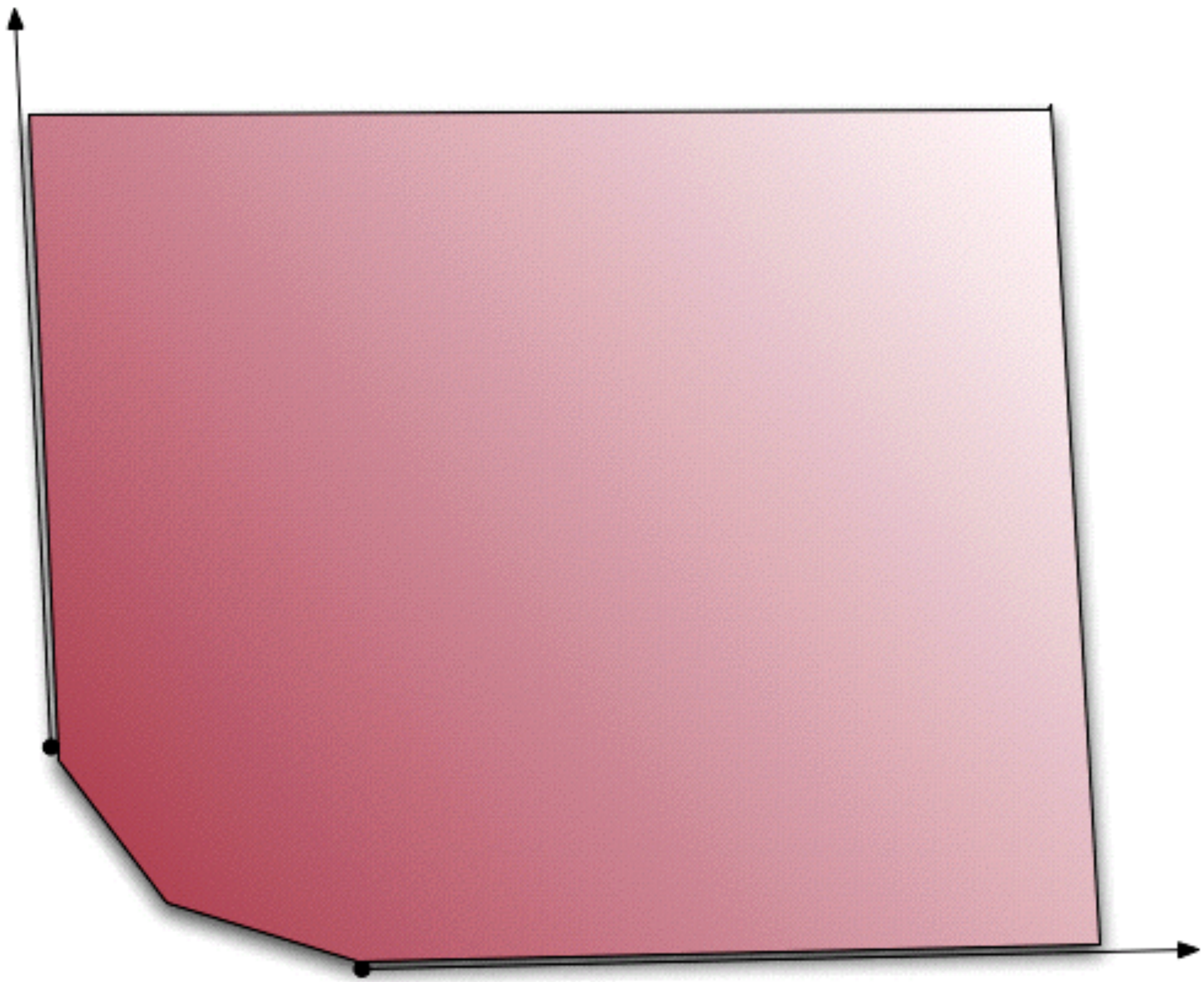
Moment map profile:



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$$[S] \cdot [S] = -k < 0$$

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- Bubble arises by rescaling region of manifold by scales $\nearrow \infty$.
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It follows that bubbling off cannot occur!

Proposition 4. *Let* $M = \mathbb{CP}_2 \# 3\overline{\mathbb{CP}}_2$ *be the blow-up of* $\mathbb{CP}_2$ *at three non-collinear points, and let* $[\omega]$ *be a Kähler class on* M *for which*

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Also works when approaching boundary of Kähler cone, but can bubble off (-1)-curves.

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Lemma. *Let* (M^4, J) *be a compact complex surface, and suppose that* h *is an Einstein metric on* M *which is Hermitian with respect to* J*:*

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In other words,

$$h = fg$$

$\exists$ Kähler metric g, smooth function $f : M \to \mathbb{R}^+$.

