On

Four-Dimensional

Einstein Manifolds

Claude LeBrun
SUNY Stony Brook

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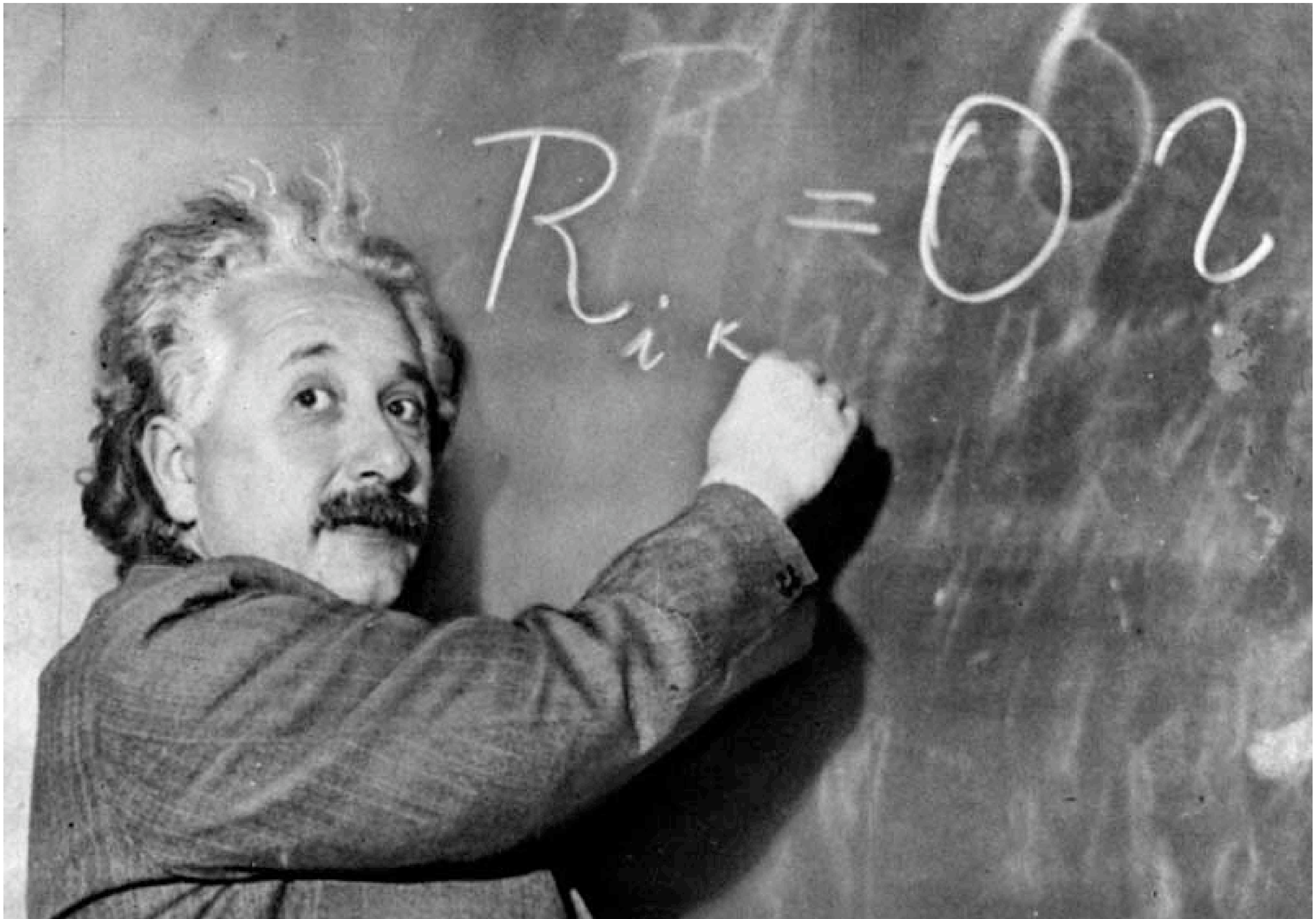

