Mass, Scalar Curvature,

Kähler Geometry, and All That

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Stony Brook University

Colloquium de Mathématiques
Institut Fourier
Université de Grenoble, 2 mai, 2019

Core results joint with

Core results joint with

Hans-Joachim Hein
Fordham University

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Mass in Kähler Geometry
Comm. Math. Phys. 347 (2016) 621–653.

Recent technical improvements:

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Mass, Kähler Manifolds,
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arXiv: 1810.11417 [math.DG]

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Ann. Glob. An. Geom. *to appear*
doi: 10.1007/s10455-019-09658-9

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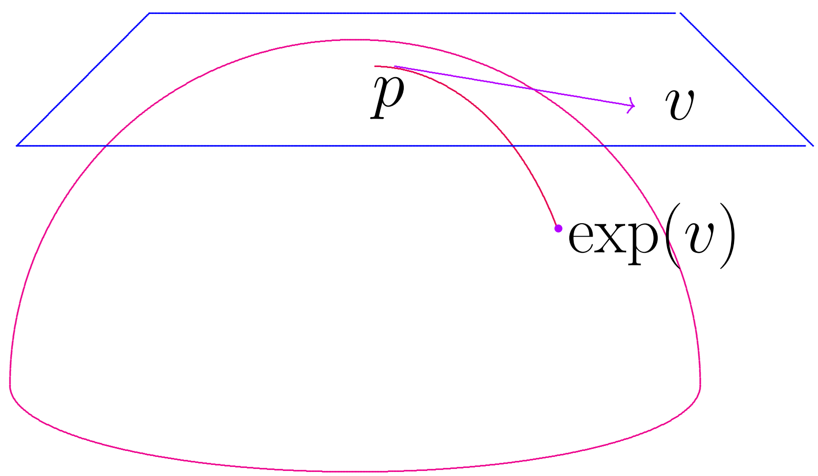

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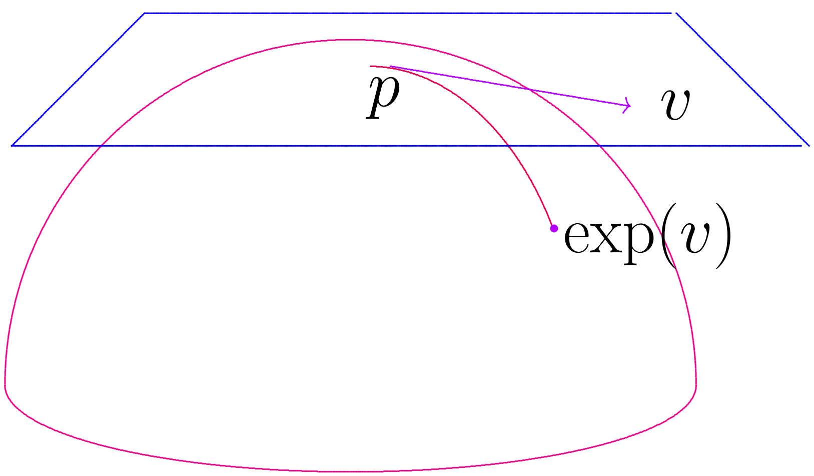

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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Components like $\mathcal{R}_{1212}$ are "sectional curvatures"...

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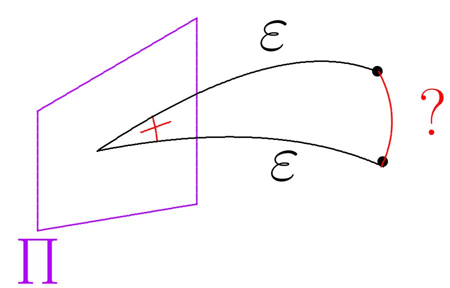

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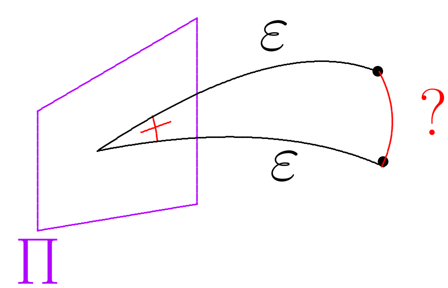

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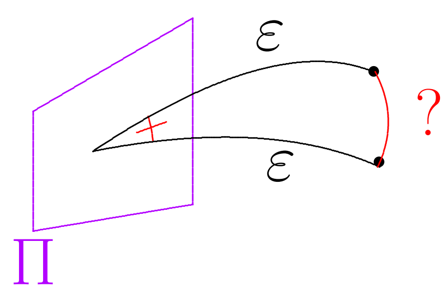

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$\Longleftrightarrow g_{jk} = \delta_{jk}$ in suitable coordinates.

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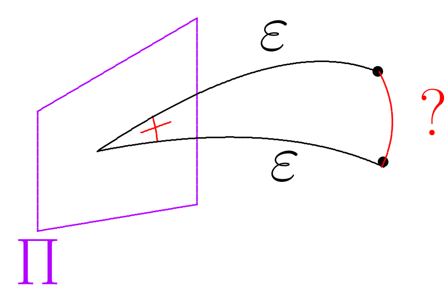

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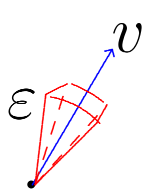

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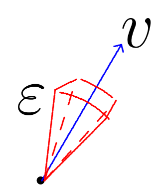

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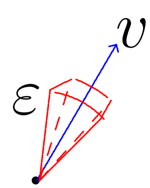

The metric g is called Ricci-flat if it satisfies $r \equiv 0$.

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But this has a simple geometric interpretation. . .

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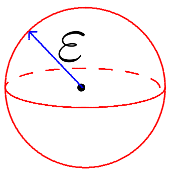

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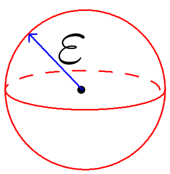

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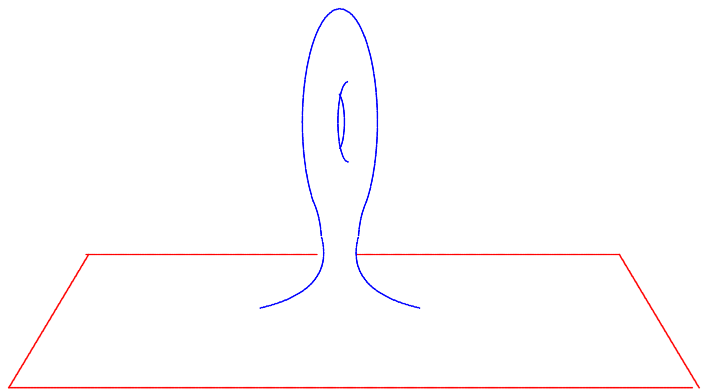

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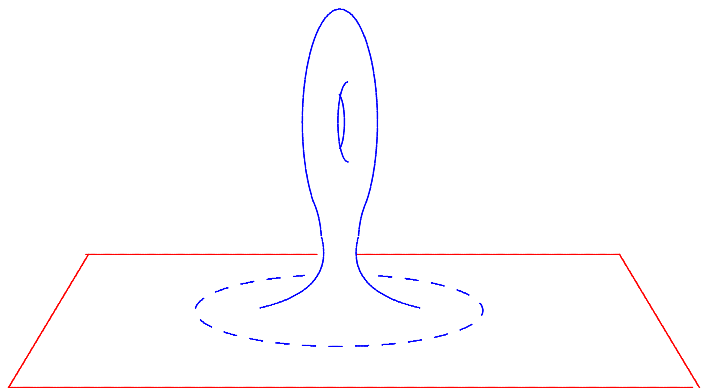
Suppose $\exists K \subset M$ compact



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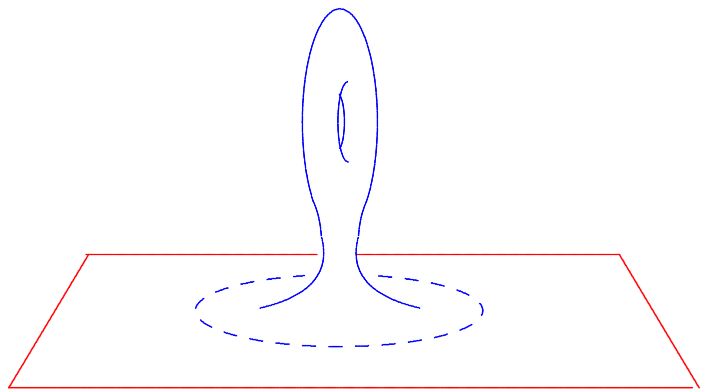

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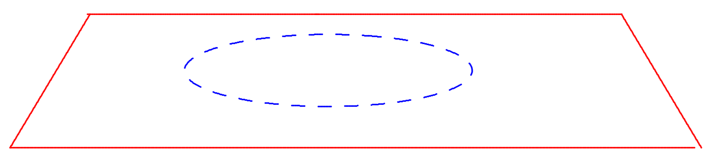

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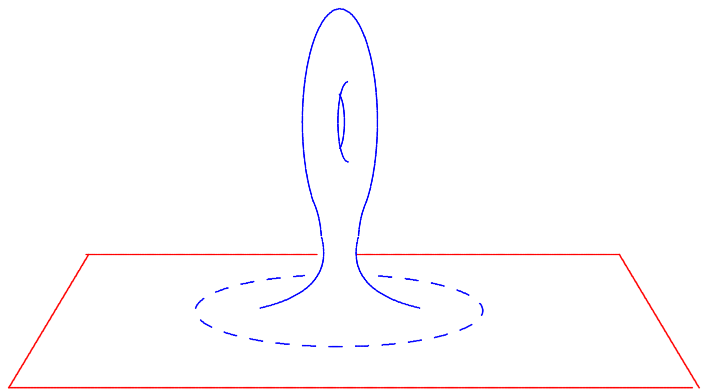

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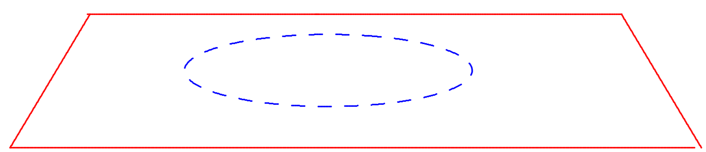

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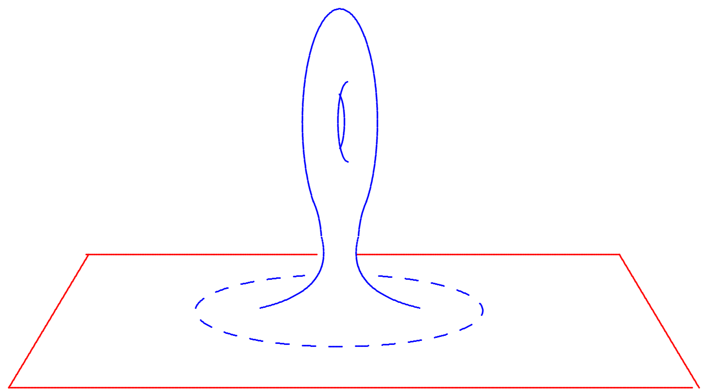

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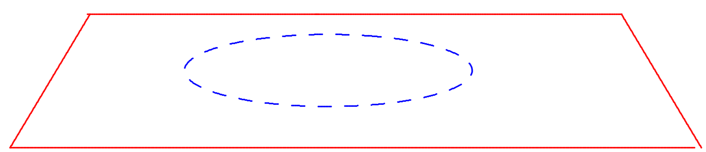

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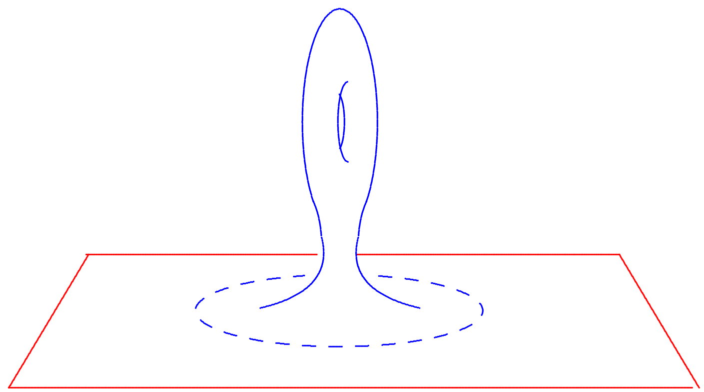

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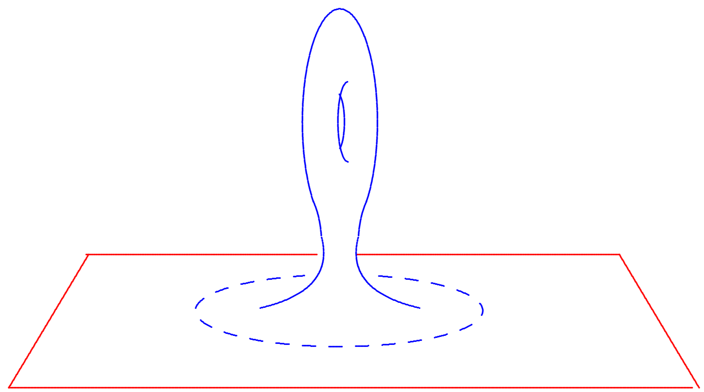

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If M has curvature ≥ 0, is it flat?

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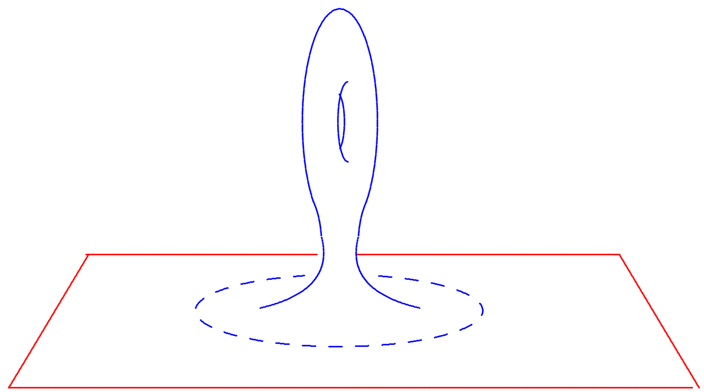

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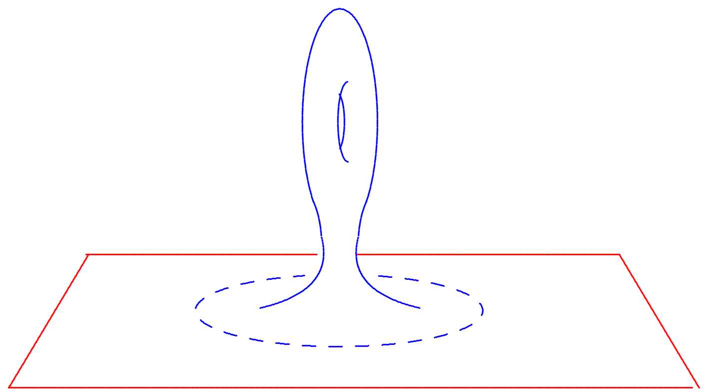

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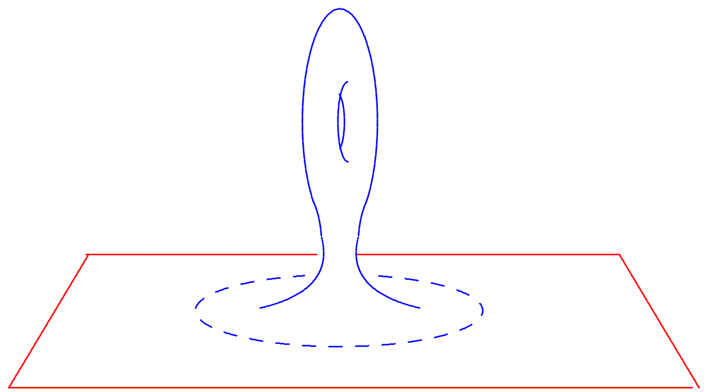

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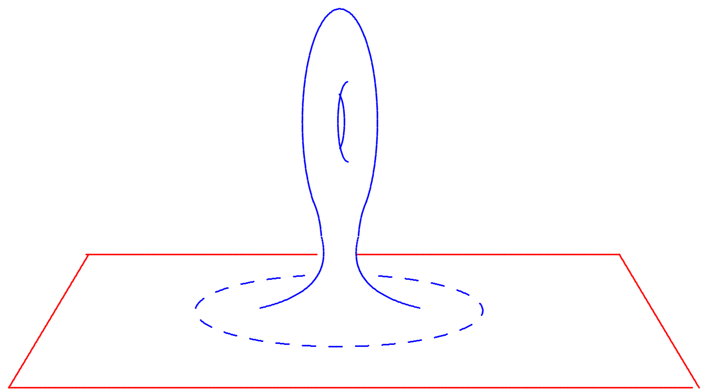

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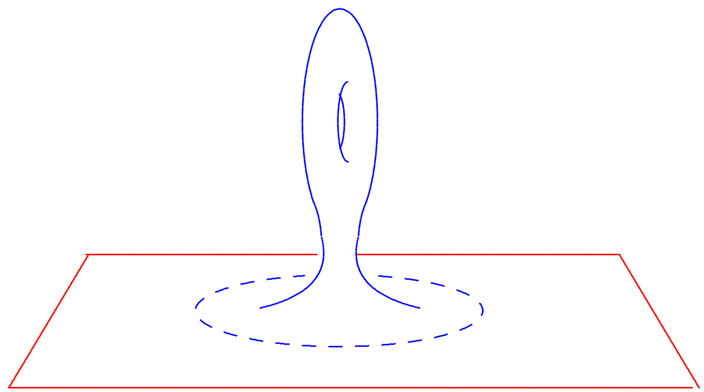