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Similarly for $S^{2n+1} \times S^{2m+1}$.

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May normalize so that either $f = s^{-2}$ or $f = 1$.

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Key step: show W_+ has a repeated eigenvalue.

Kähler case:

$$\Lambda^{1,1} = \mathbb{R}\omega \oplus \Lambda^{-}$$

$$\Lambda^{+} = \mathbb{R}\omega \oplus \Re e(\Lambda^{2,0})$$

$$\nabla J = 0 \Longrightarrow \mathcal{R} \in \mathrm{End}(\Lambda^{1,1}) \Longrightarrow$$

$$W_{+} = \begin{pmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{pmatrix}$$

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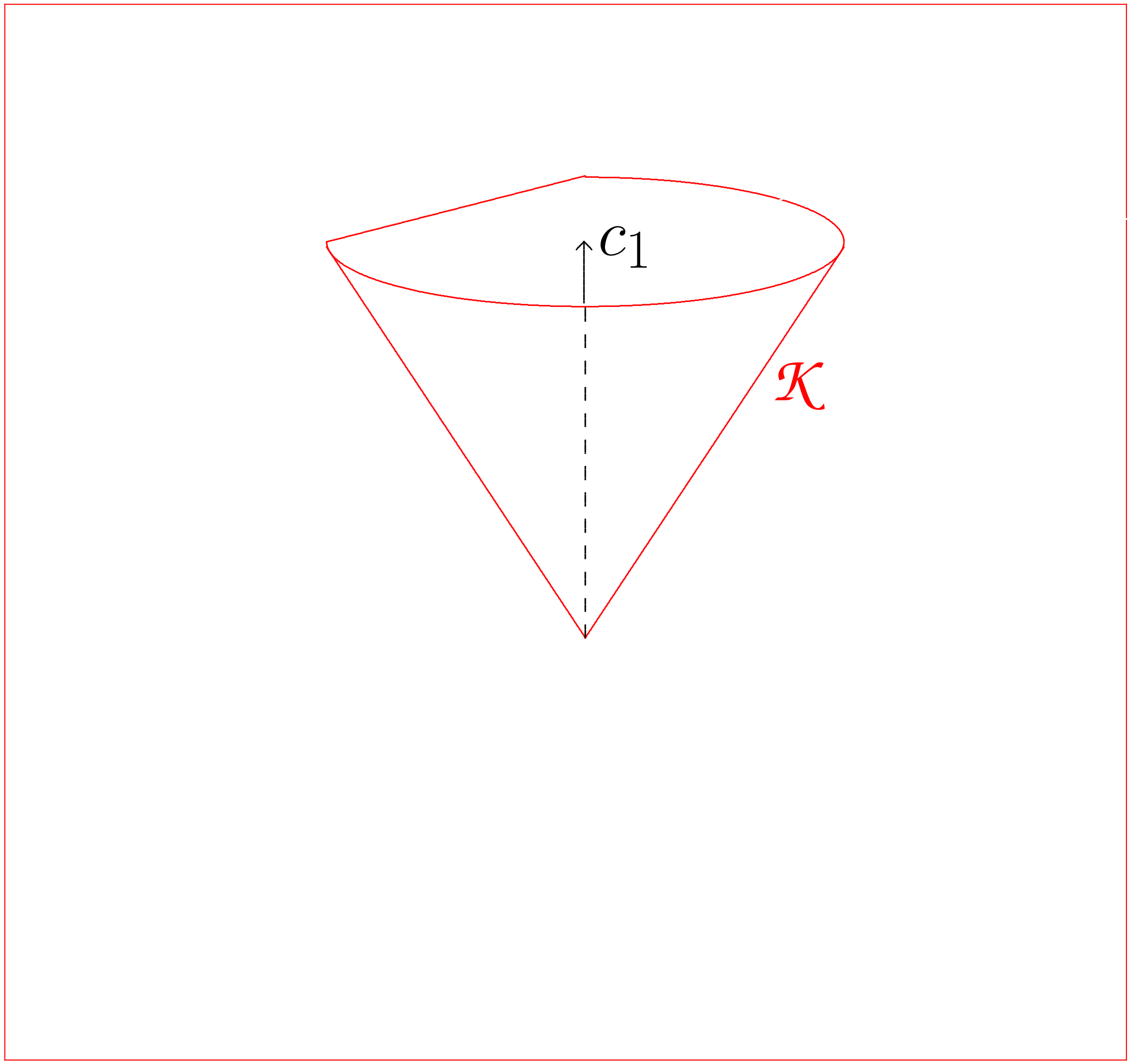

