Twistor Correspondences,

Zoll Phenomena, &

Holomorphic Disks

(Lecture VII)

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Stony Brook University

Autumn School on Holomorphic Disks
Schloss Rauischholzhausen, November 18, 2018

Joint work with

Lionel Mason
Oxford University

Zoll Metrics, Branched Covers,
and Holomorphic Disks,
Comm. An. Geom. 18 (2010) 475–502.

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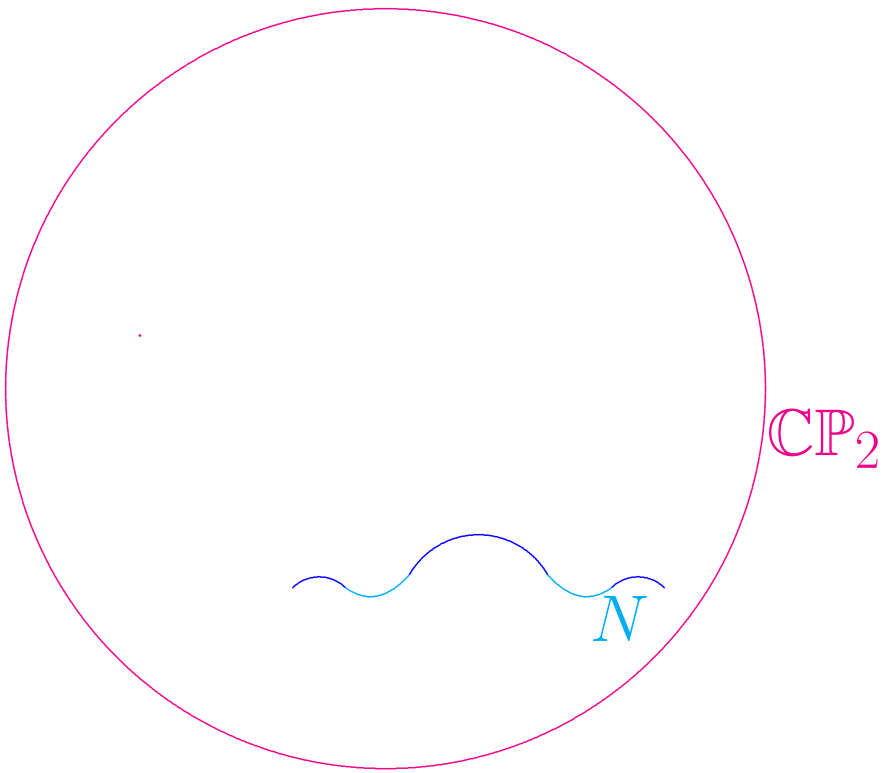
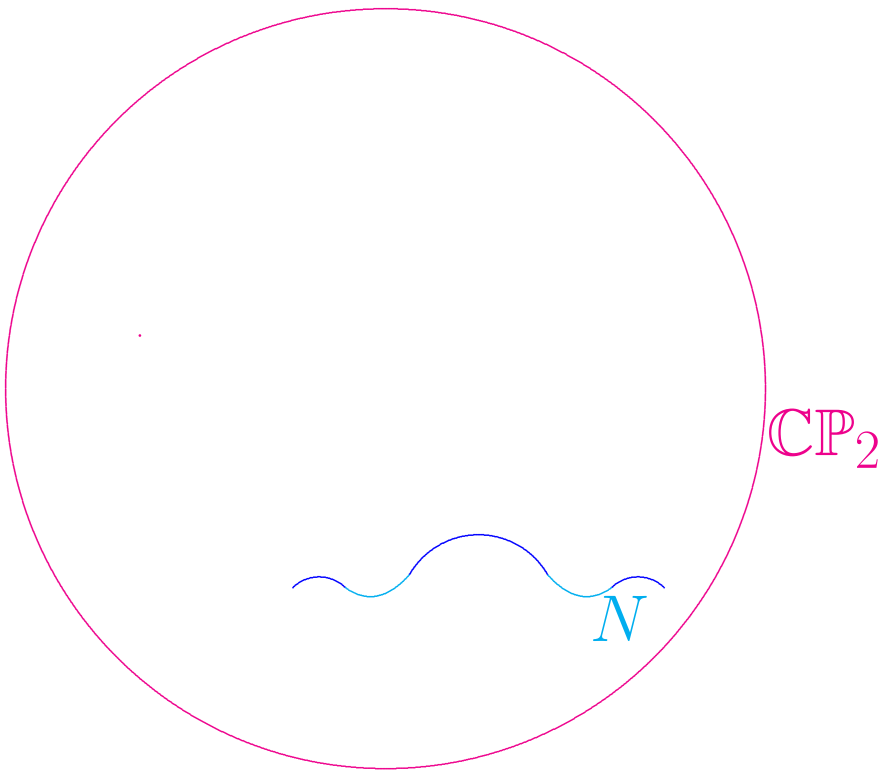