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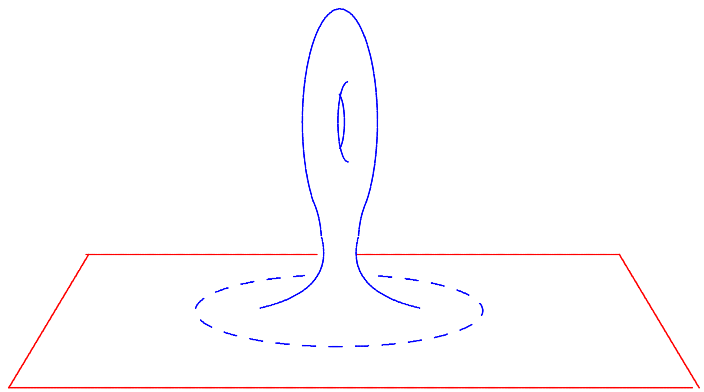

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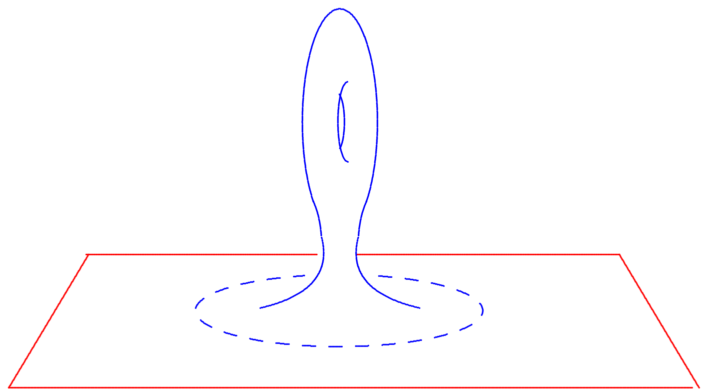

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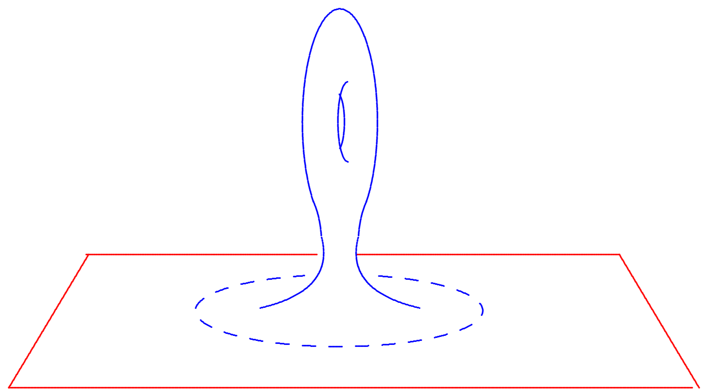

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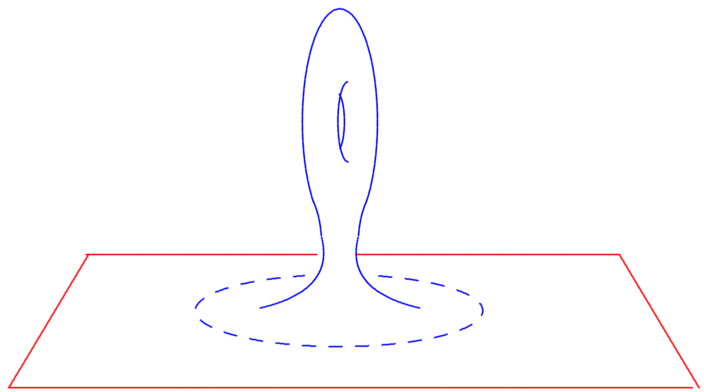

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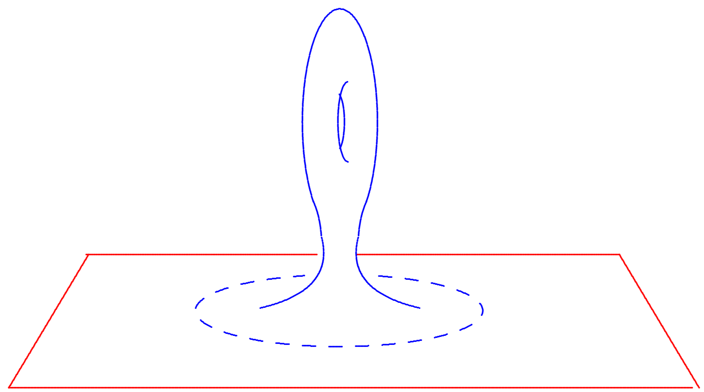

and an isometry $M - K \to \mathbb{R}^n - D^n$. (Euclidean)

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Much harder!

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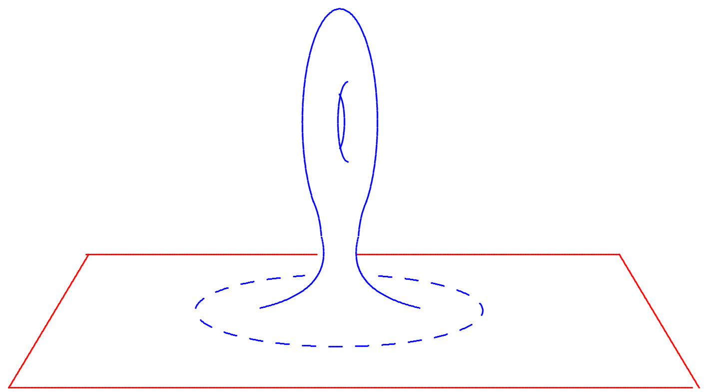

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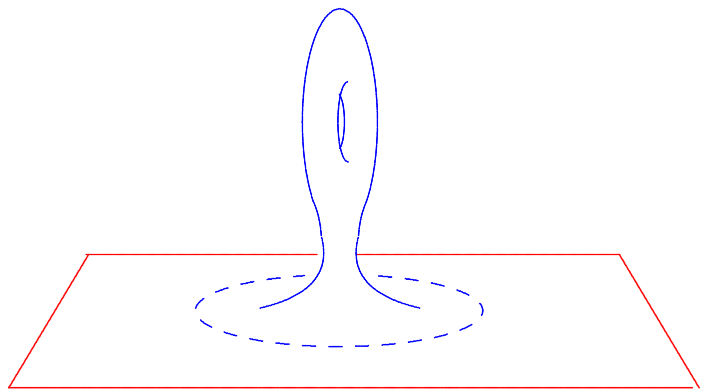

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This time, the inspiration comes from physics!

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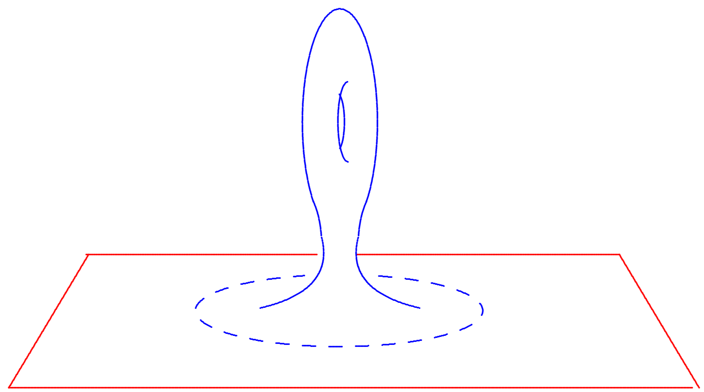

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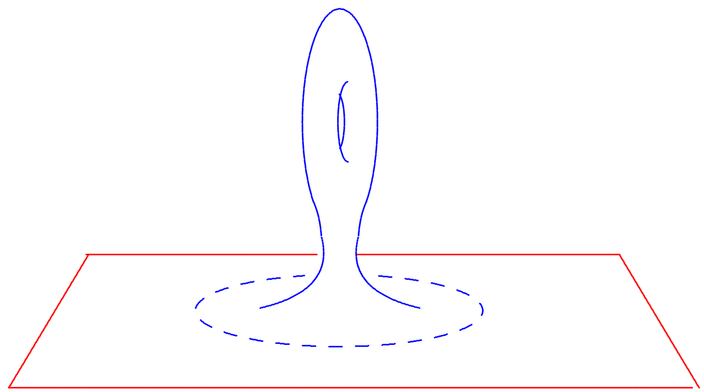

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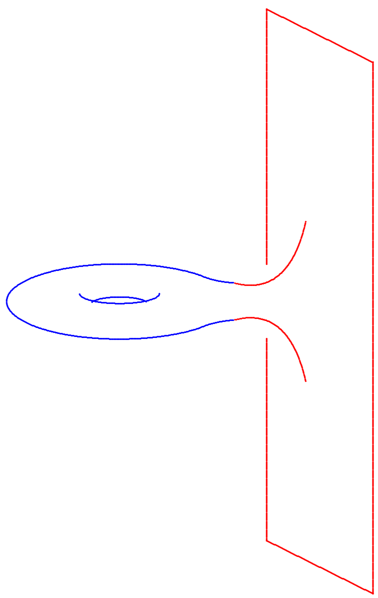
"Positive Mass Theorem"

Get result even with appropriate fall-off to Euclidean. . .

Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g)



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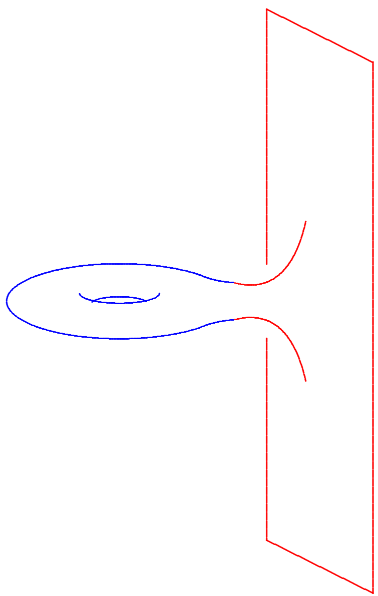

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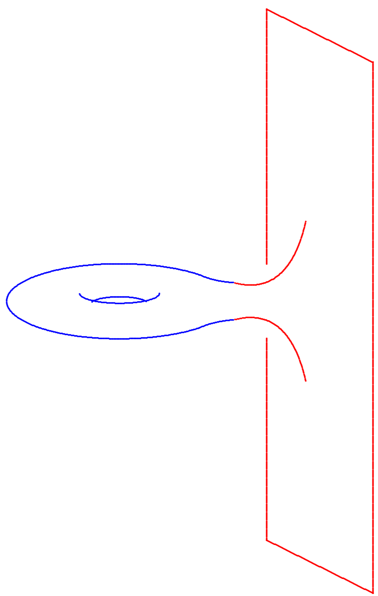

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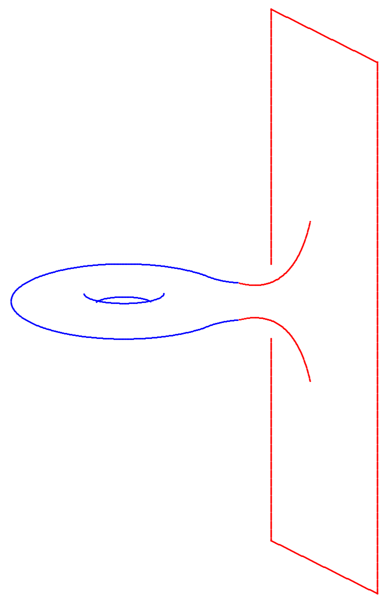

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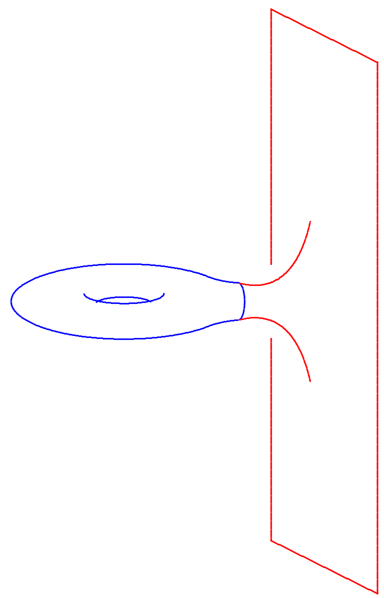

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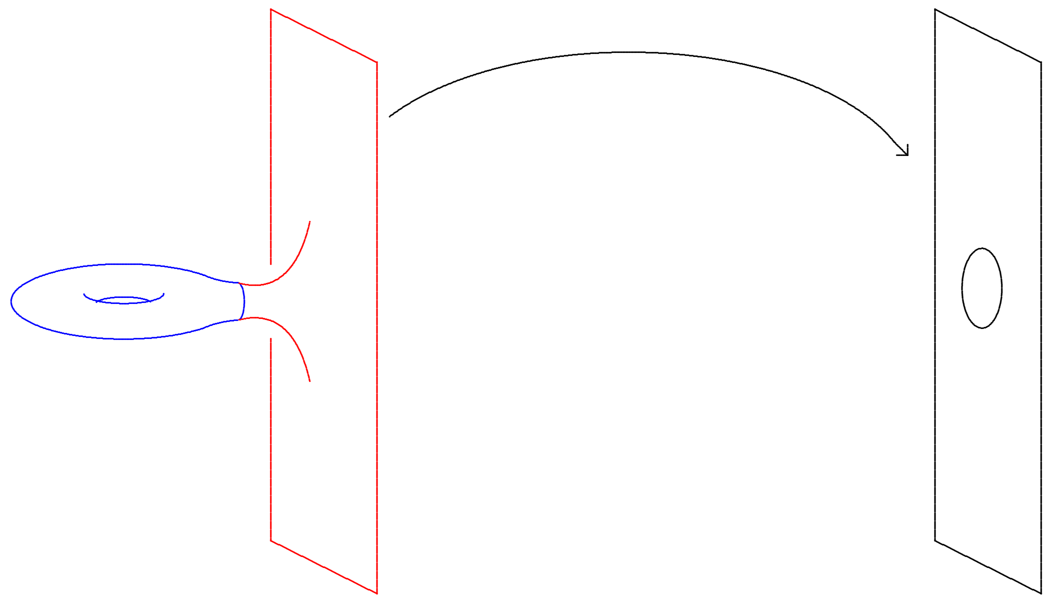

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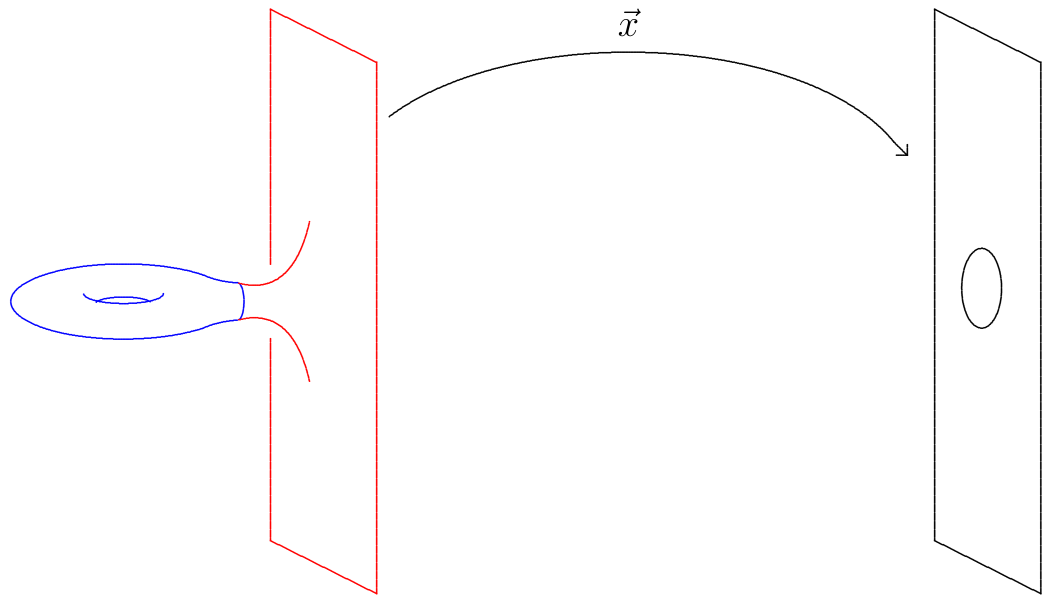

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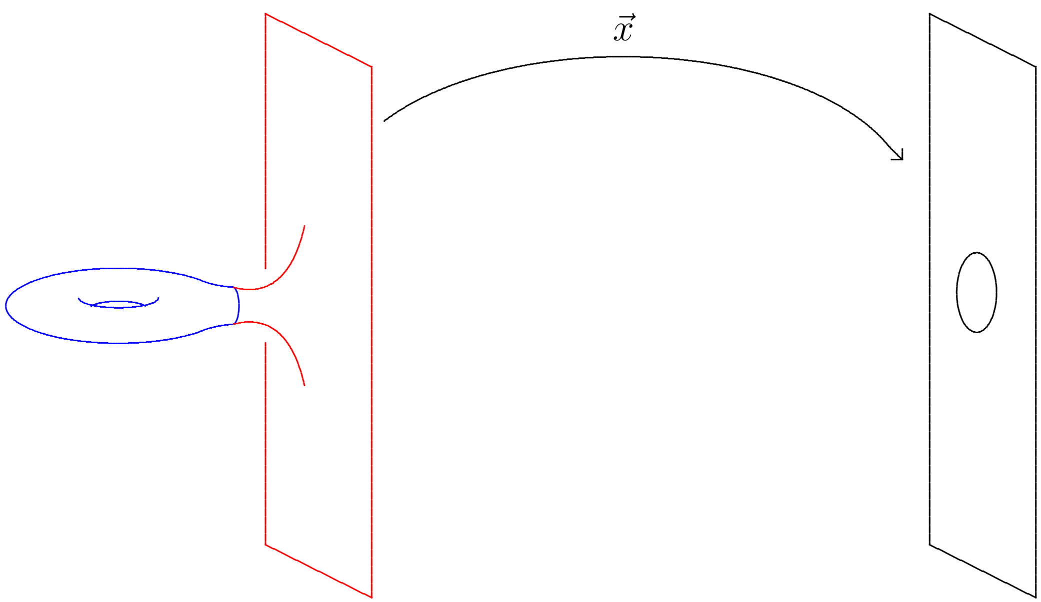