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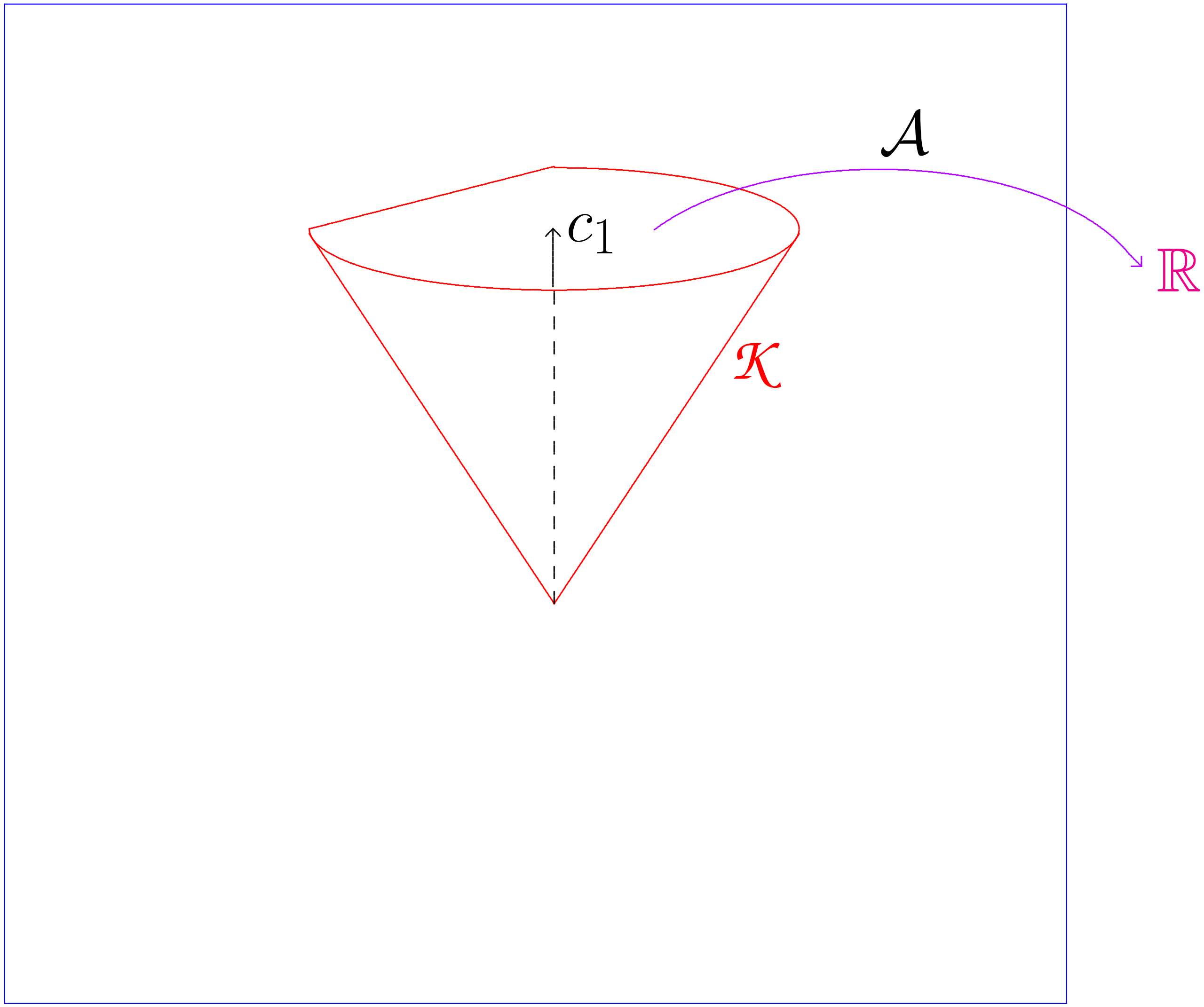

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$$=: \mathcal{A}([\omega])$$

where $\mathcal{F}$ is Futaki invariant.

$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.

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$$0 = 6s^{-1}B = \mathring{r} + 2s^{-1}\mathrm{Hess}_0(s)$$

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$$\rho + 2i\partial\bar{\partial}\log s > 0.$$

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Hermitian, Einstein metric then given by

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and uniqueness Theorem B follows.