Definition. *A compact connected smoothly embedded 2-manifold* $N \subset \mathbb{CP}_2$ *will be called a docile surface*



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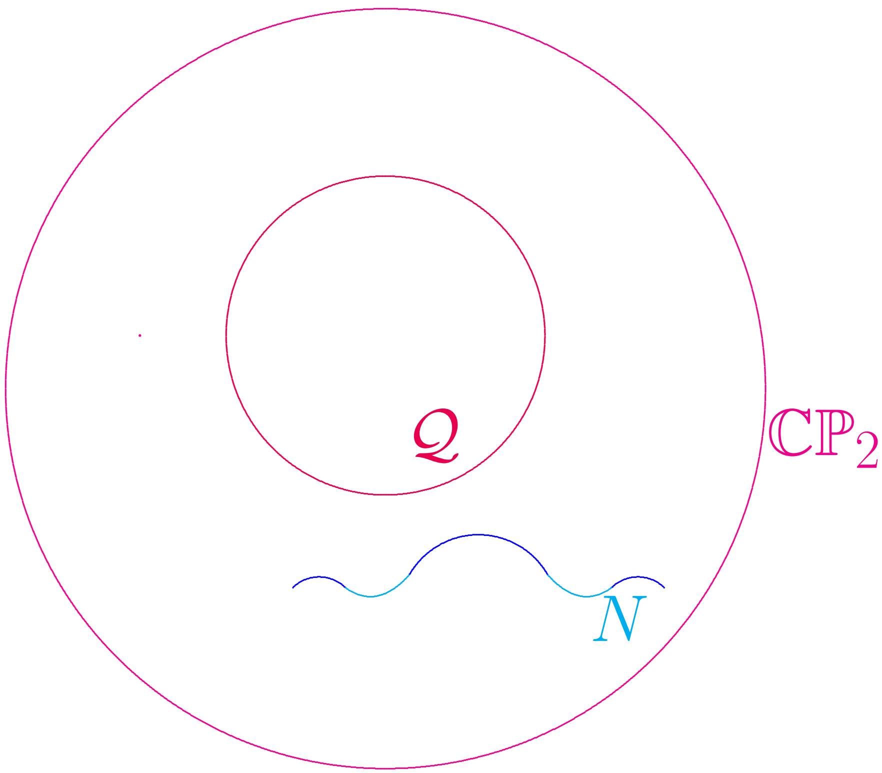

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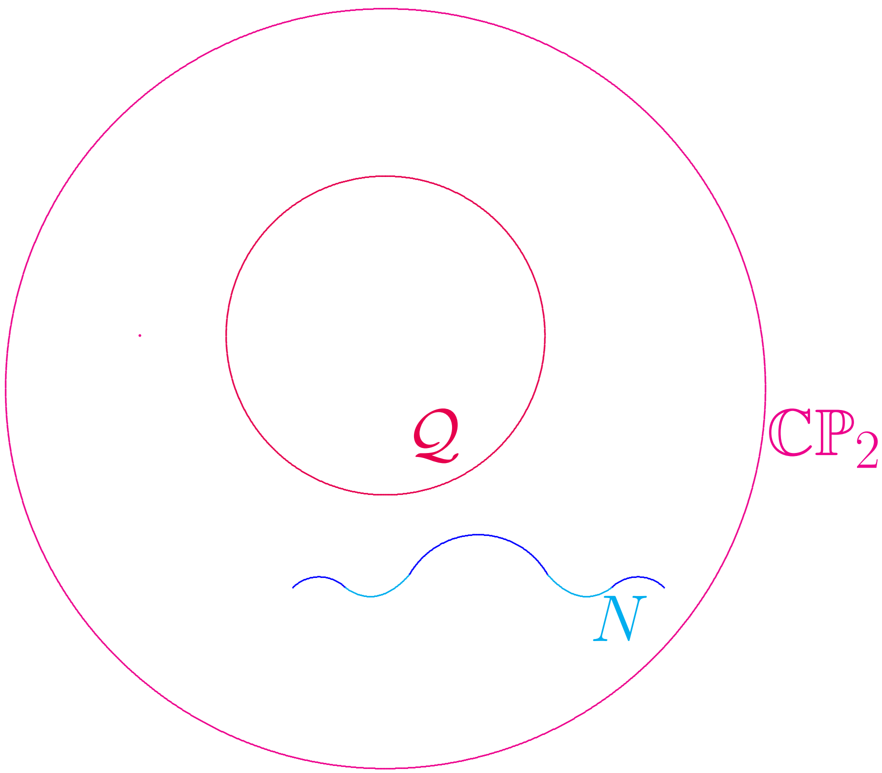