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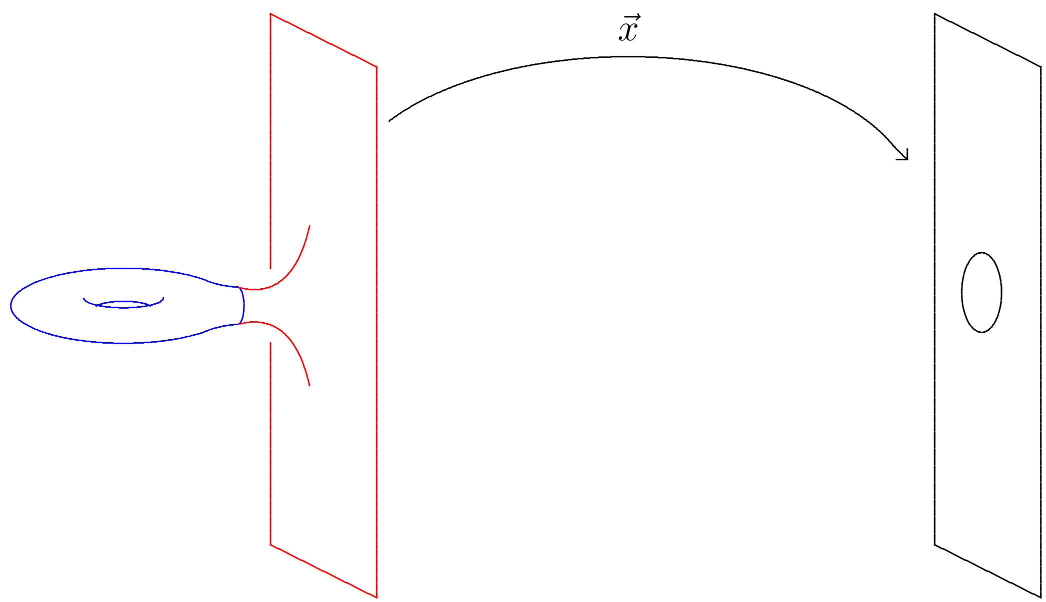

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Chruściel-type fall-off:

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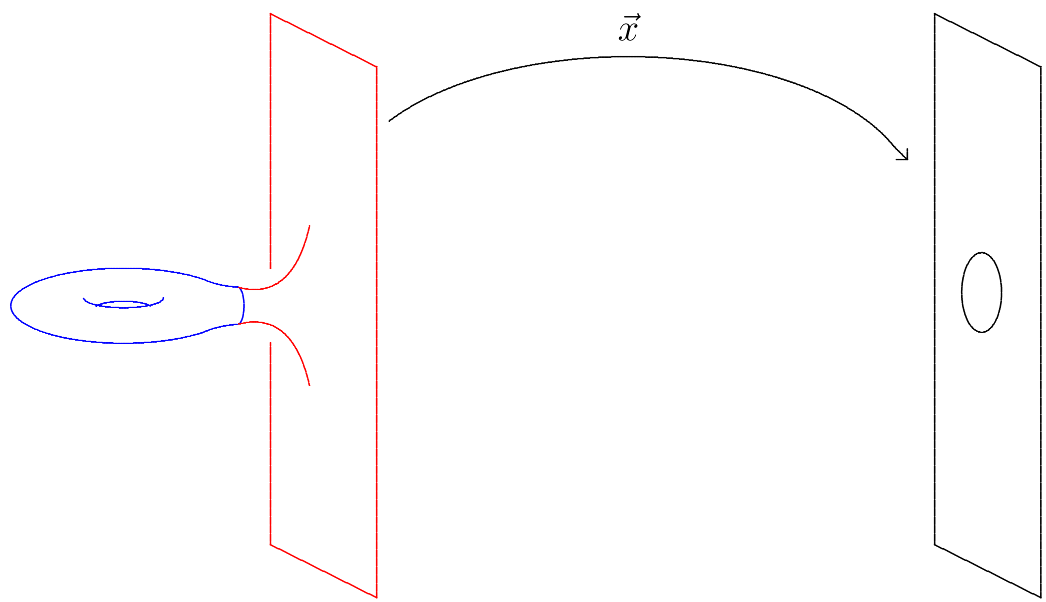

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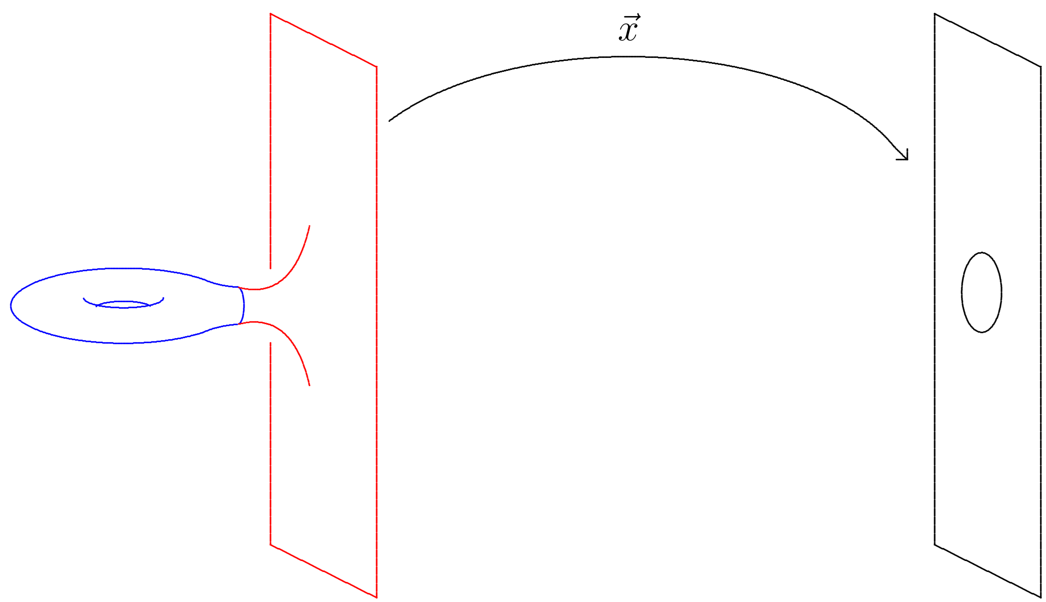

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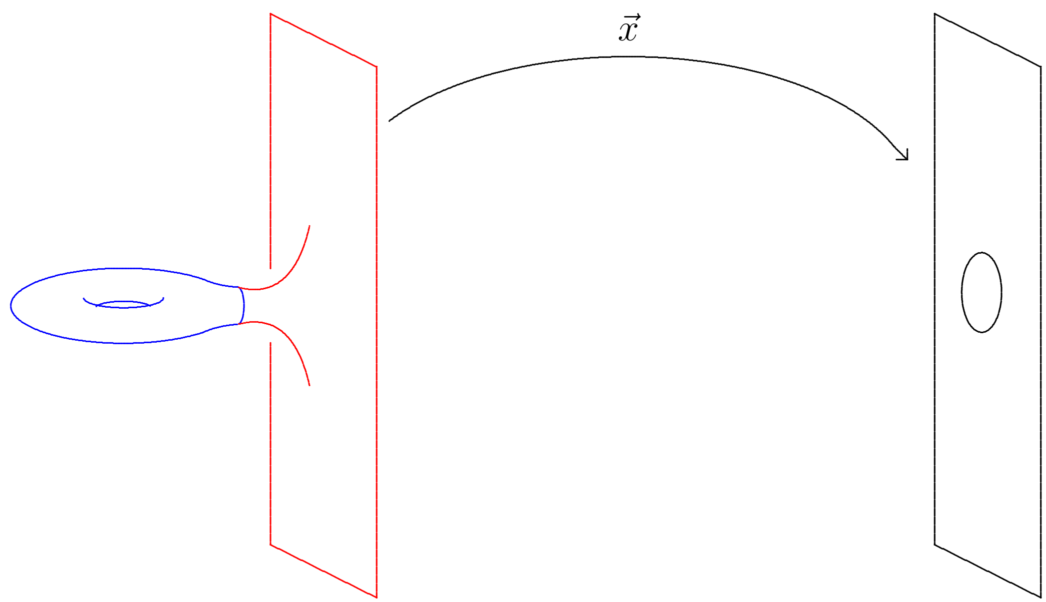

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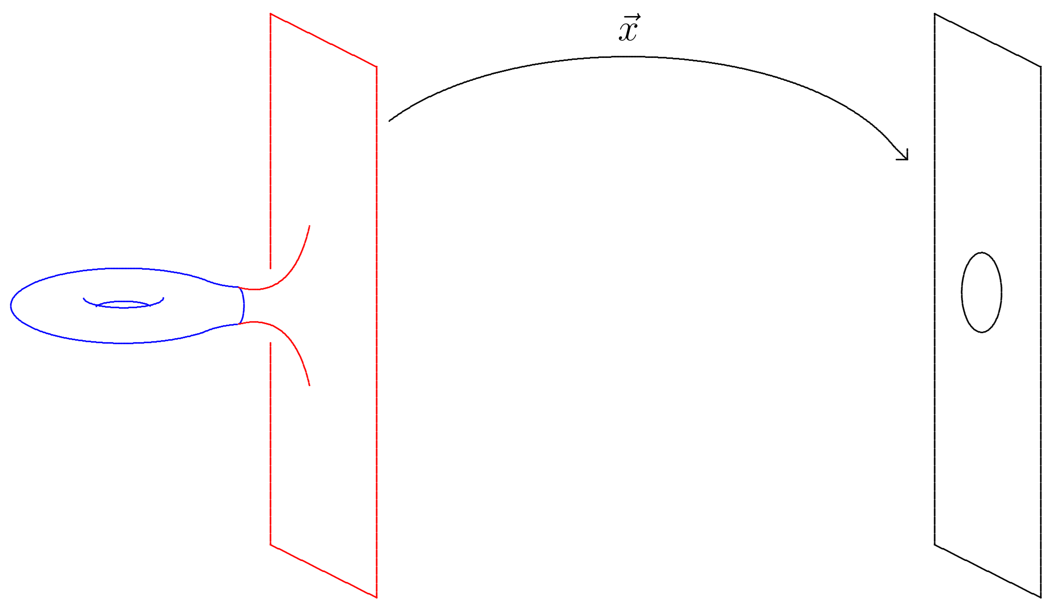

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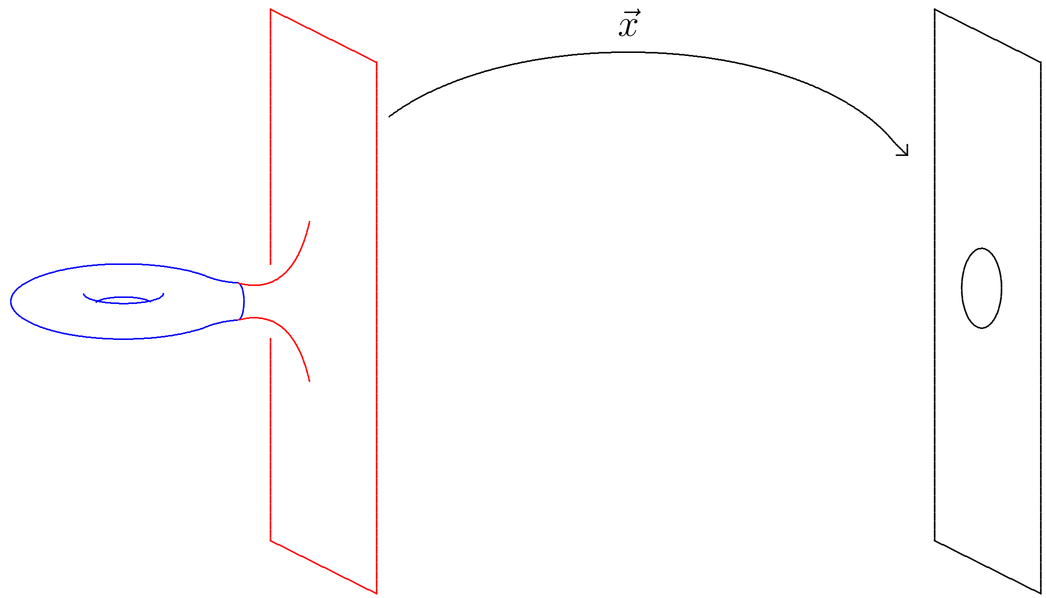

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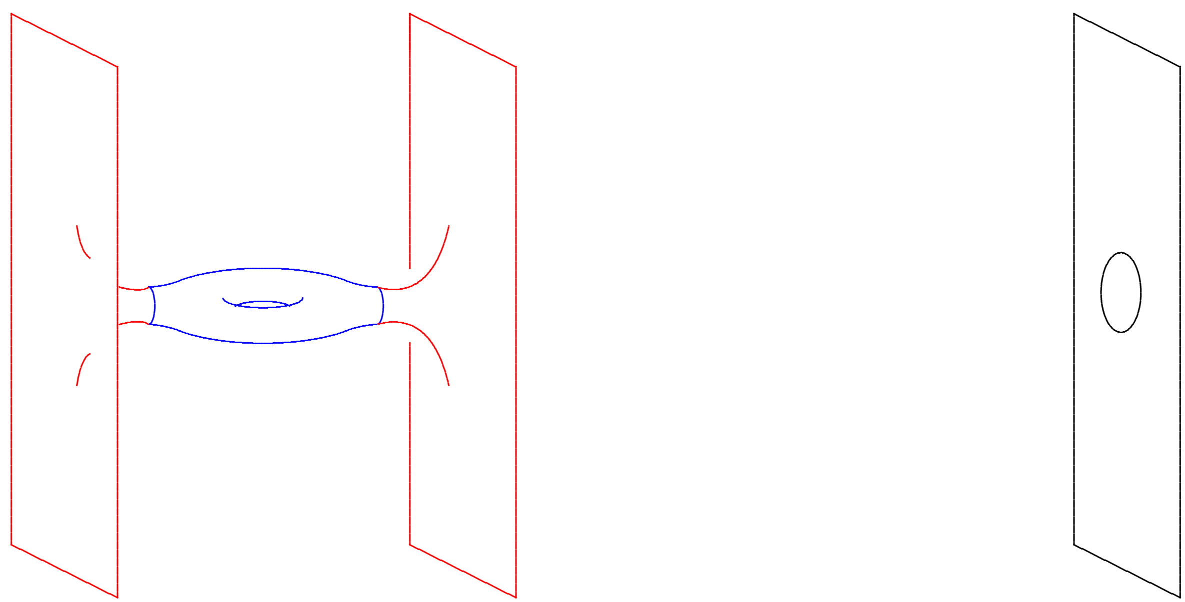

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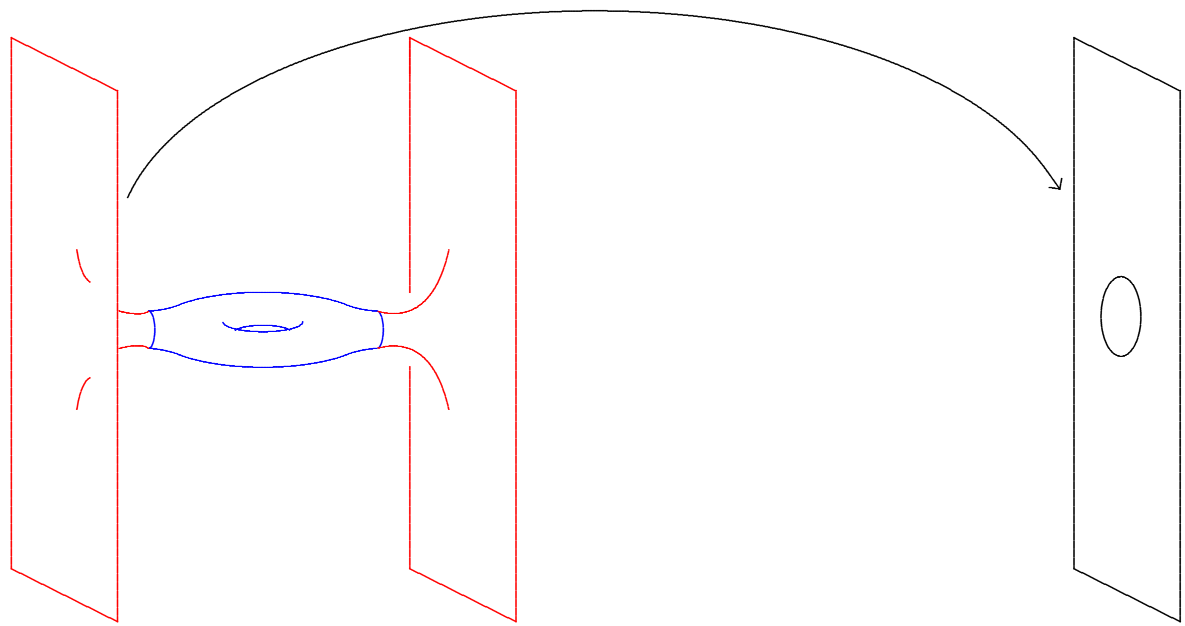