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Ricci curvature measures
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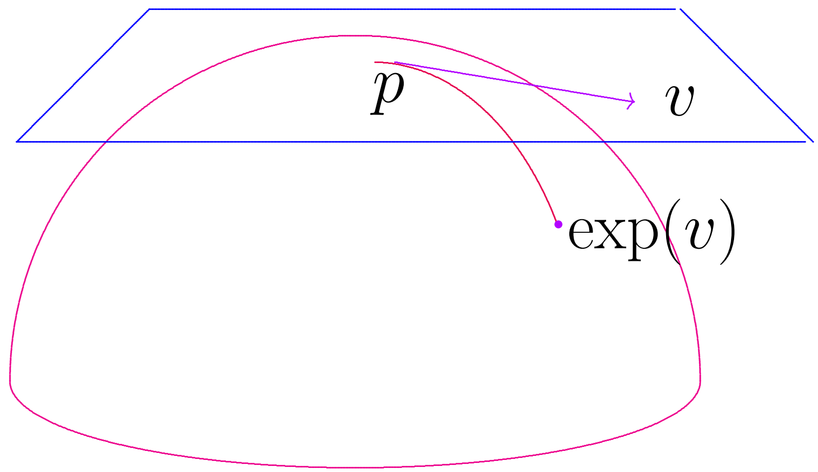

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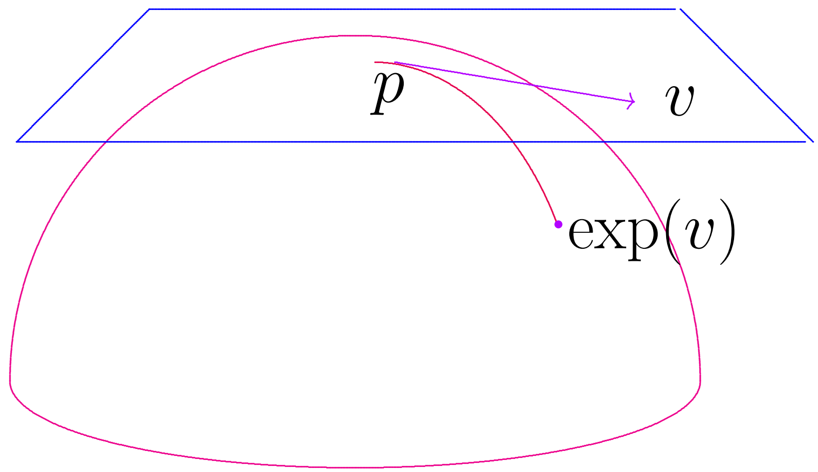

In "geodesic normal coordinates"
metric volume measure is

$$d\mu_g = \left[1 - \frac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\right] d\mu_{\mathrm{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i{}_{jik}$.

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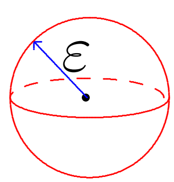
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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Many geometric and differential-topological phenomena have been discovered that are specific to the $n = 4$ case.

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Symplectic Analog. *If M^4 is a smooth compact 4-manifold ectic form ω, when does M^4 admit Einstein metrics?*

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(Warning: In rare circumstances, h could still be Kähler for some $\tilde{J} \neq J$!)

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In $\lambda < 0$ case, corresponding questions still open. Will try to briefly indicate what's currently known.

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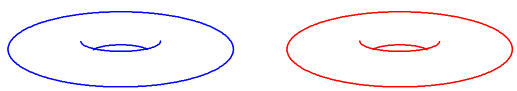

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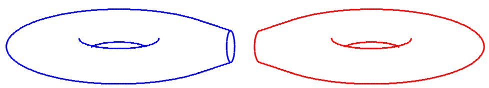

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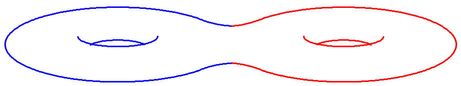

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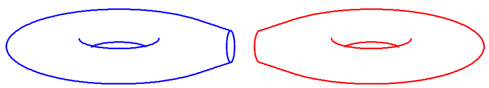