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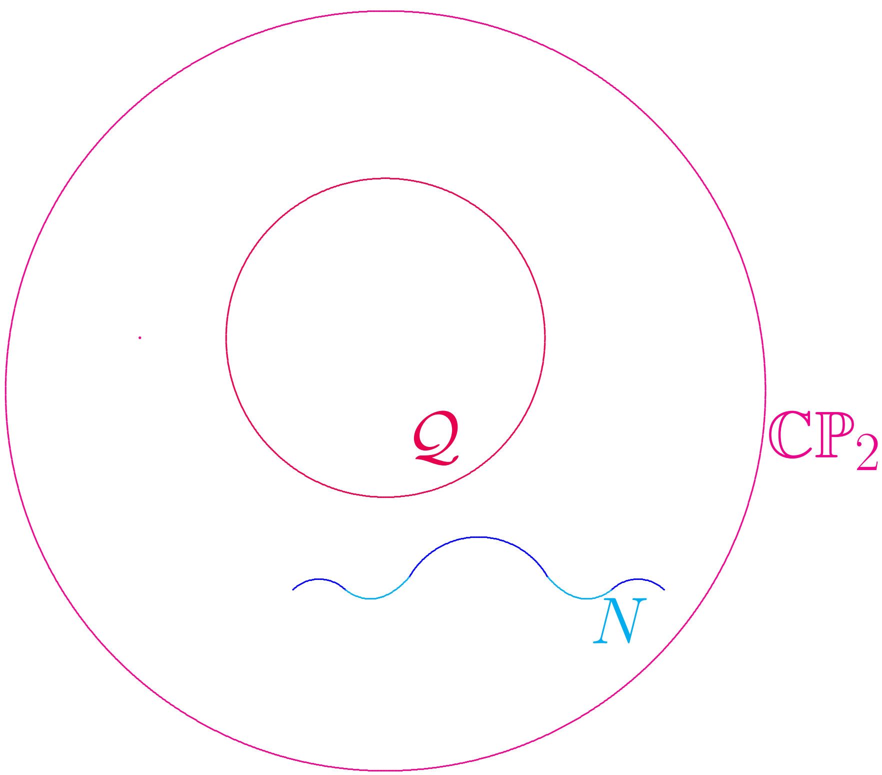

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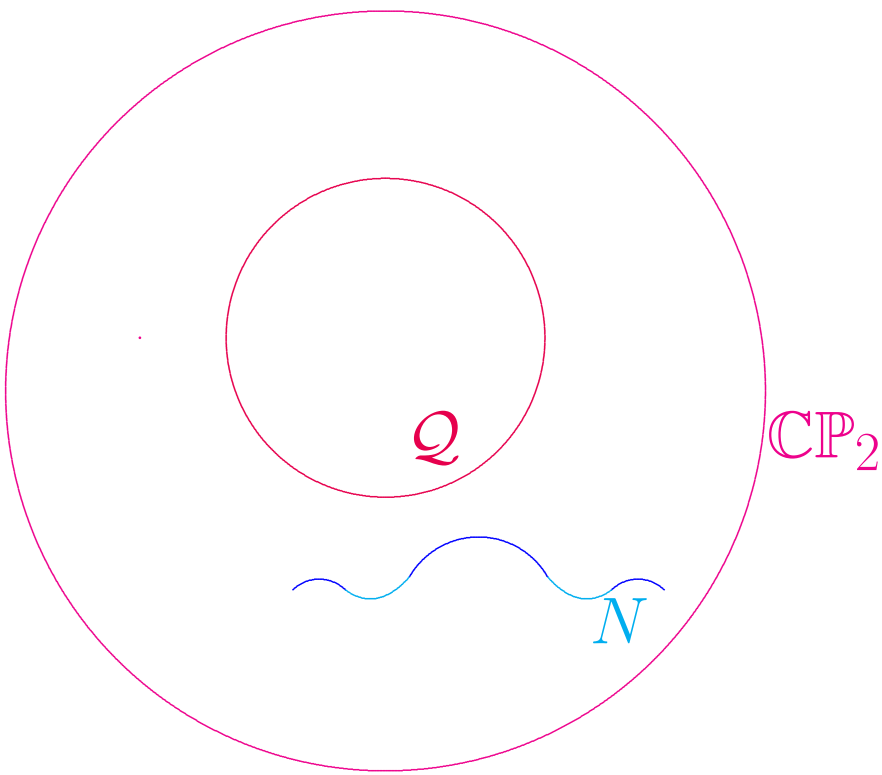