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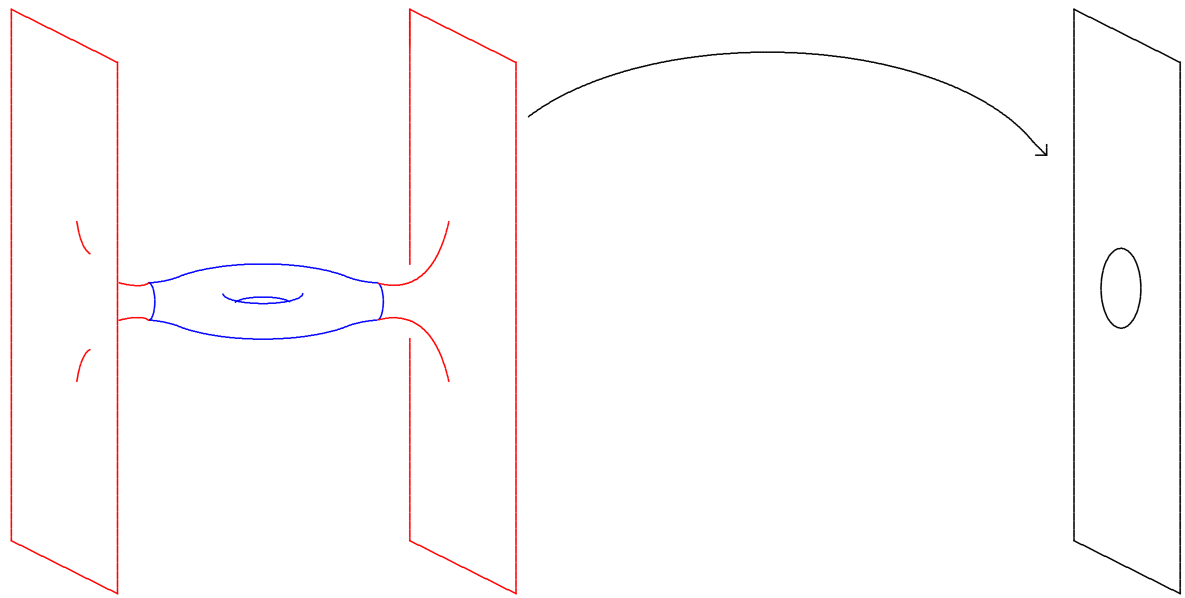

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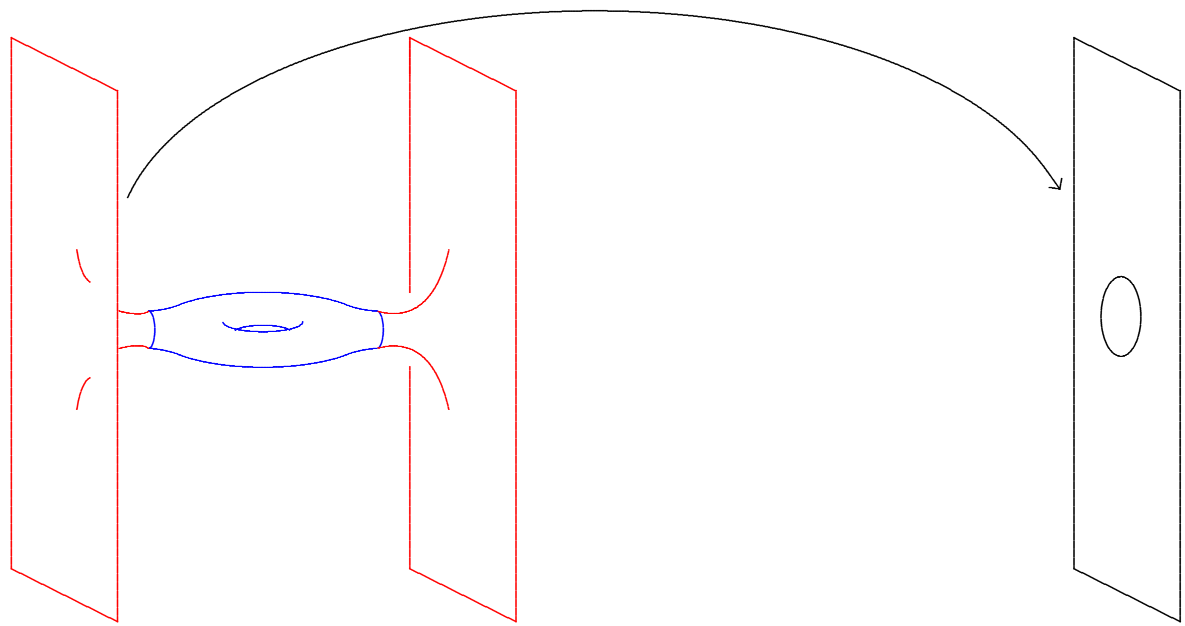

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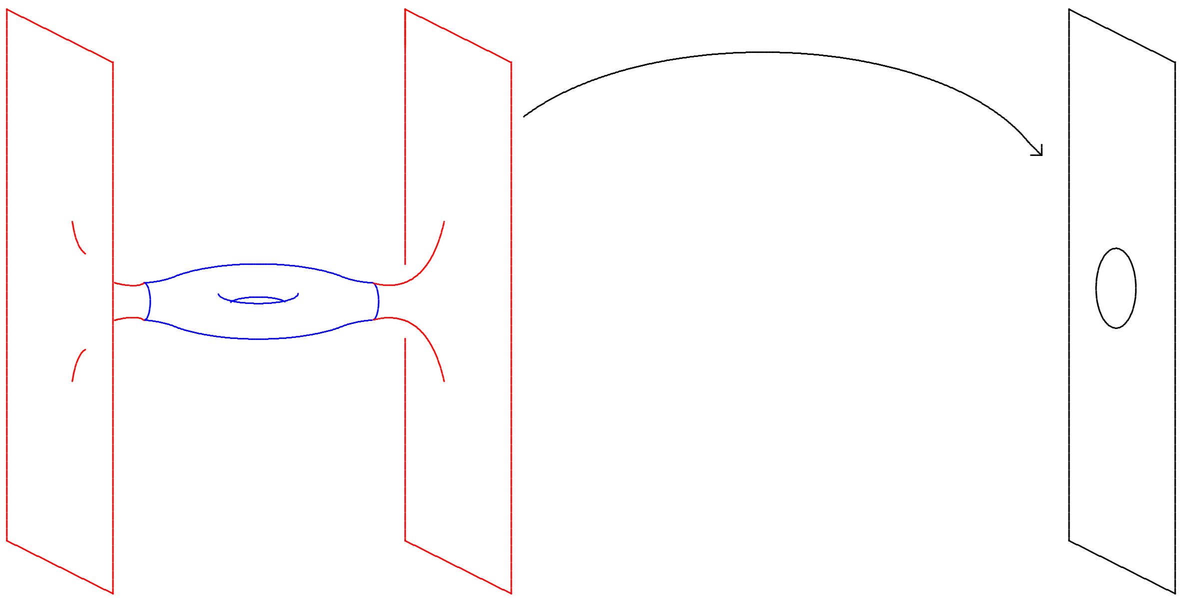

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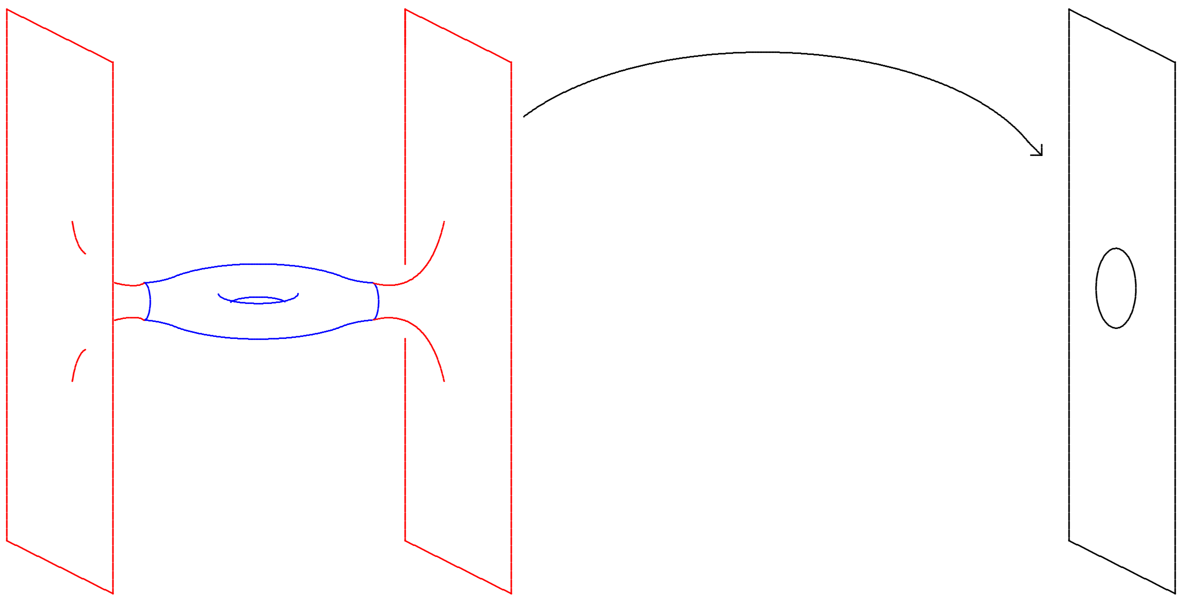

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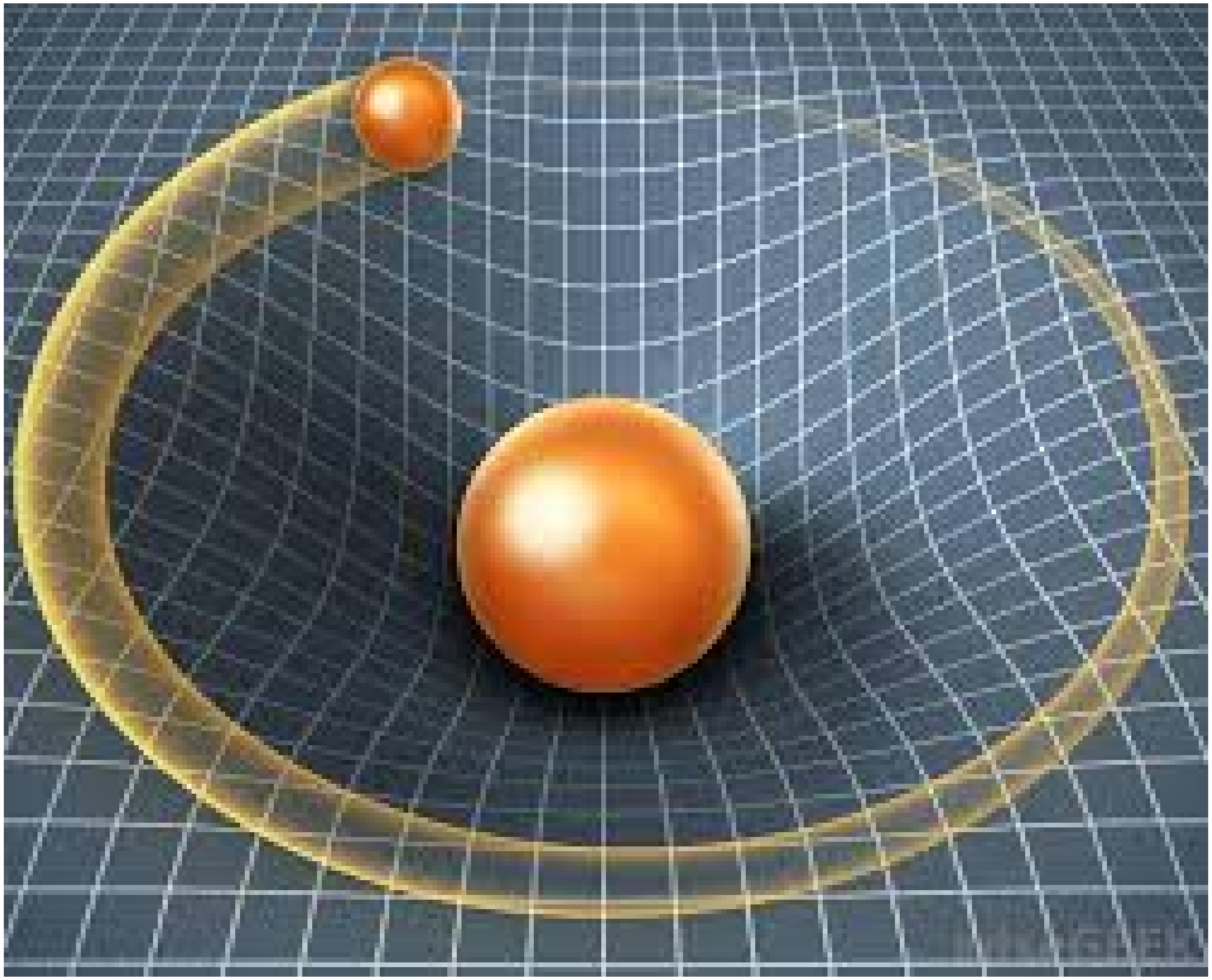

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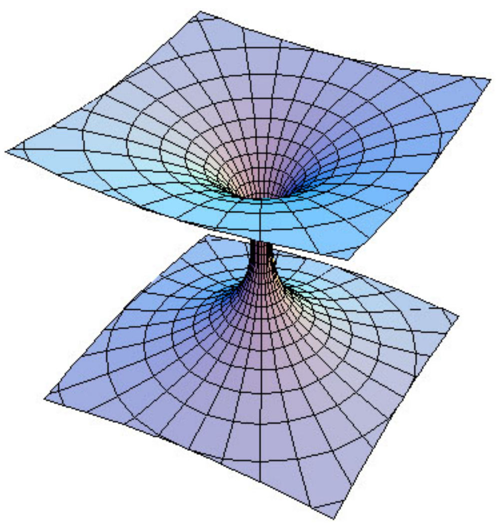
Reads off "apparent mass" from strength of the gravitational field far from an isolated source.

In any dimension, reproduces "mass" of $t = 0$ hypersurface in $(n+1)$-dimensional Schwarzschild

$$g = \left(1 - \frac{2m}{\varrho^{n-2}}\right)^{-1} d\varrho^2 + \varrho^2 h_{S^{n-1}}$$

Two such regions fit together to form the wormhole metric.



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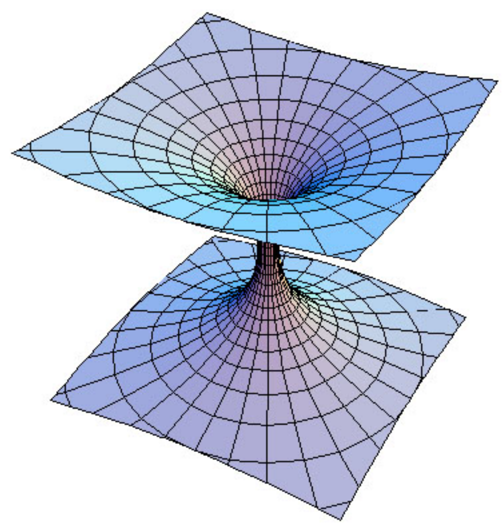

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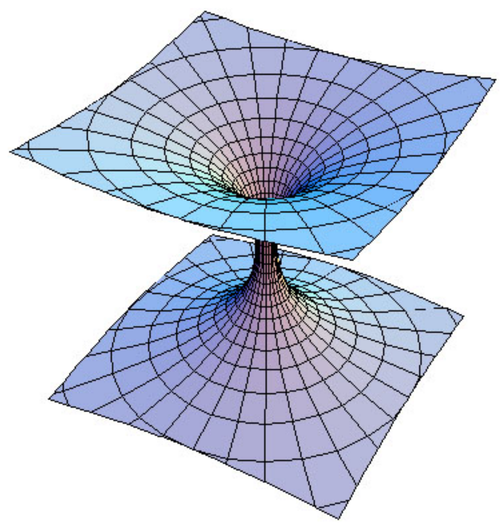

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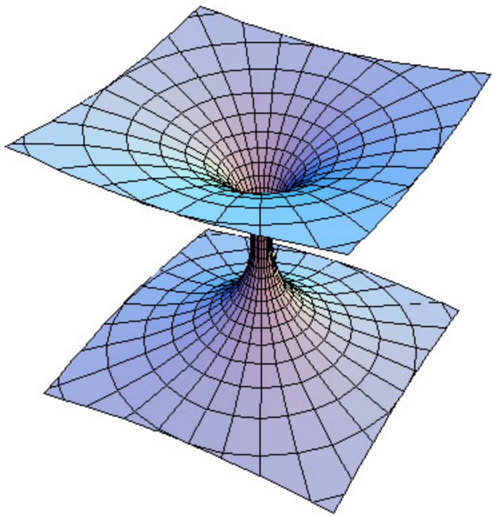

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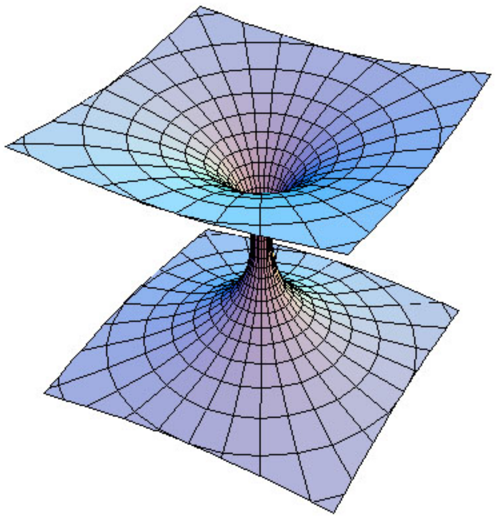

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$$g = \left(1 + \frac{m/2}{r^{n-2}}\right)^{4/(n-2)} \left[\sum (dx^j)^2\right]$$

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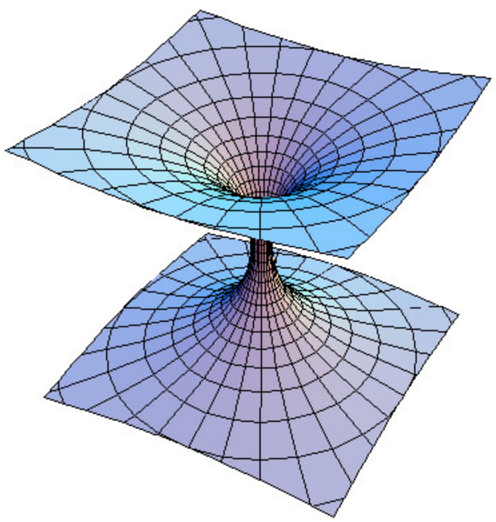

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$$g_{jk} = \left(1 + \frac{2m}{(n-2)r^{n-2}}\right)\delta_{jk} + \cdots$$