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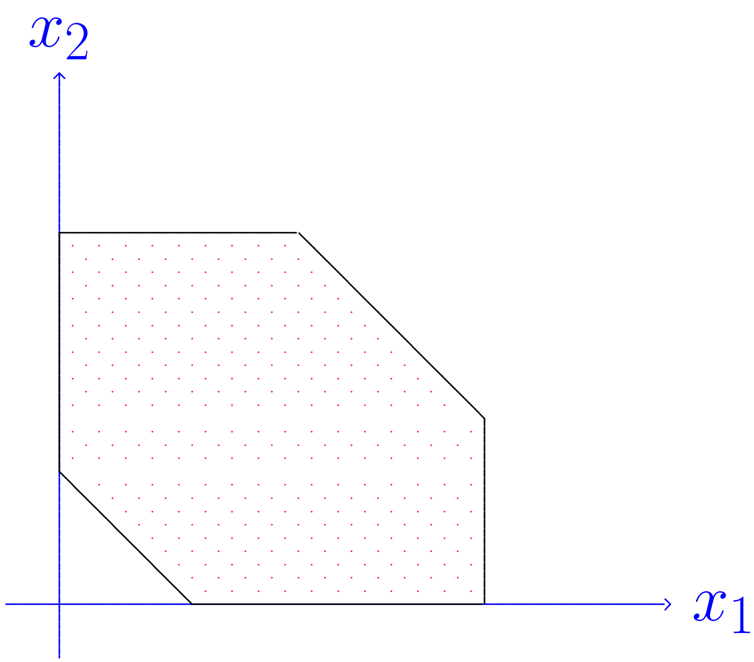
Only three cases are non-trivial:

$$\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \qquad k = 1, 2, 3.$$

The non-trivial cases are toric, and the action $\mathcal{A}$ can be directly computed from moment polygon.



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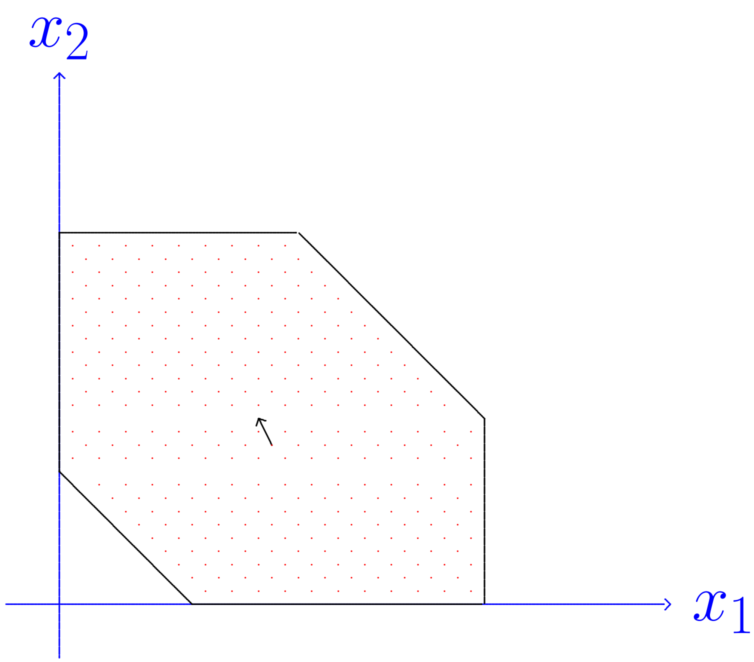

$$\mathcal{A}([\omega]) = \frac{|\partial P|^2}{2}\left(\frac{1}{|P|} + \vec{\mathfrak{D}} \cdot \Pi^{-1}\vec{\mathfrak{D}}\right)$$

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$\mathcal{A}$ is explicit rational function —

$$3\Big[3+28\gamma+96\gamma^2+168\gamma^3+164\gamma^4+80\gamma^5+16\gamma^6+16\beta^6(1+\gamma)^4+16\alpha^6(1+\beta+\gamma)^4+16\beta^5(5+24\gamma+43\gamma^2+37\gamma^3+15\gamma^4+2\gamma^5)+4\beta^4(41+228\gamma+478\gamma^2+496\gamma^3+263\gamma^4+$$

$$60\gamma^5+4\gamma^6)+8\beta^3(21+135\gamma+326\gamma^2+392\gamma^3+248\gamma^4+74\gamma^5+8\gamma^6)+4\beta(7+58\gamma+176\gamma^2+270\gamma^3+228\gamma^4+96\gamma^5+16\gamma^6)+4\beta^2(24+176\gamma+479\gamma^2+652\gamma^3+478\gamma^4+$$