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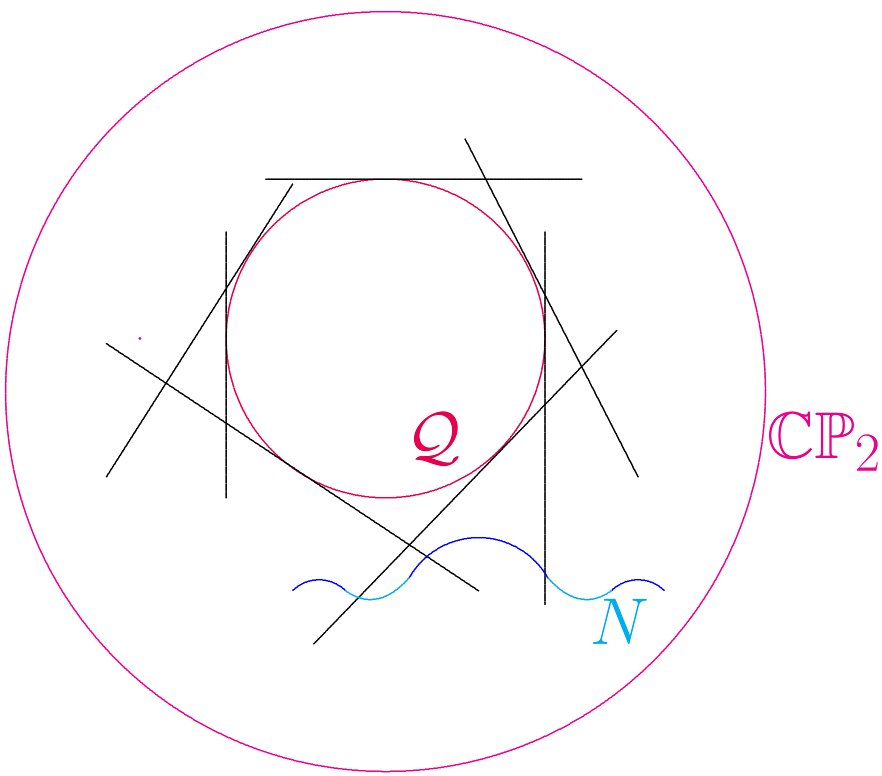
- *N is transverse to each tangent projective line of the conic $\mathcal{Q}$.*



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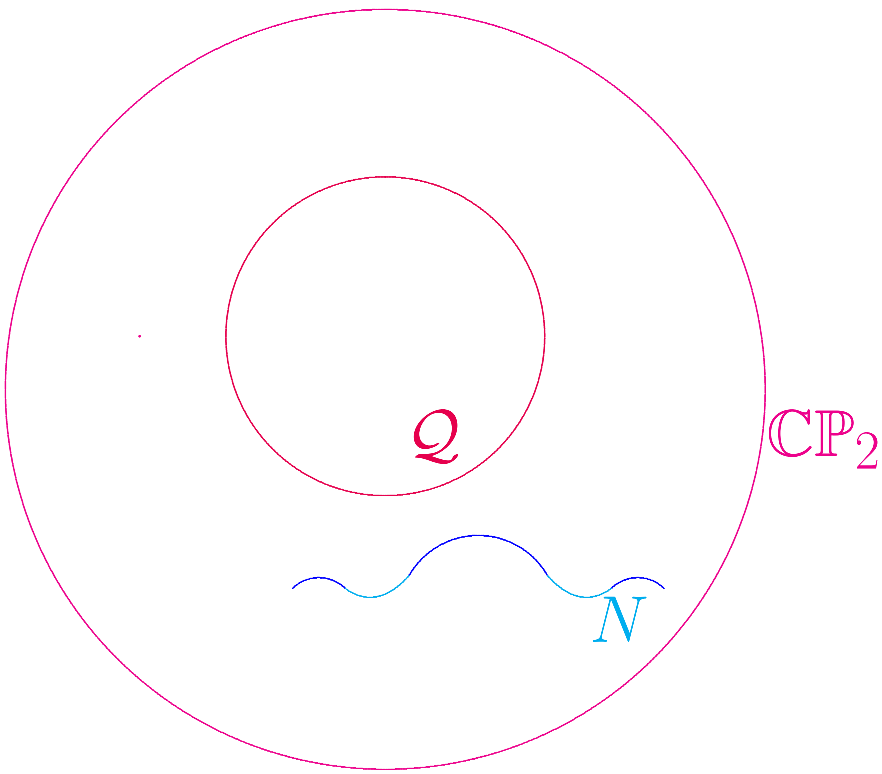
Proposition.