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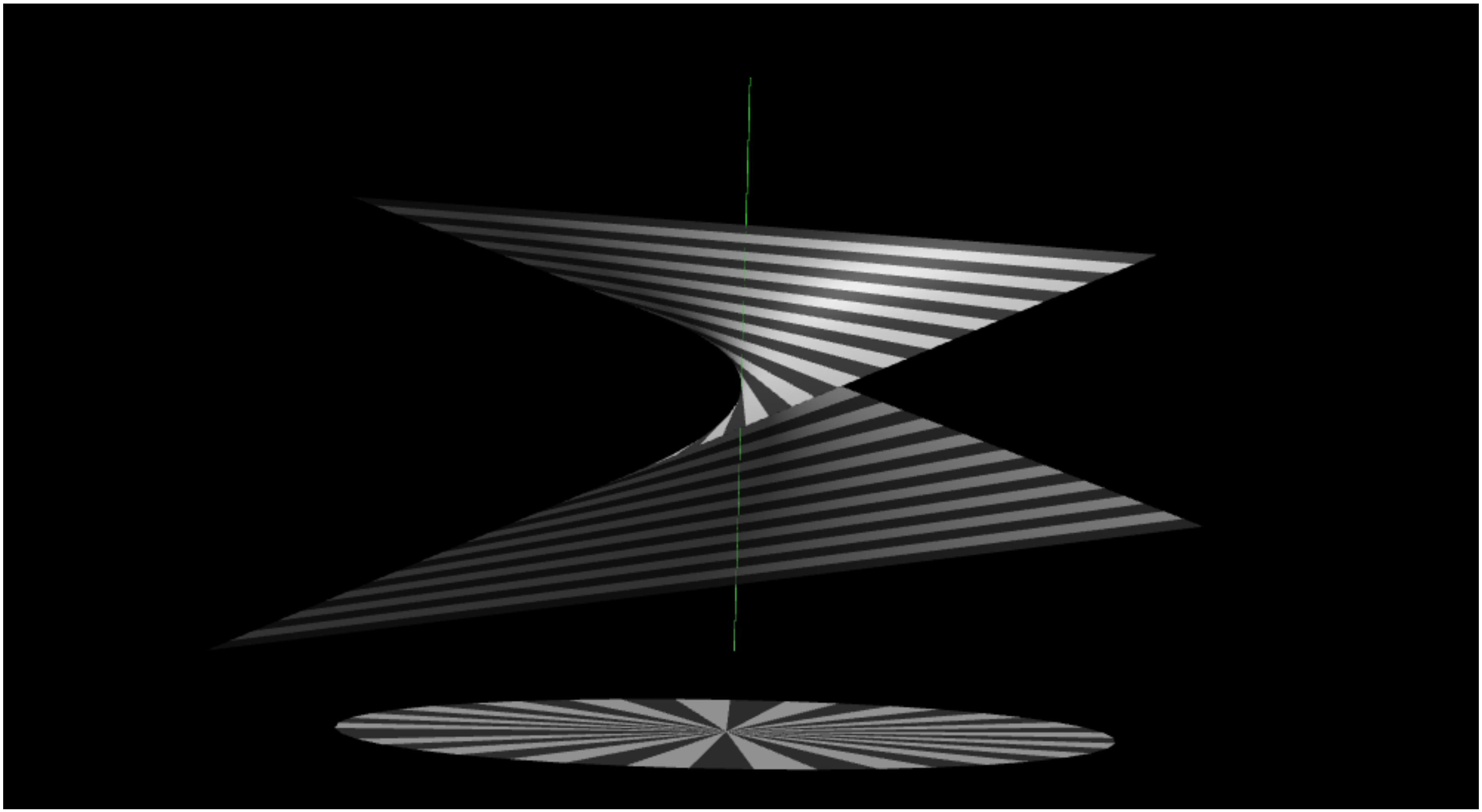

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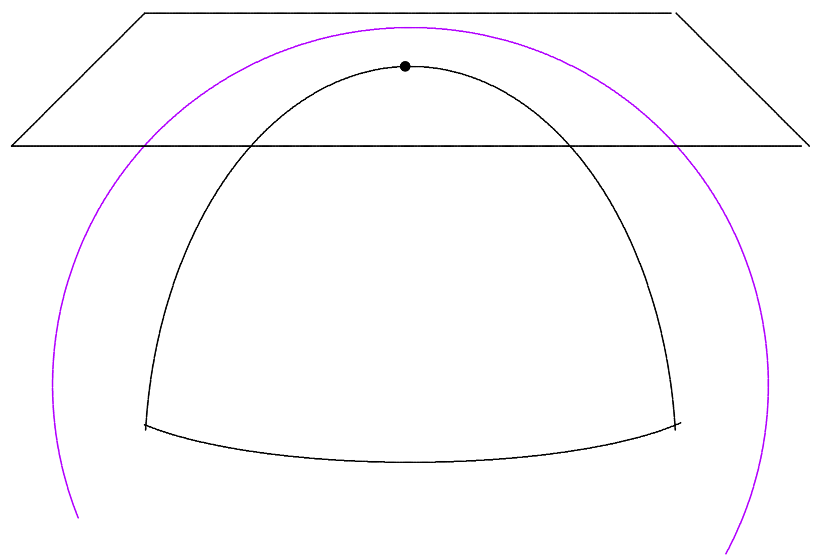

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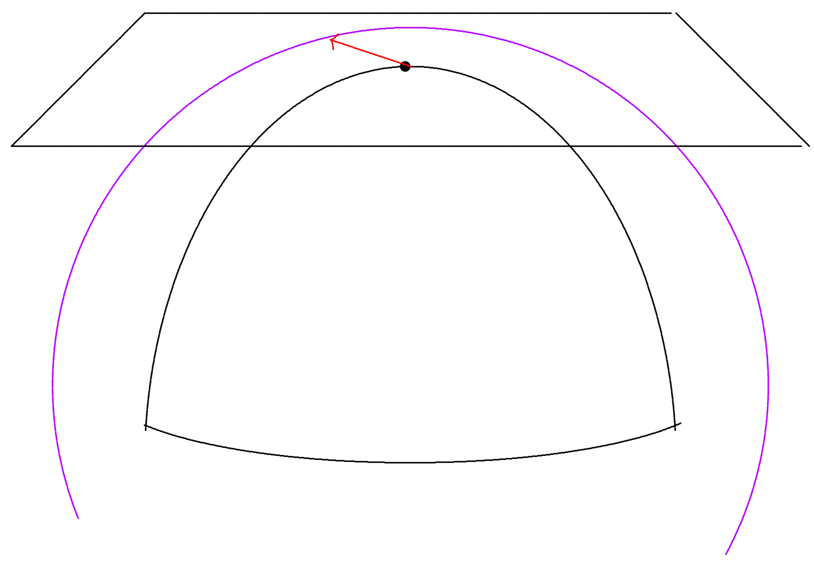

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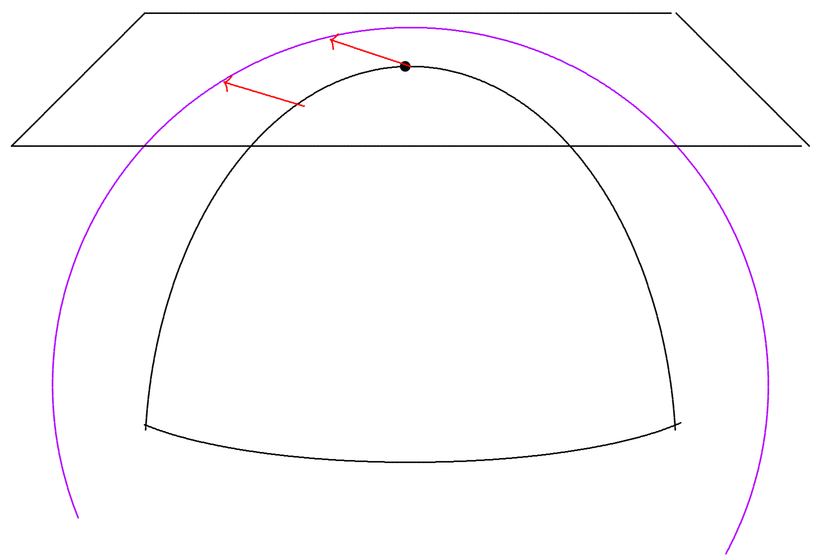

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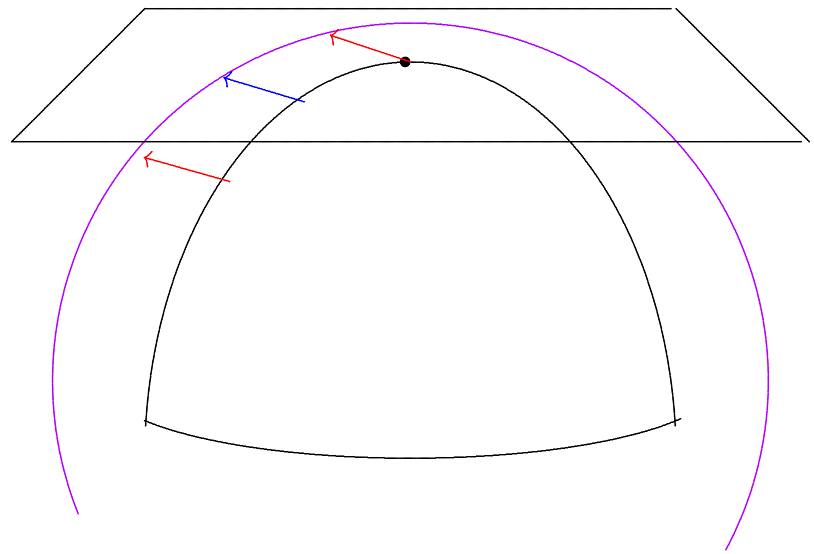

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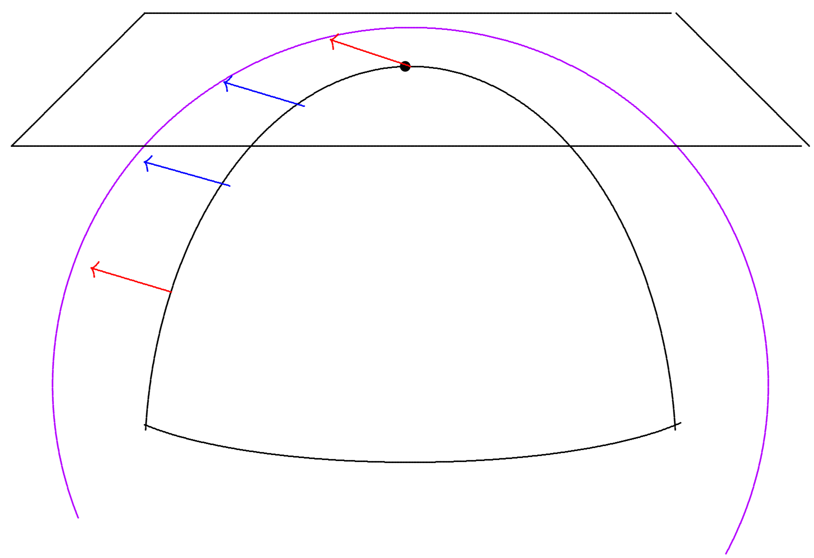

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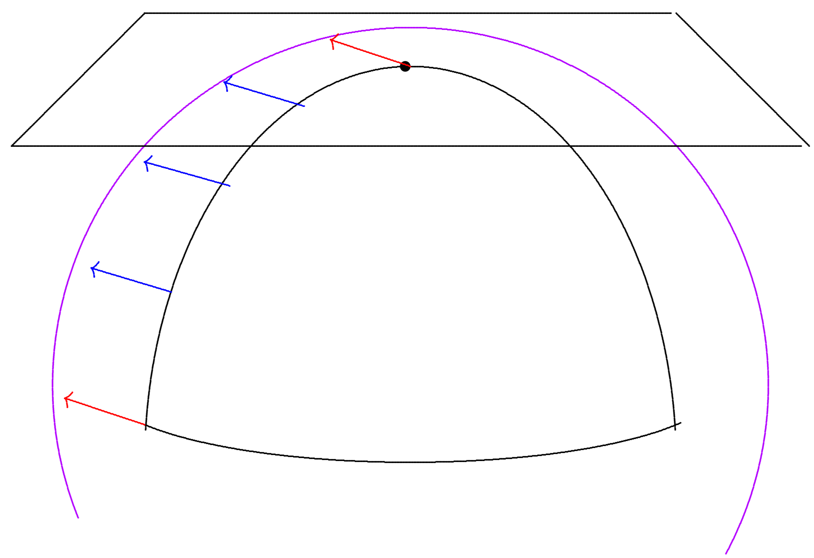

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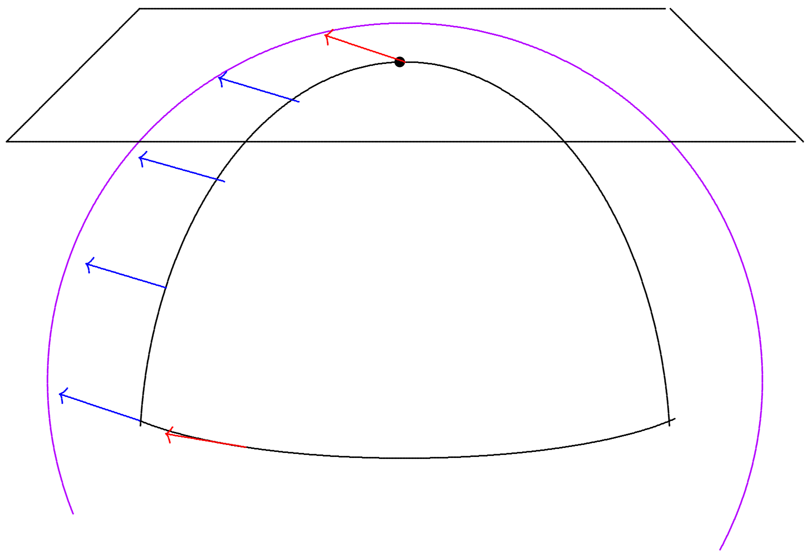

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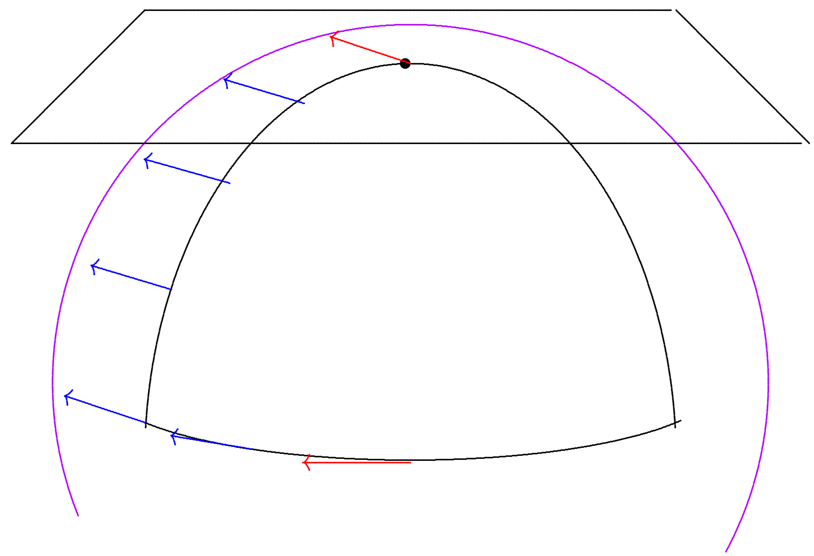

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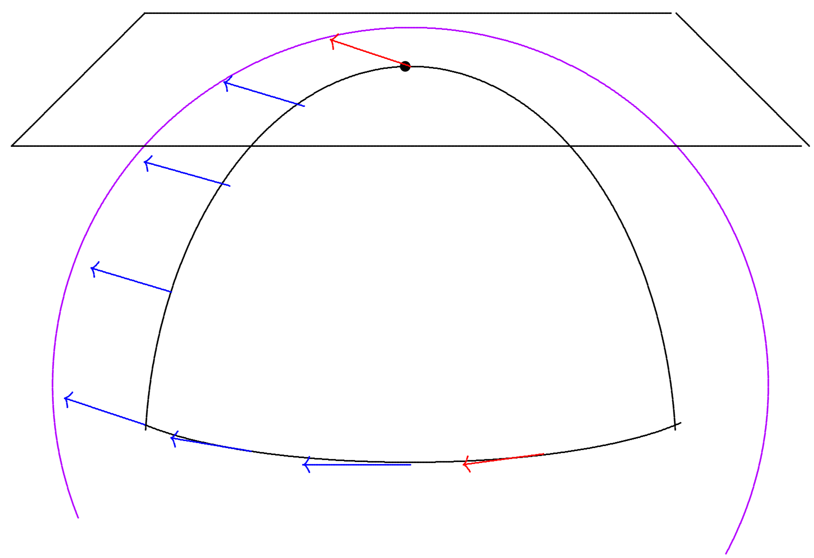

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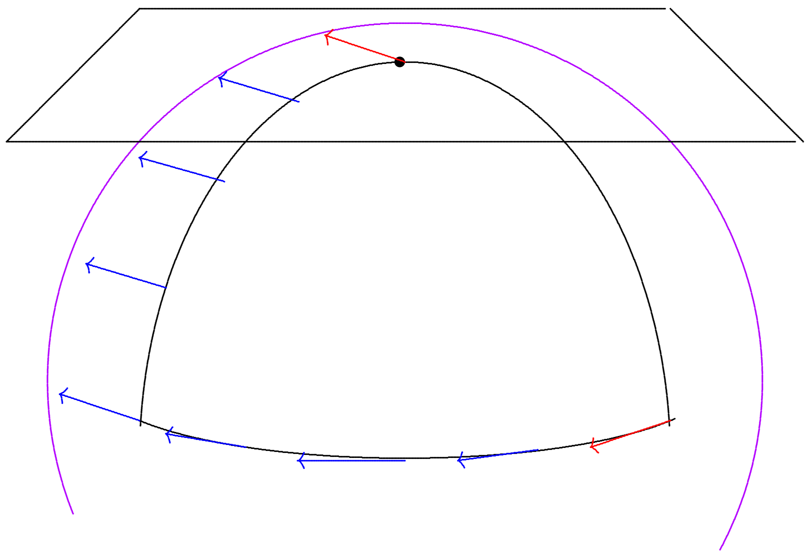