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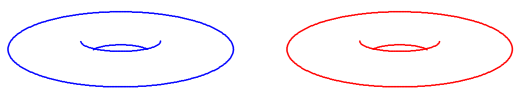

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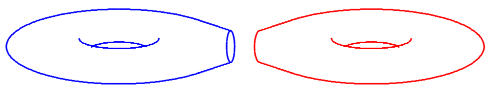

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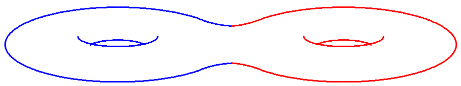

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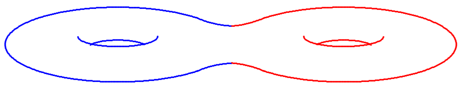

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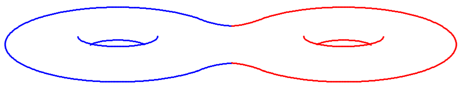

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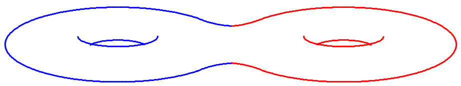

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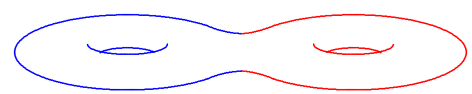

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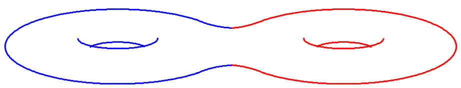

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in which new $\mathbb{CP}_1$ has self-intersection -1.

Theorem. *Suppose that* M *is a smooth compact oriented* 4*-manifold which admits a complex structure* J*. Then* M *also admits an (unrelated) Einstein metric* g *with* $\lambda > 0$

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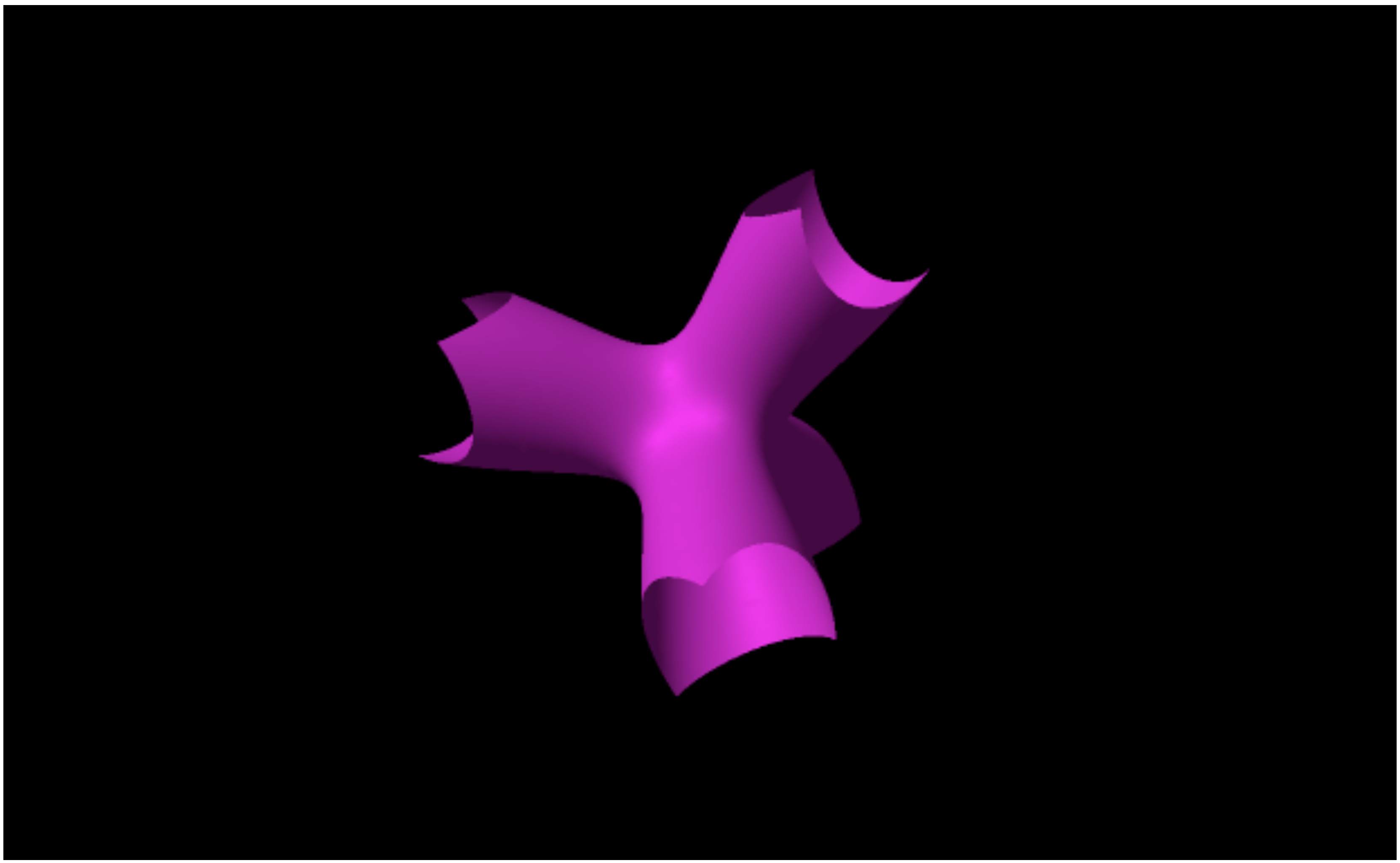