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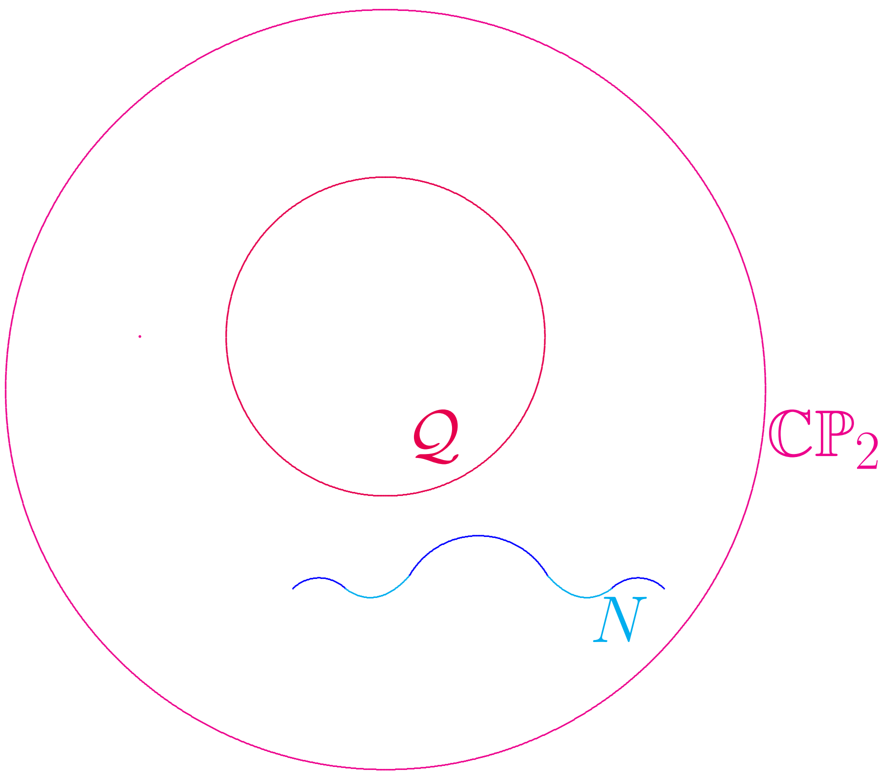

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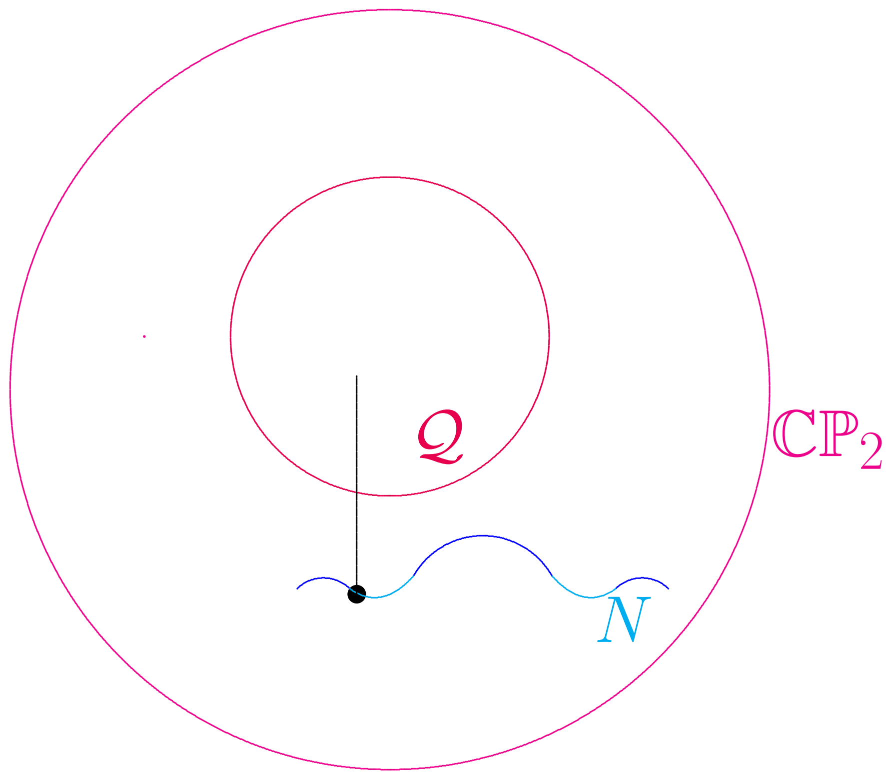
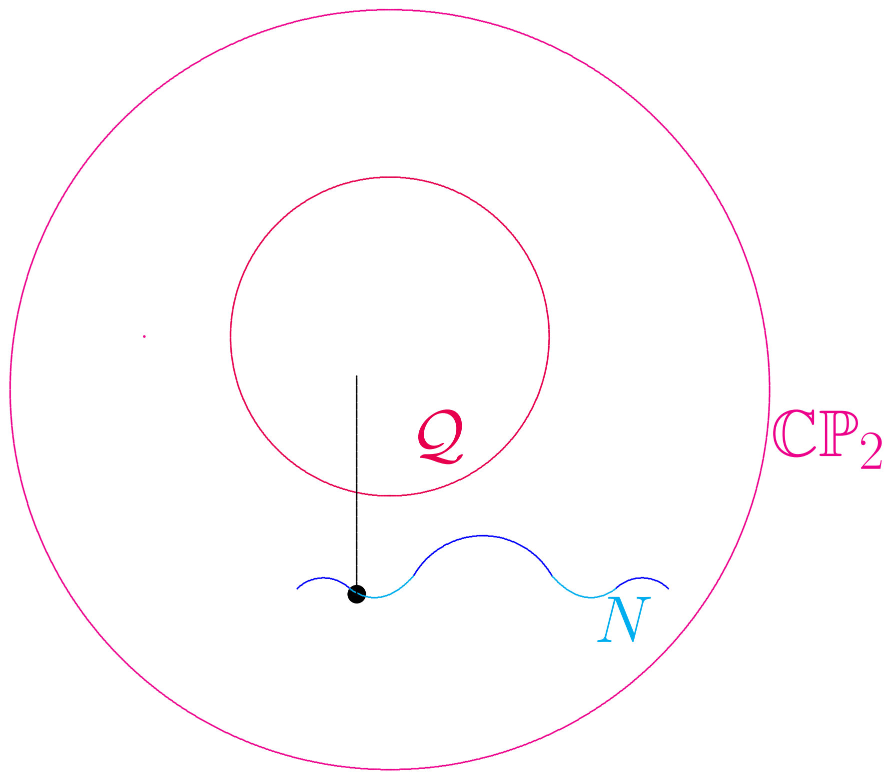