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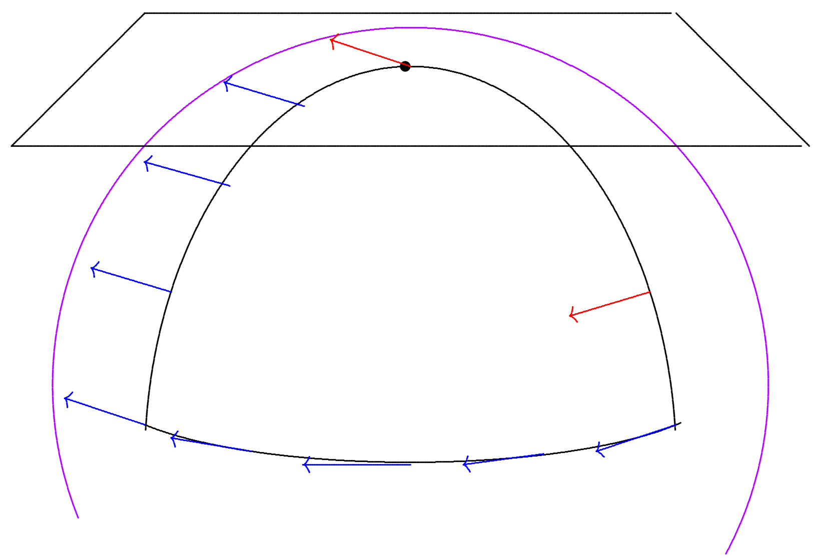

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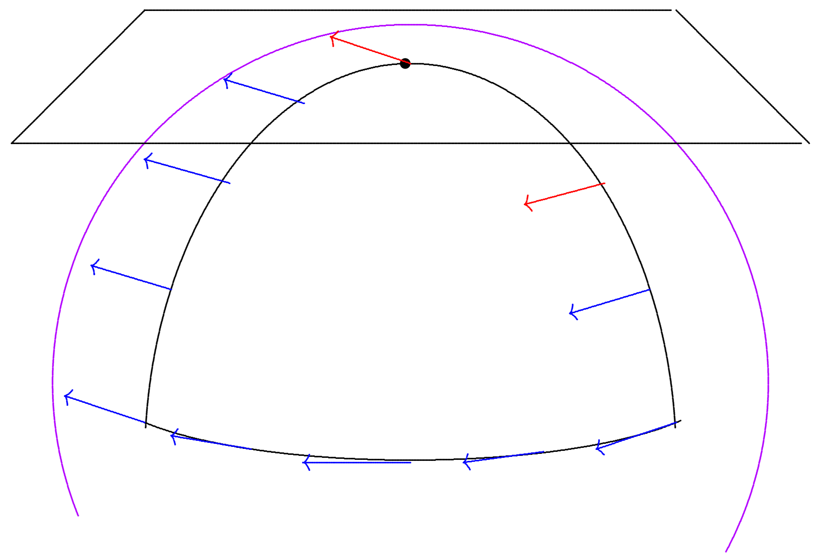

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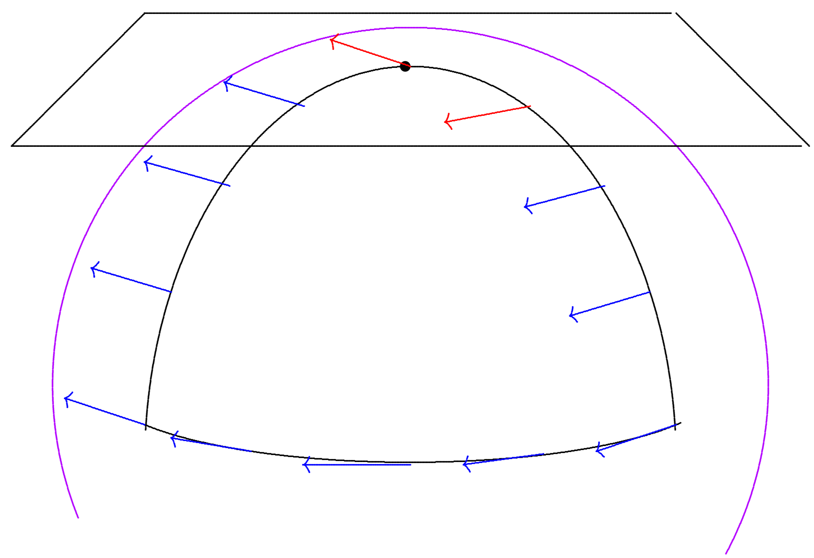

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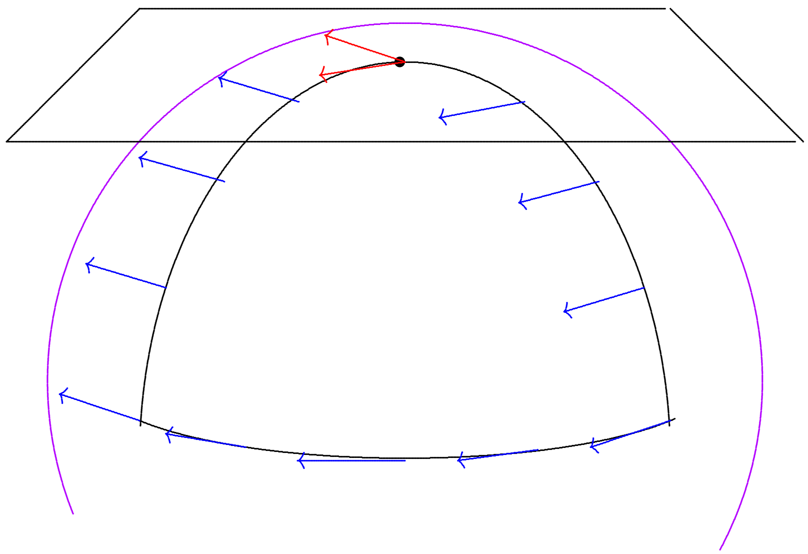

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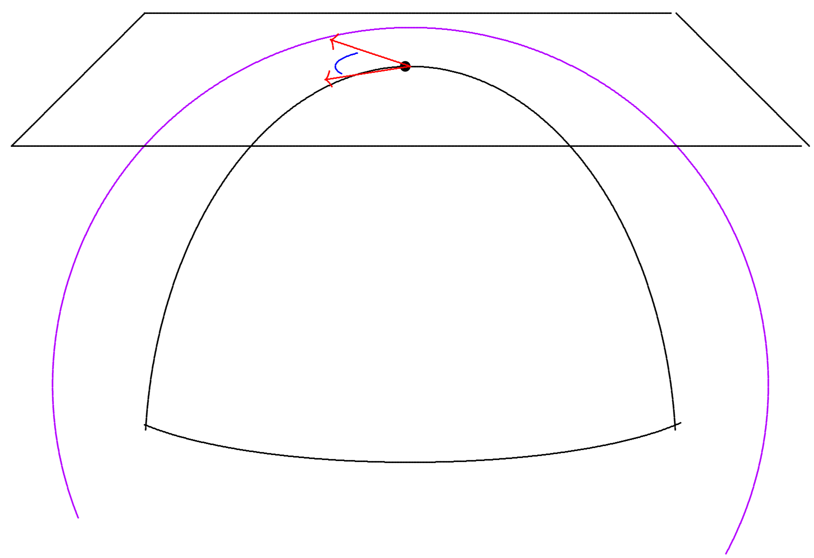
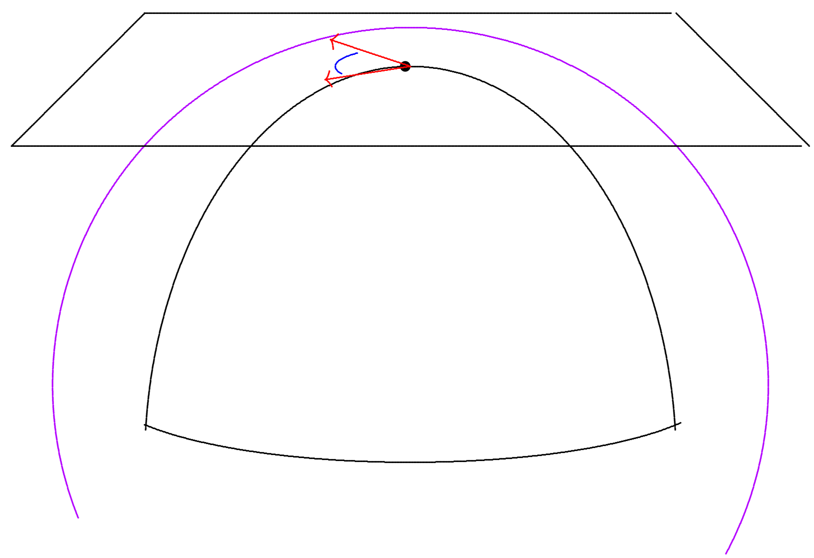
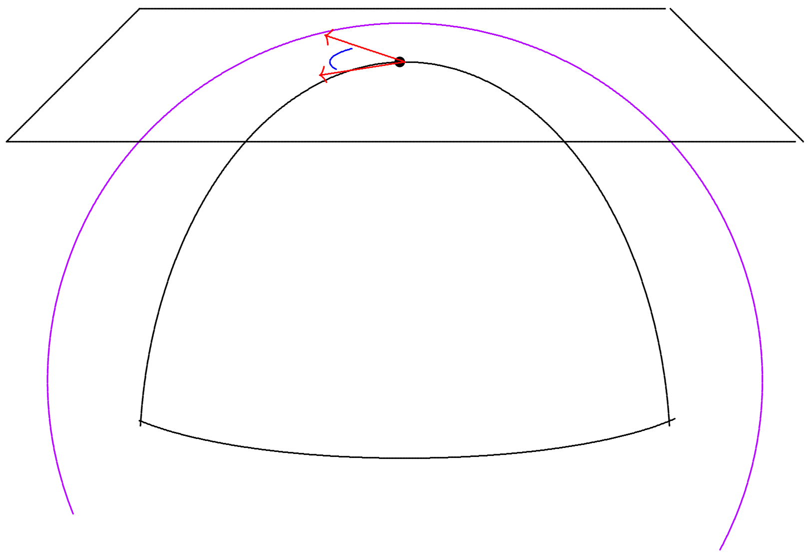


(M^n, g): holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

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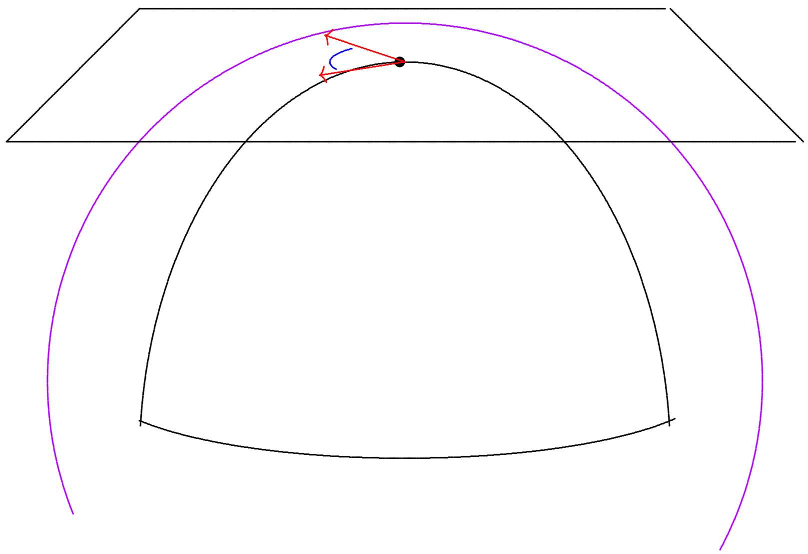

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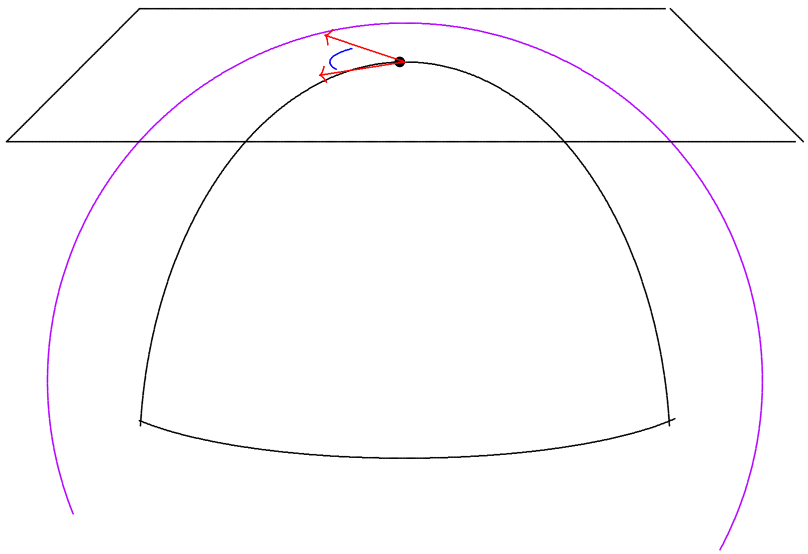

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

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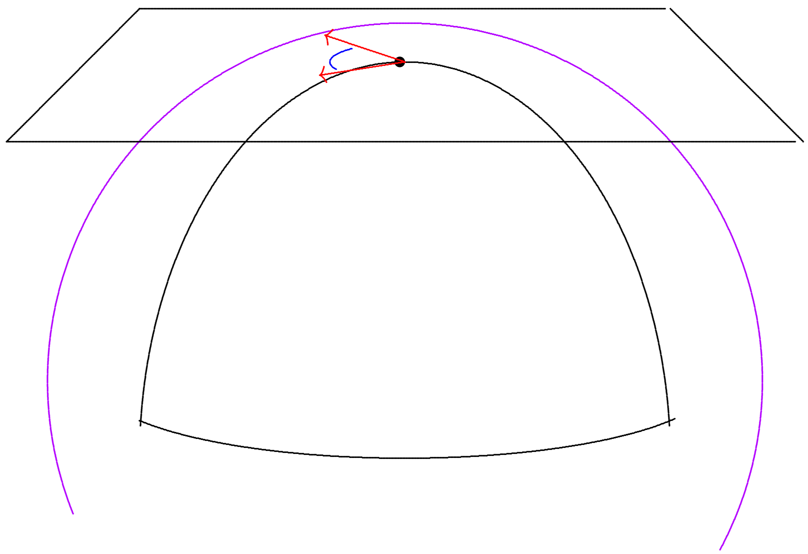

Makes tangent space a complex vector space!

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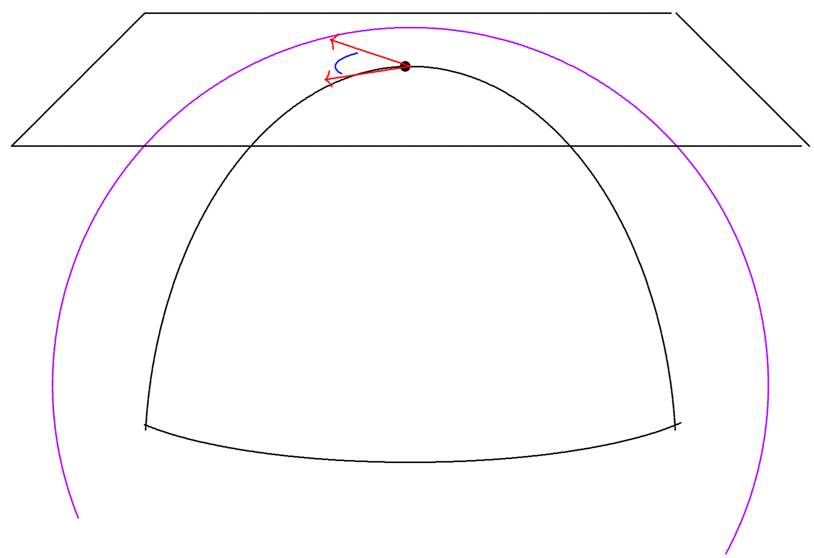

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$$J : TM \to TM, \qquad J^2 = -\text{identity}$$

"almost-complex structure"

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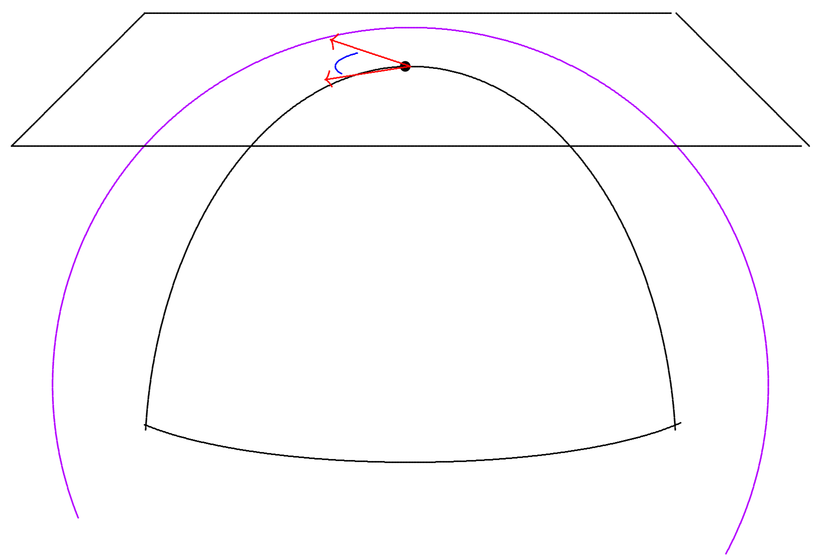

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Invariant under parallel transport!

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$$\omega = i \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \, dz^j \wedge d\bar{z}^k$$

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Kähler magic:

$$r = -\sum_{j,k=1}^{m} \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det[g_{p\bar{q}}] \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j\right]$$

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Kähler magic:

If we define the Ricci form by

$$\rho = r(J\cdot, \cdot)$$

then $i\rho$ is curvature of canonical line bundle $\Lambda^{m,0}$.