$$172\gamma^5+24\gamma^6)+16\alpha^5(5+2\beta^5+24\gamma+43\gamma^2+37\gamma^3+15\gamma^4+2\gamma^5+\beta^4(15+14\gamma)+\beta^3(37+70\gamma+30\gamma^2)+\beta^2(43+123\gamma+108\gamma^2+30\gamma^3)+\beta(24+92\gamma+123\gamma^2+70\gamma^3+$$

$$14\gamma^4))+4\alpha^4(41+4\beta^6+228\gamma+478\gamma^2+496\gamma^3+263\gamma^4+60\gamma^5+4\gamma^6+\beta^5(60+56\gamma)+\beta^4(263+476\gamma+196\gamma^2)+8\beta^3(62+169\gamma+139\gamma^2+35\gamma^3)+2\beta^2(239+876\gamma+1089\gamma^2+$$

$$556\gamma^3+98\gamma^4)+4\beta(57+263\gamma+438\gamma^2+338\gamma^3+119\gamma^4+14\gamma^5))+8\alpha^3(21+135\gamma+326\gamma^2+392\gamma^3+248\gamma^4+74\gamma^5+8\gamma^6+8\beta^6(1+\gamma)+2\beta^5(37+70\gamma+30\gamma^2)+4\beta^4(62+$$

$$169\gamma+139\gamma^2+35\gamma^3)+4\beta^3(98+353\gamma+428\gamma^2+210\gamma^3+35\gamma^4)+2\beta^2(163+735\gamma+1179\gamma^2+856\gamma^3+278\gamma^4+30\gamma^5)+\beta(135+736\gamma+1470\gamma^2+1412\gamma^3+676\gamma^4+140\gamma^5+$$

$$8\gamma^6))+4\alpha(7+58\gamma+176\gamma^2+270\gamma^3+228\gamma^4+96\gamma^5+16\gamma^6+16\beta^6(1+\gamma)^3+4\beta^5(24+92\gamma+123\gamma^2+70\gamma^3+14\gamma^4)+4\beta^4(57+263\gamma+438\gamma^2+338\gamma^3+119\gamma^4+14\gamma^5)+$$

$$2\beta^3(135+736\gamma+1470\gamma^2+1412\gamma^3+676\gamma^4+140\gamma^5+8\gamma^6)+4\beta^2(44+278\gamma+645\gamma^2+735\gamma^3+438\gamma^4+123\gamma^5+12\gamma^6)+2\beta(29+210\gamma+556\gamma^2+736\gamma^3+526\gamma^4+184\gamma^5+$$

$$24\gamma^6))+4\alpha^2(24+176\gamma+479\gamma^2+652\gamma^3+478\gamma^4+172\gamma^5+24\gamma^6+24\beta^6(1+\gamma)^2+4\beta^5(43+123\gamma+108\gamma^2+30\gamma^3)+2\beta^4(239+876\gamma+1089\gamma^2+556\gamma^3+98\gamma^4)+4\beta^3(163+$$

$$735\gamma+1179\gamma^2+856\gamma^3+278\gamma^4+30\gamma^5)+4\beta(44+278\gamma+645\gamma^2+735\gamma^3+438\gamma^4+123\gamma^5+12\gamma^6)+\beta^2(479+2580\gamma+5058\gamma^2+4716\gamma^3+2178\gamma^4+432\gamma^5+24\gamma^6))\Big]\Big/$$

$$\Big[1+10\gamma+36\gamma^2+64\gamma^3+60\gamma^4+24\gamma^5+24\beta^5(1+\gamma)^5+24\alpha^5(1+\beta+\gamma)^5+12\beta^4(1+\gamma)^2(5+20\gamma+23\gamma^2+10\gamma^3)+16\beta^3(4+28\gamma+72\gamma^2+90\gamma^3+57\gamma^4+15\gamma^5)+$$

$$12\beta^2(3+24\gamma+69\gamma^2+96\gamma^3+68\gamma^4+20\gamma^5)+2\beta(5+45\gamma+144\gamma^2+224\gamma^3+180\gamma^4+60\gamma^5)+12\alpha^4(1+\beta+\gamma)^2(5+20\gamma+23\gamma^2+10\gamma^3+10\beta^3(1+\gamma)+\beta^2(23+46\gamma+$$

$$16\gamma^2)+2\beta(10+30\gamma+23\gamma^2+5\gamma^3))+16\alpha^3(4+28\gamma+72\gamma^2+90\gamma^3+57\gamma^4+15\gamma^5+15\beta^5(1+\gamma)^2+3\beta^4(19+57\gamma+50\gamma^2+13\gamma^3)+3\beta^3(30+120\gamma+155\gamma^2+78\gamma^3+$$