Proposition. *Let $N \subset \mathbb{CP}_2$ be any docile surface, and suppose that f is a parameterized holomorphic disk*



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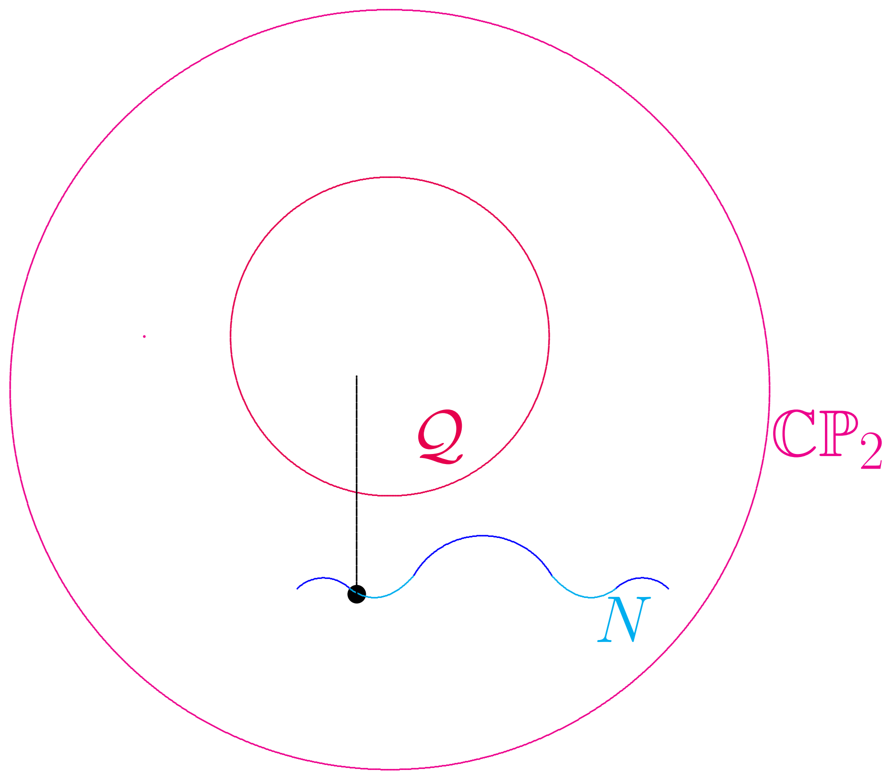

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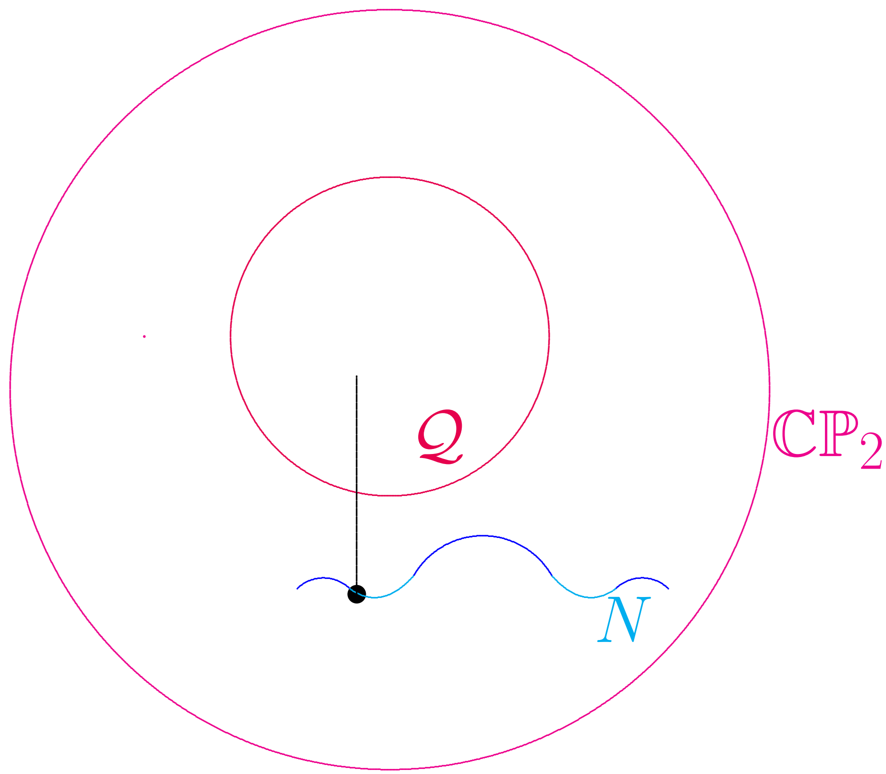