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$$[\omega] \in H^2(M)$$

"Kähler class"

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ω non-degenerate closed 2-form:

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ω non-degenerate closed 2-form: symplectic form

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Scalar-flat-Kähler Burns metric on $\widetilde{\mathbb{C}^2} \subset \mathbb{C}^2 \times \mathbb{CP}_1$:

Restrict Euclidean $\times$ round metric to $\begin{vmatrix} z_1 & z_2 \\ \zeta_1 & \zeta_2 \end{vmatrix} = 0.$

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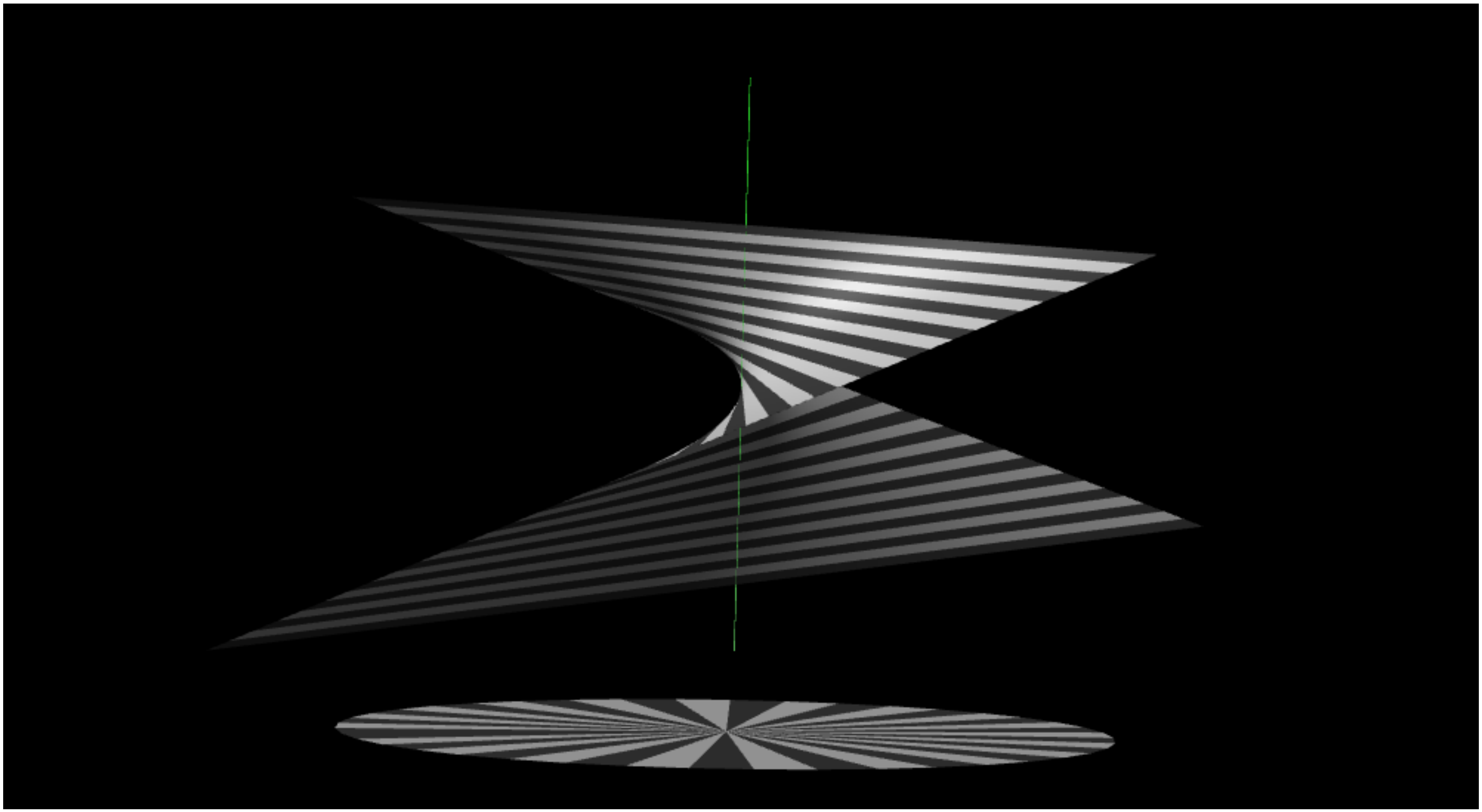

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Scalar-flat-Kähler Burns metric on $\widetilde{\mathbb{C}^2} \subset \mathbb{C}^2 \times \mathbb{CP}_1$:

Restrict Euclidean $\times$ round metric to $\begin{vmatrix} z_1 & z_2 \\ \zeta_1 & \zeta_2 \end{vmatrix} = 0.$

Motivation:

When $n = 3$, ADM mass in general relativity.

Reads off "apparent mass" from strength of the gravitational field far from an isolated source.

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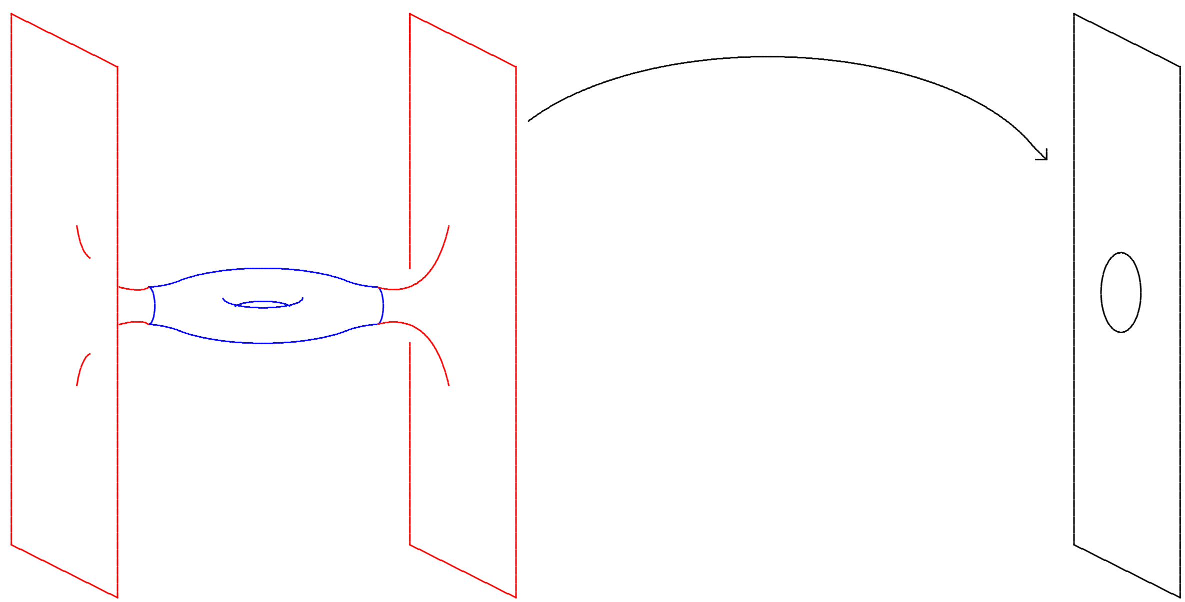
Definition. *A complete, non-compact Riemannian n-manifold (M^n, g) is called* asymptotically Euclidean *(AE) if there is a compact set $K \subset M$ such that each* "end" *is diffeomorphic to $\mathbb{R}^n - D^n$ in such a manner that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

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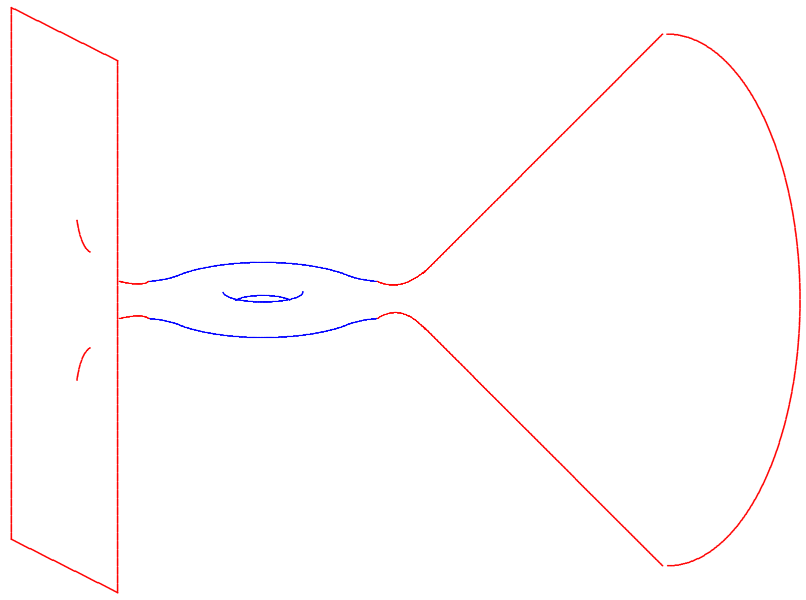
Interesting generalization. . .

Definition. *Complete, non-compact* n*-manifold* (M^n, g) *is* asymptotically locally Euclidean



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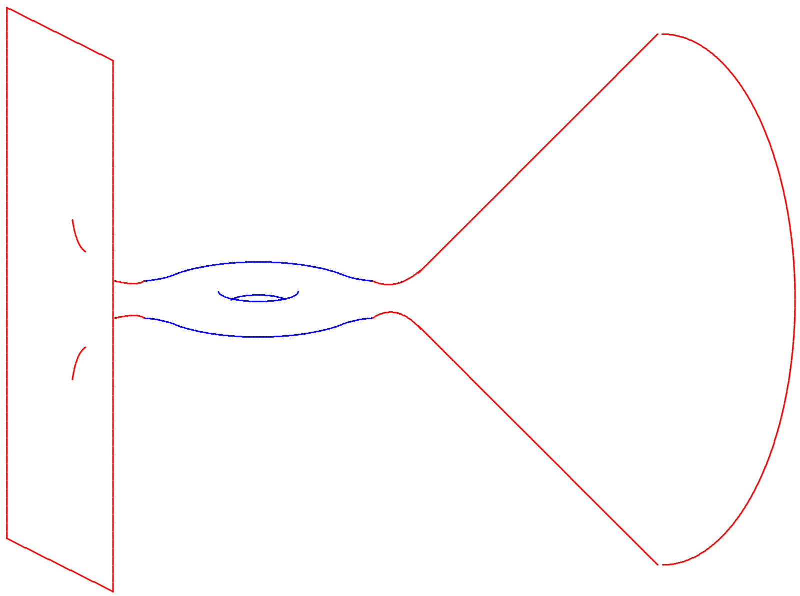

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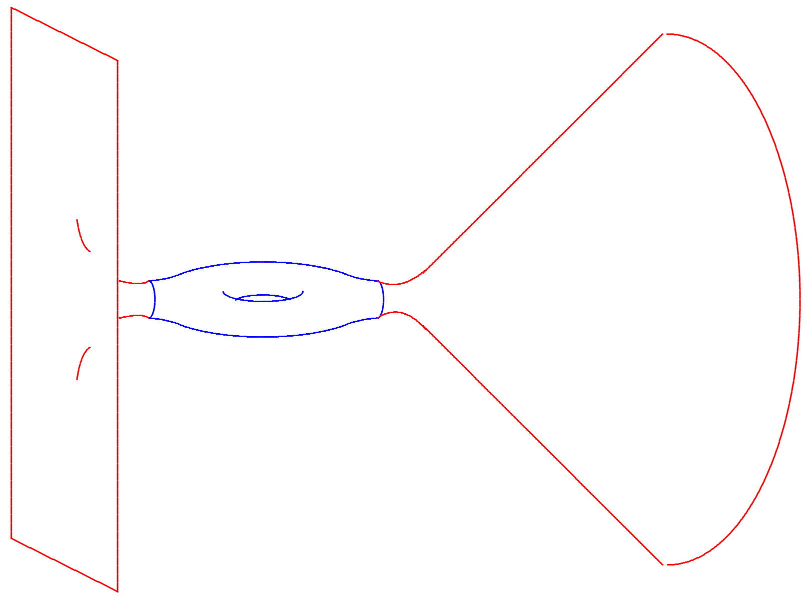

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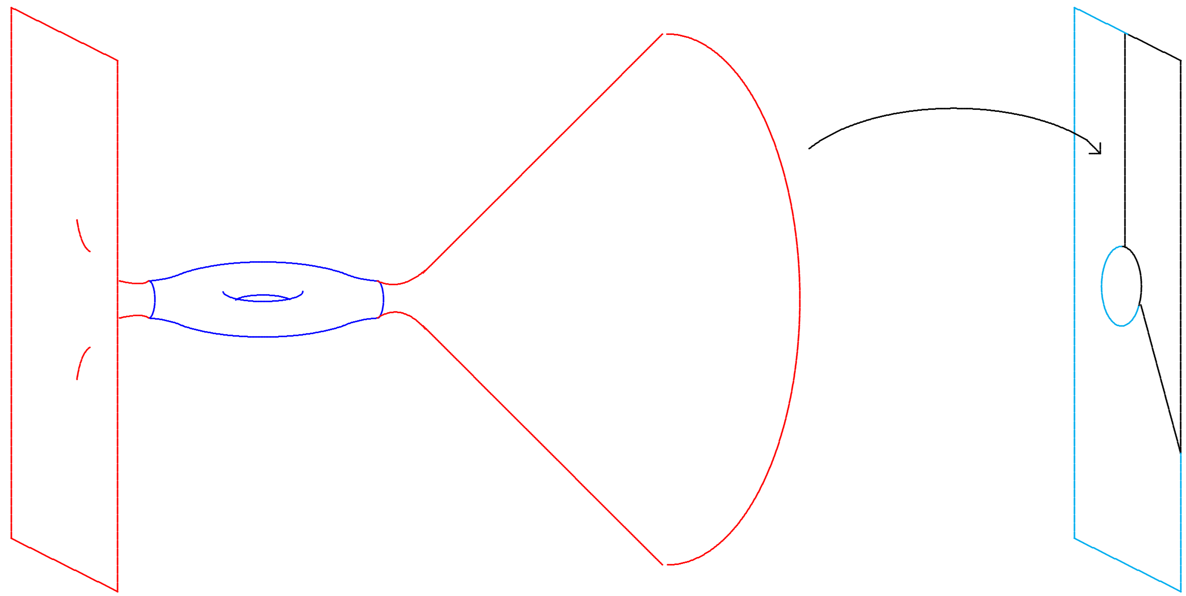

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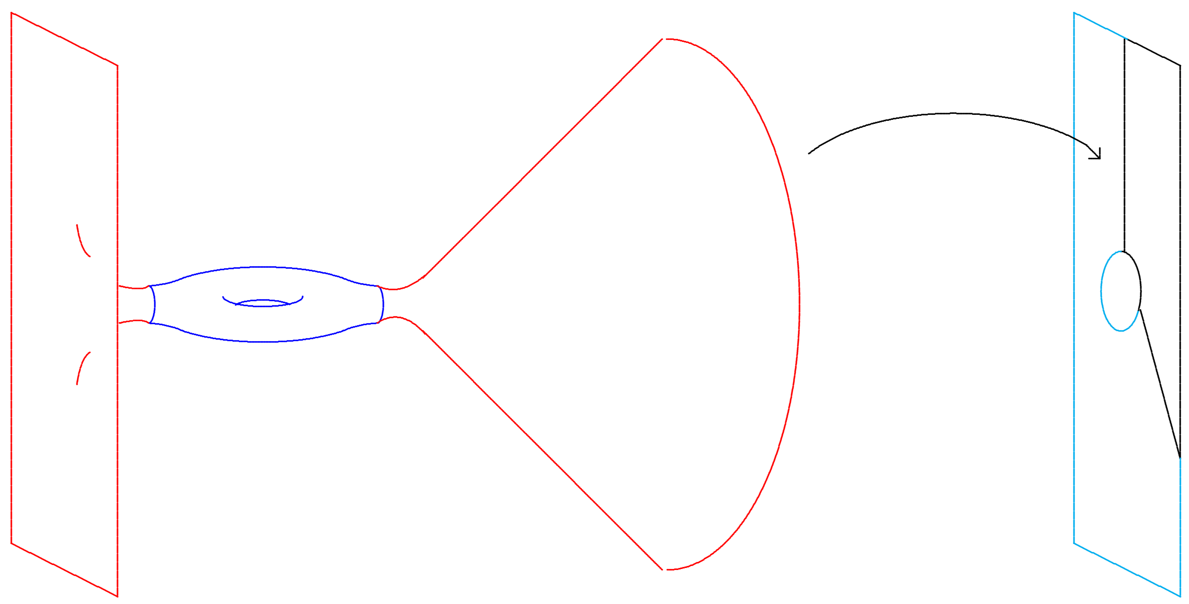