$$13\gamma^4)+3\beta^2(24+120\gamma+206\gamma^2+155\gamma^3+50\gamma^4+5\gamma^5)+\beta(28+168\gamma+360\gamma^2+360\gamma^3+171\gamma^4+30\gamma^5))+12\alpha^2(3+24\gamma+69\gamma^2+96\gamma^3+68\gamma^4+20\gamma^5+20\beta^5(1+\gamma)^3+$$

$$\beta^4(68+272\gamma+366\gamma^2+200\gamma^3+36\gamma^4)+4\beta^3(24+120\gamma+206\gamma^2+155\gamma^3+50\gamma^4+5\gamma^5)+2\beta(12+84\gamma+207\gamma^2+240\gamma^3+136\gamma^4+30\gamma^5)+\beta^2(69+414\gamma+864\gamma^2+$$

$$824\gamma^3+366\gamma^4+60\gamma^5))+2\alpha(5+45\gamma+144\gamma^2+224\gamma^3+180\gamma^4+60\gamma^5+60\beta^5(1+\gamma)^4+12\beta^4(15+75\gamma+136\gamma^2+114\gamma^3+43\gamma^4+5\gamma^5)+12\beta^2(12+84\gamma+207\gamma^2+$$

$$240\gamma^3+136\gamma^4+30\gamma^5)+8\beta^3(28+168\gamma+360\gamma^2+360\gamma^3+171\gamma^4+30\gamma^5)+3\beta(15+120\gamma+336\gamma^2+448\gamma^3+300\gamma^4+80\gamma^5))\Big]$$

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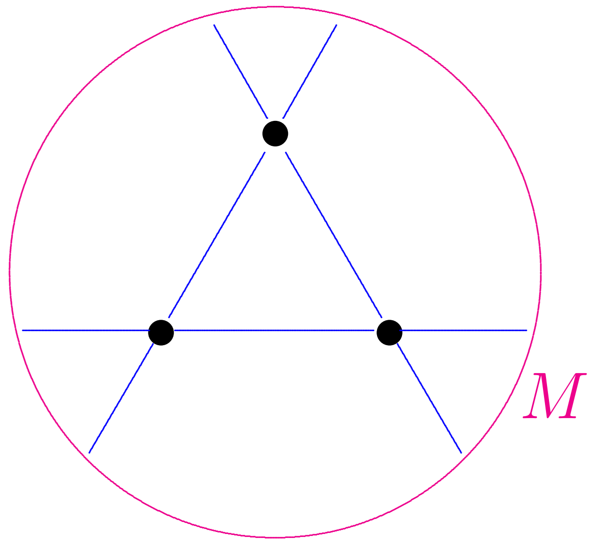
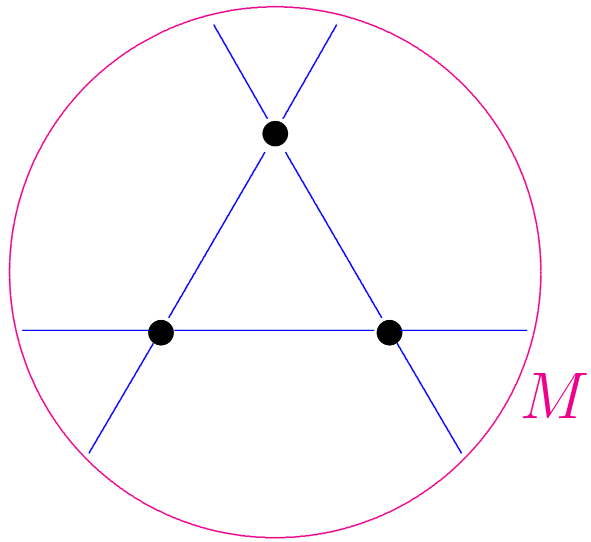