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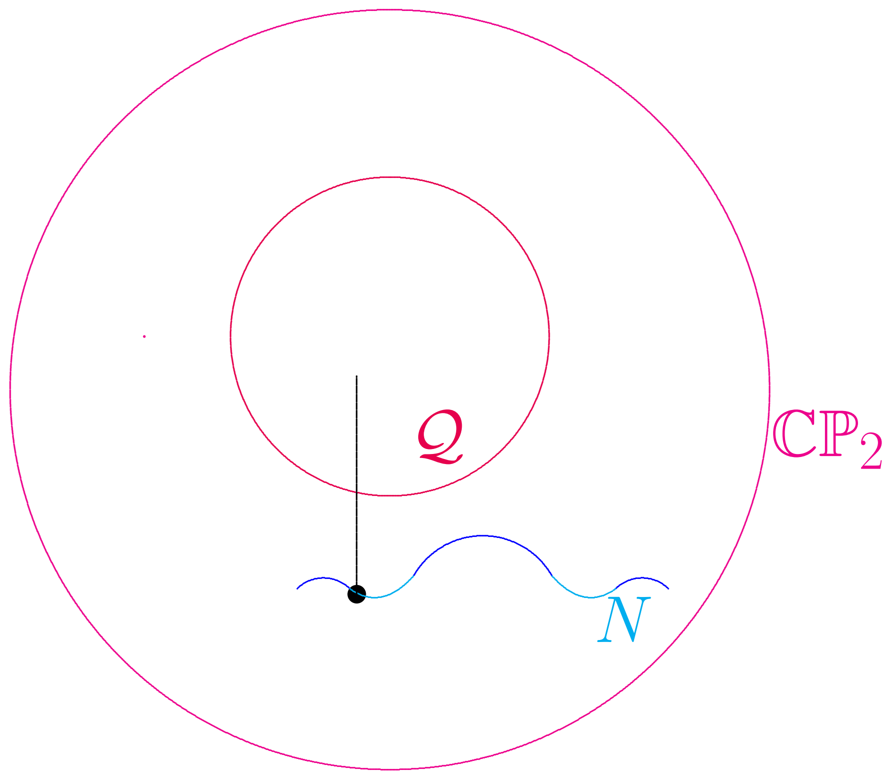

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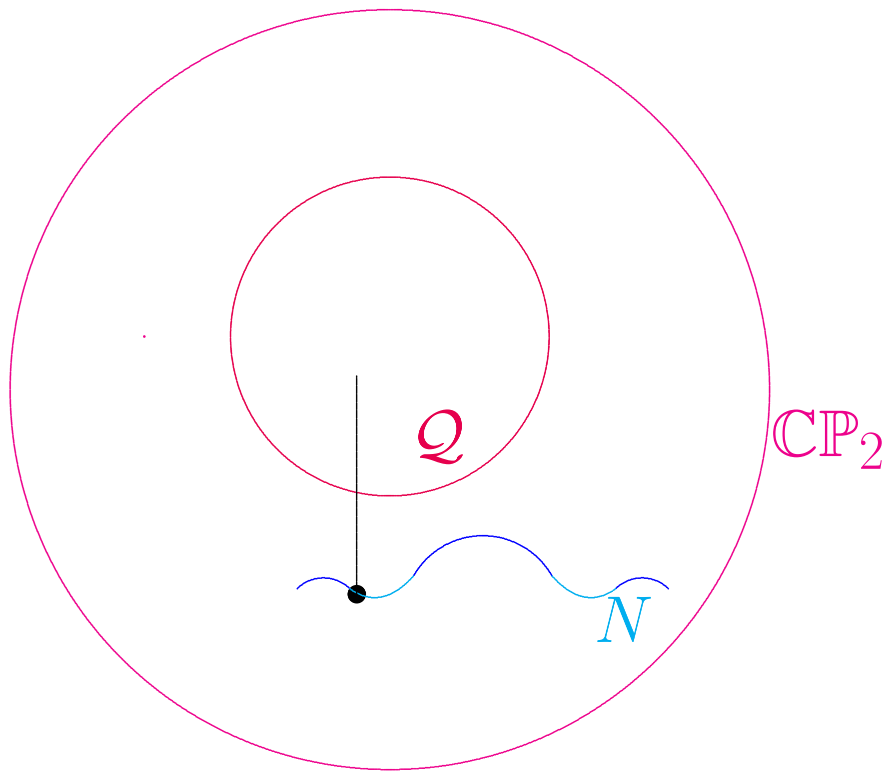