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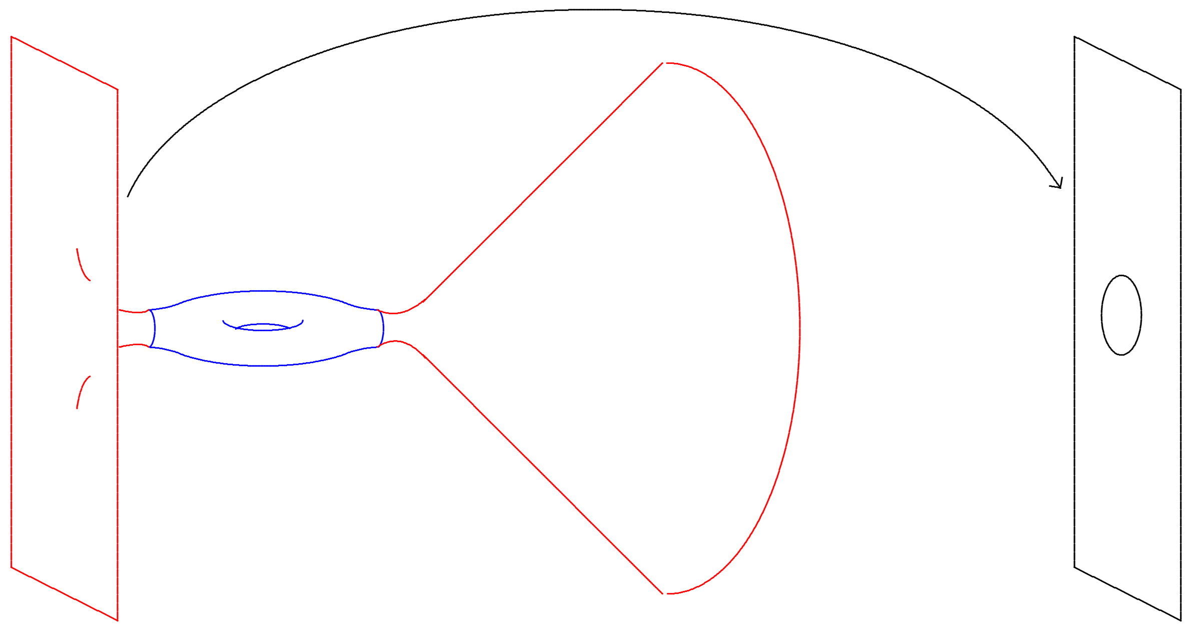

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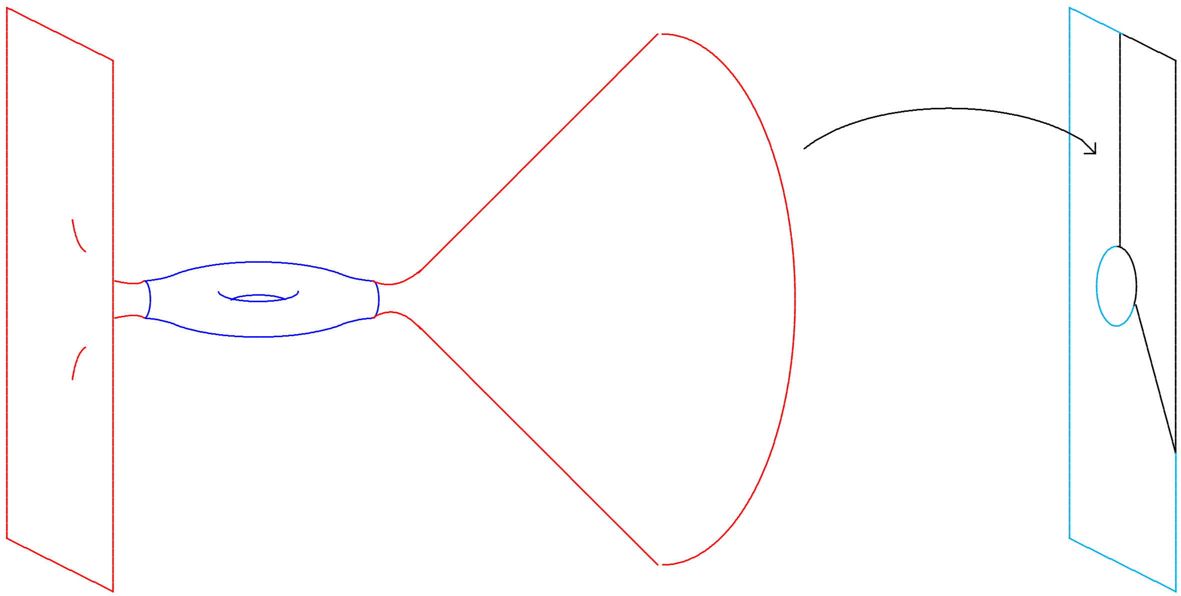

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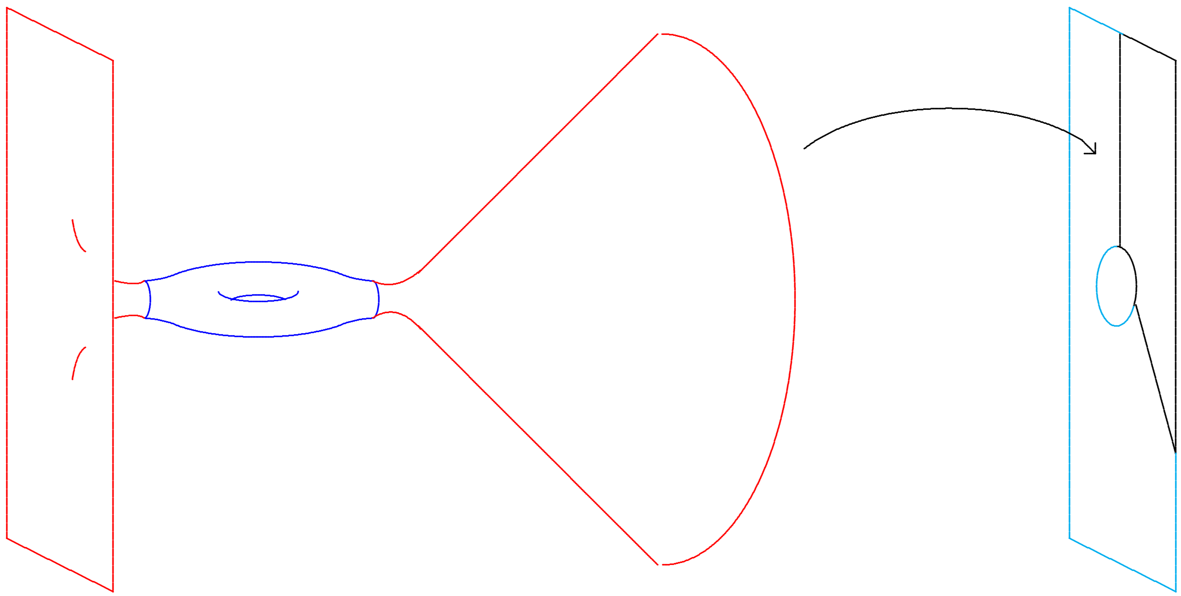

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Why consider ALE spaces?

Key examples:

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Gravitational Instantons:

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ALE Ricci-flat Kähler manifolds.

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Bubbling modes for sequences of Einstein metrics.

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Chen-L-Weber '08.

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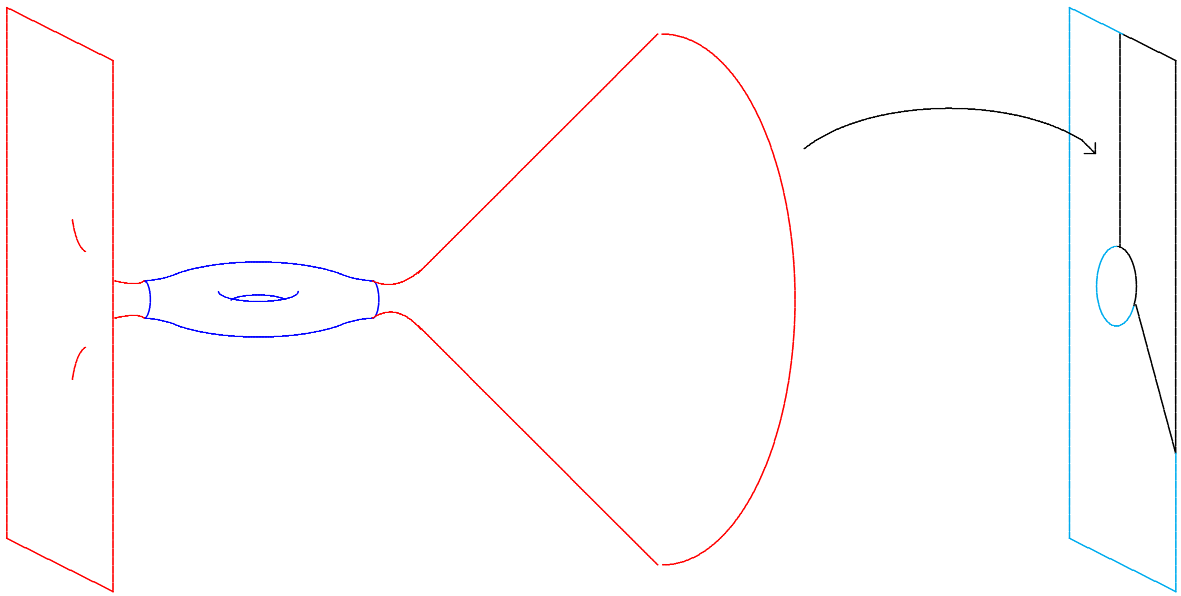

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Mass still meaningful in this context. . .

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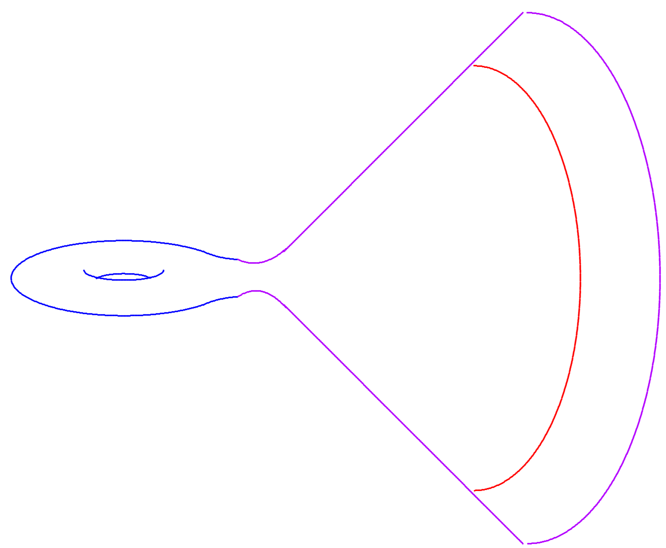

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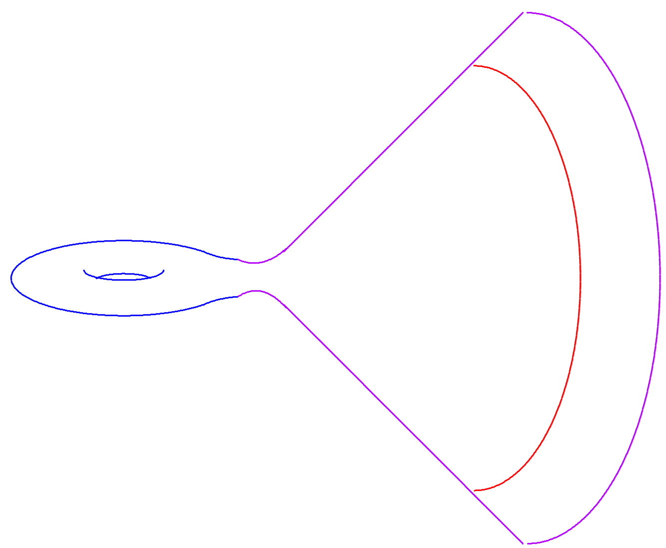

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Bartnik/Chruściel (1986): With weak fall-off conditions, the mass is well-defined & coordinate independent.

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General case in arbitrary dimension n.

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on line bundles $L \to \mathbb{CP}_1$ of Chern-class ≤ -3.

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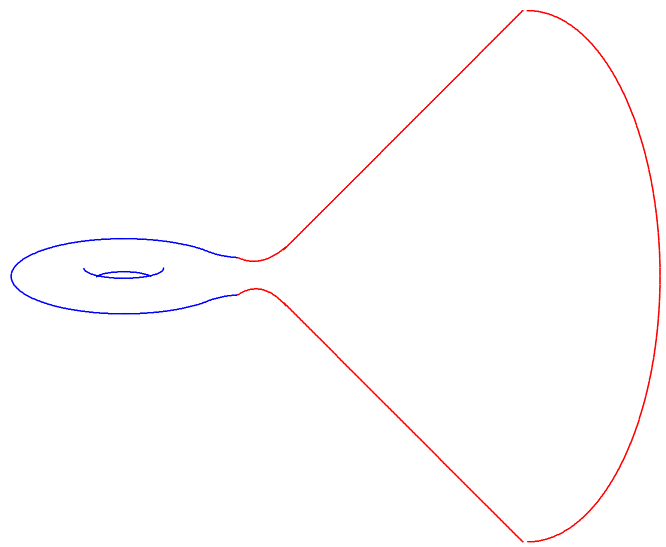

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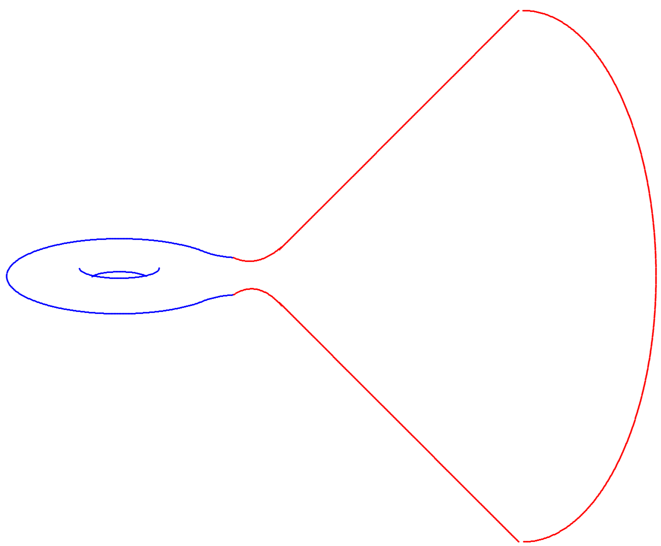

$$n = 2m \geq 4$$

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Proof uses methods of symplectic topology.

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Does not depend on the choice of an end!

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Theorem A.

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

The explicit formula reproduces the mass in cases where it previously had been laboriously computed from the definition.

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minimal: containing no (-1)-curves

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Non-minimal resolutions typically admit families of such metrics for which the mass can be continuously deformed from negative to positive.

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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So **Theorem A** is an immediate consequence!

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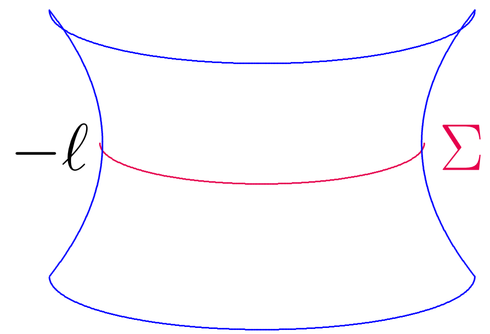

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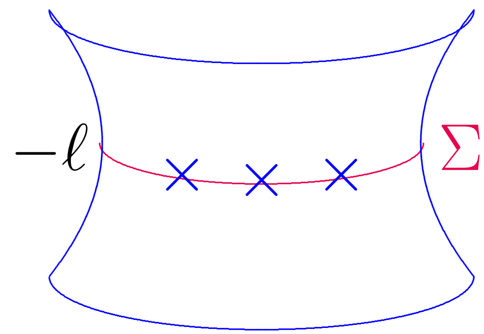

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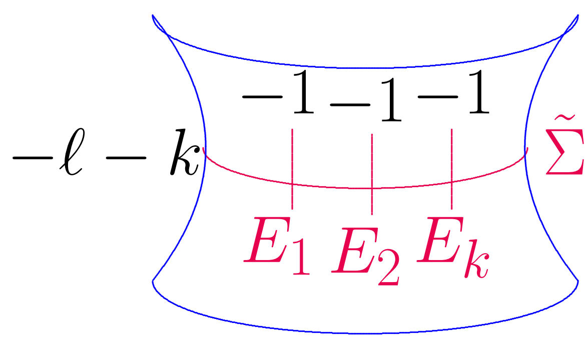

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Proof actually shows something stronger!

Theorem E (Penrose Inequality)**.**

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numerically canonical $\Longleftrightarrow$ Poincaré dual to $-c_1$.

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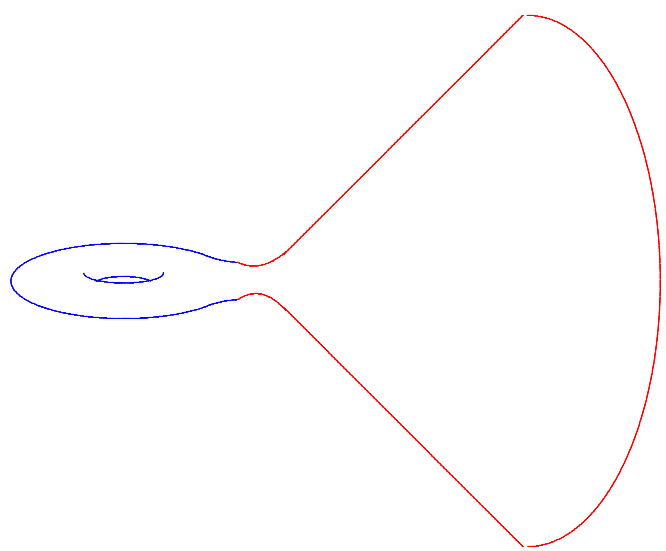

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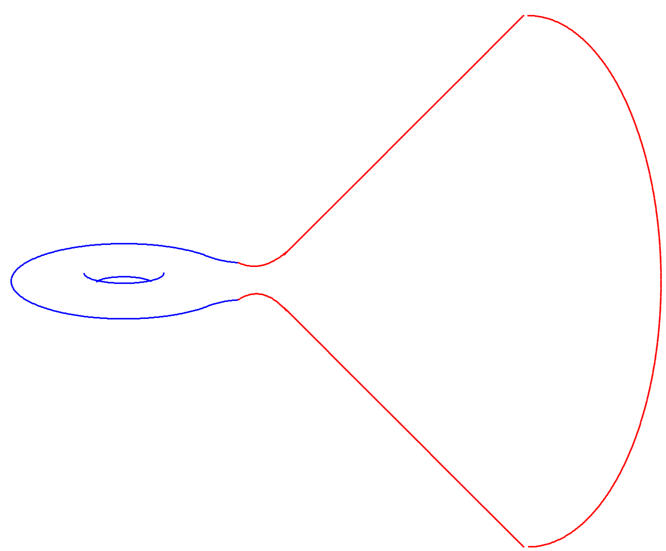

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