M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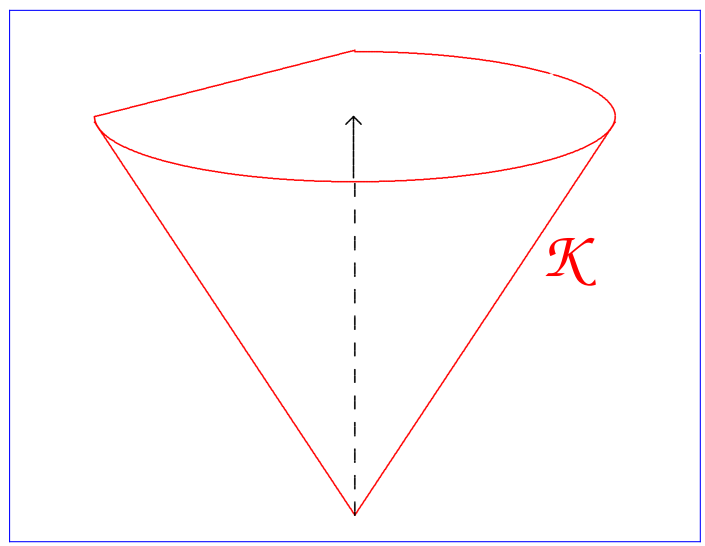
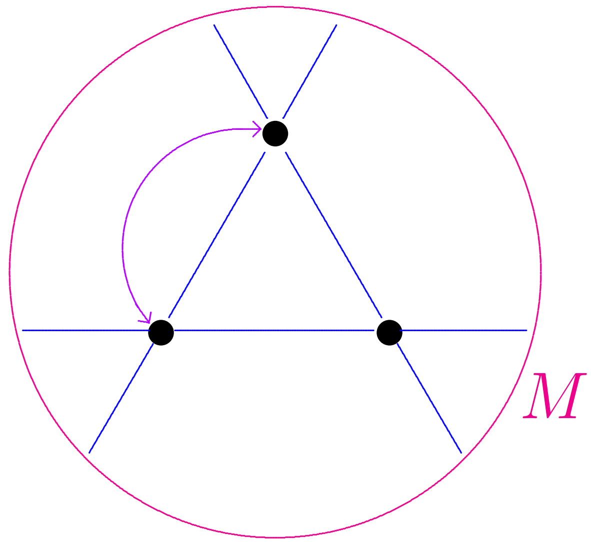
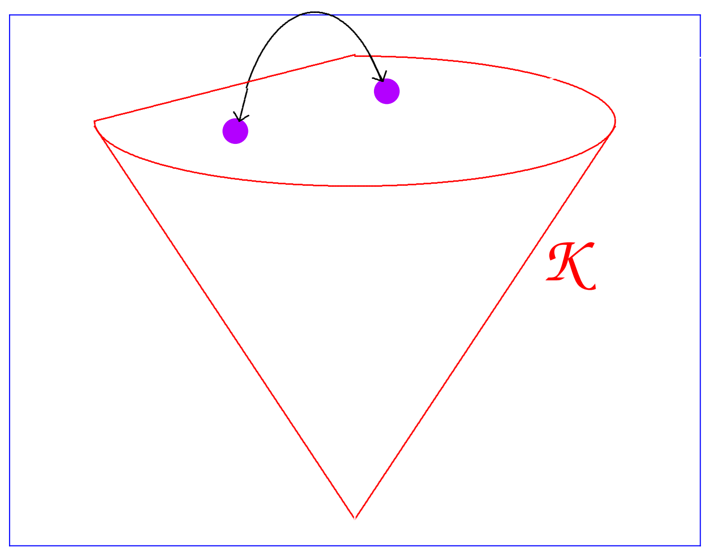