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Differentiable model for relevant $\mathbb{Z}_2$-action:

$$(t, u, v, w) \mapsto \overline{(t, u, v, w)}$$

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In cases other than Del Pezzo surfaces:
also know moduli space of all Einstein metrics.

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We begin with existence.

Einstein metrics which are Kähler

Kähler-Einstein metrics

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Of course, $\mathbb{CP}_2$ and $S^2 \times S^2$ also admit K-E metrics with $\lambda > 0$ — namely, obvious homogeneous ones!

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Find Kähler metric which minimizes

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Einstein metric is $g = s^{-2}h$.

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We've discussed existence of Einstein metrics.
Will now discuss obstructions to Einstein metrics.

Special character of dimension 4:

On oriented (M^4, g),

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

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$$\star : \Lambda^2 \to \Lambda^2,$$

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Λ^+ self-dual 2-forms.
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(M, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$

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for signature $\tau(M) = b_+(M) - b_-(M)$.

Hitchin-Thorpe Inequality:

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Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

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with equality only if g-induced connection on $\Lambda^+ \to M$ is flat.

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Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

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Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

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In the ${c_1}^2(M) > 0$ case, there is then a well-defined Seiberg-Witten invariant of M, for the spin^c structure induced by J or ω.

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generalized Kähler geometry of non-Kähler 4-manifolds.

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But $\bar{\partial} + \bar{\partial}^*$ does generalize:

spin^c Dirac operator, preferred connection on L.

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Every unitary connection A on L induces spin^c Dirac operator

$$D_A : \Gamma(\mathbb{V}_+) \to \Gamma(\mathbb{V}_-)$$

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Non-linear, but elliptic once ‘gauge-fixing’

$$d^*(A - A_0) = 0$$

imposed to eliminate automorphisms of $L \longrightarrow M$.

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Del Pezzo by Taubes, Gromov, McDuff, Liu.

Theorem. *Suppose that M is a smooth compact oriented 4-manifold which admits either a complex structure or a symplectic structure. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if M is diffeomorphic to*

- *a Del Pezzo surface,*
- *a K3 surface,*
- *an Enriques surface,*
- *an Abelian surface, or*
- *a hyper-elliptic surface.*

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Compact complex surface (M^4, J) general type if

$$\dim \Gamma(M, \mathcal{O}(K^{\otimes \ell})) \sim a\ell^2, \qquad \ell \gg 0,$$

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Theorem (L '01). *Let* X *be a minimal surface of general type, and let*

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(Better than Hitchin-Thorpe by a factor of 3.)

So being "very" non-minimal is an obstruction.

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If so, quite different from Kähler-Einstein metrics!