$\mathcal{K}$


M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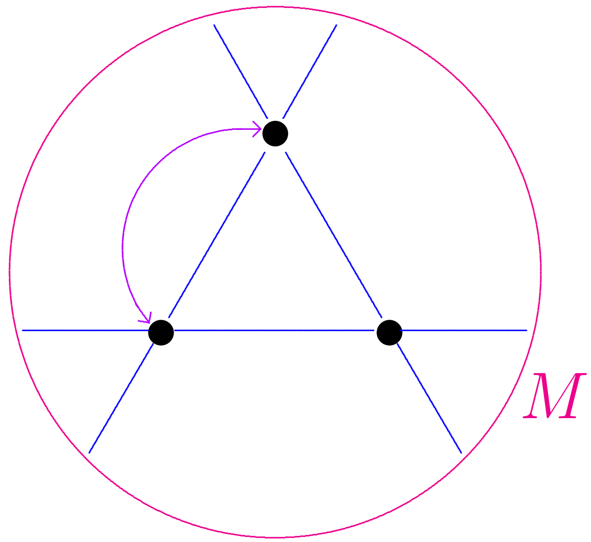
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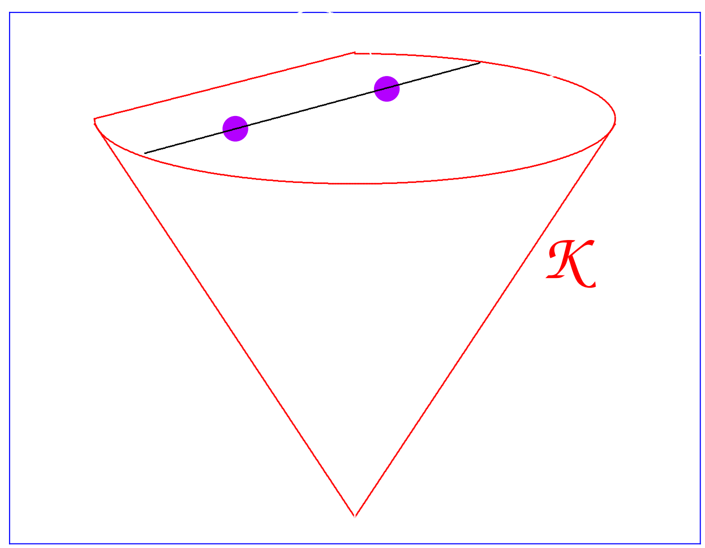
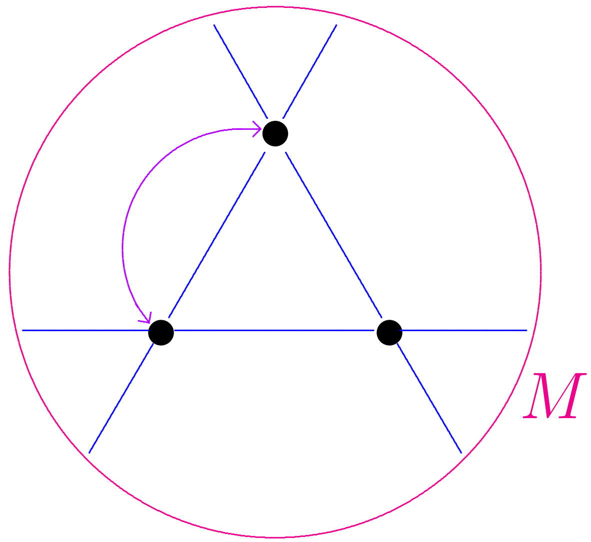
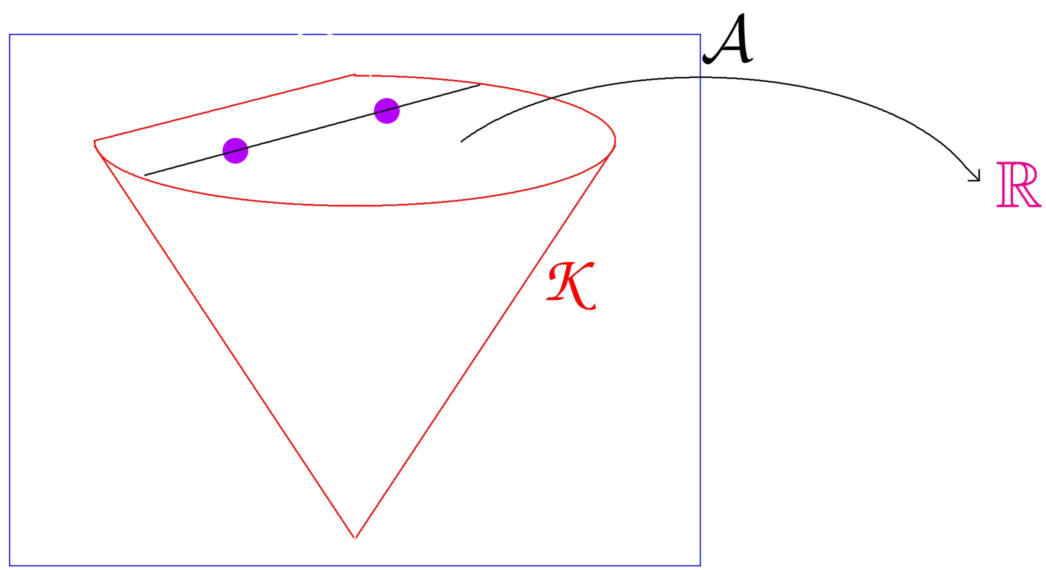

$\mathcal{K}$



M



$\mathcal{A}$
$\mathbb{R}$
$\mathcal{K}$

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Final step then just calculus in one variable...

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Proposition. *Modulo rescalings and biholomorphisms, there is only one conformally Kähler, Einstein metric* h *on* $M = \mathbb{CP}_2 \# 3\overline{\mathbb{CP}}_2$. *This metric is actually Kähler-Einstein, and is exactly the metric discovered by Siu.*

Theorem B. *Let* (M^4, J) *be a compact complex surface, and suppose that* h *is an Einstein metric on* M *which is Hermitian with respect to* J*:*

$$h(J\cdot, J\cdot) = h.$$

Then either

- (M, J, h) *is Kähler-Einstein; or*
- $M \approx \mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$, *and* h *is a constant times the Page metric; or*
- $M \approx \mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$ *and* h *is a constant times the CLW metric.*