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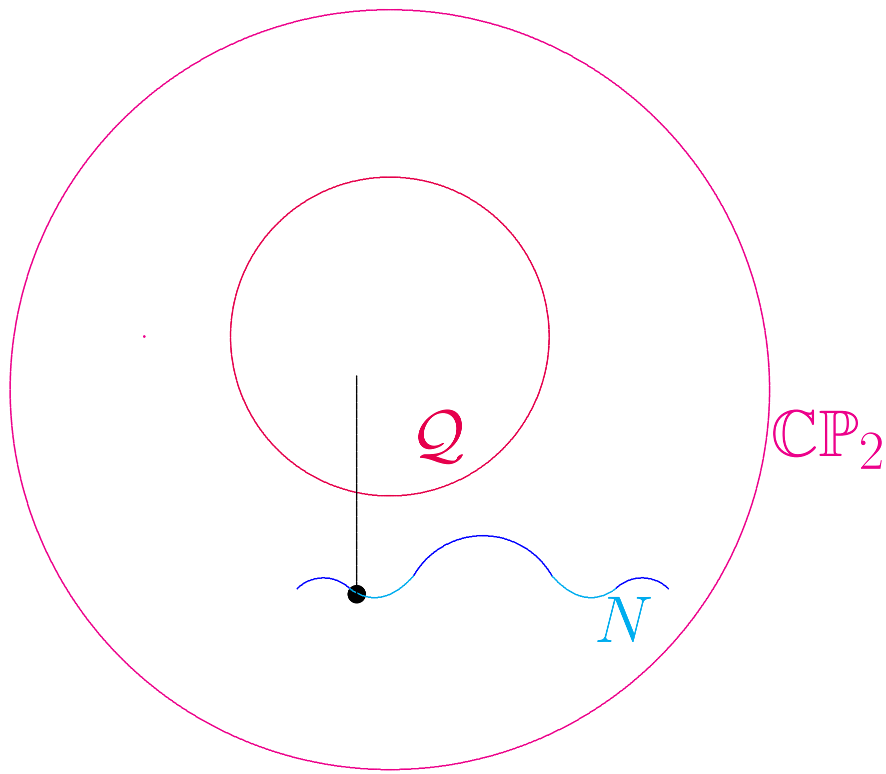

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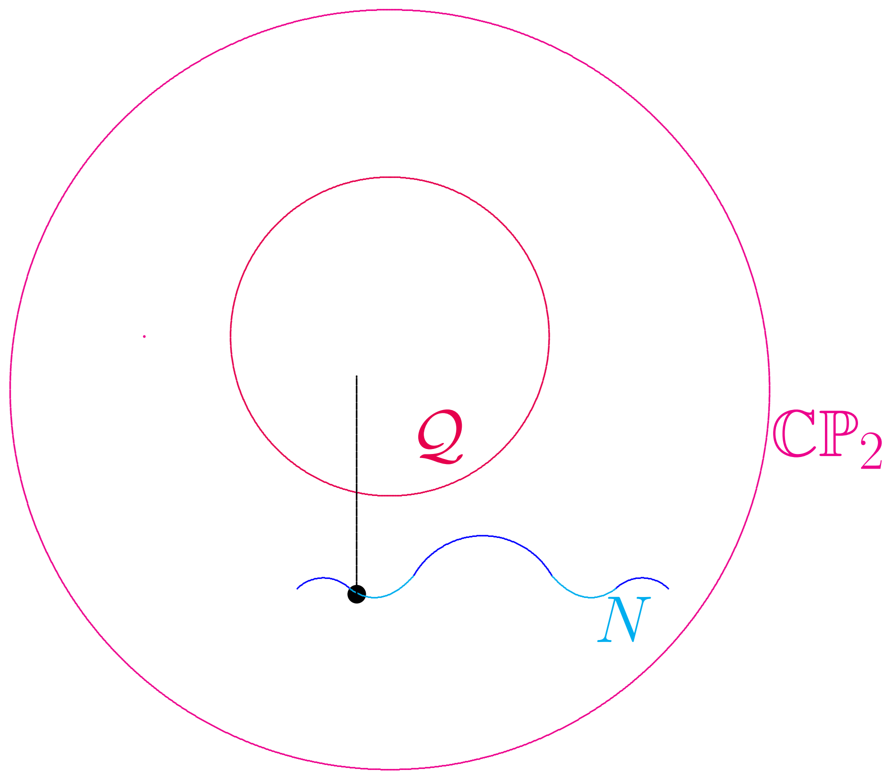

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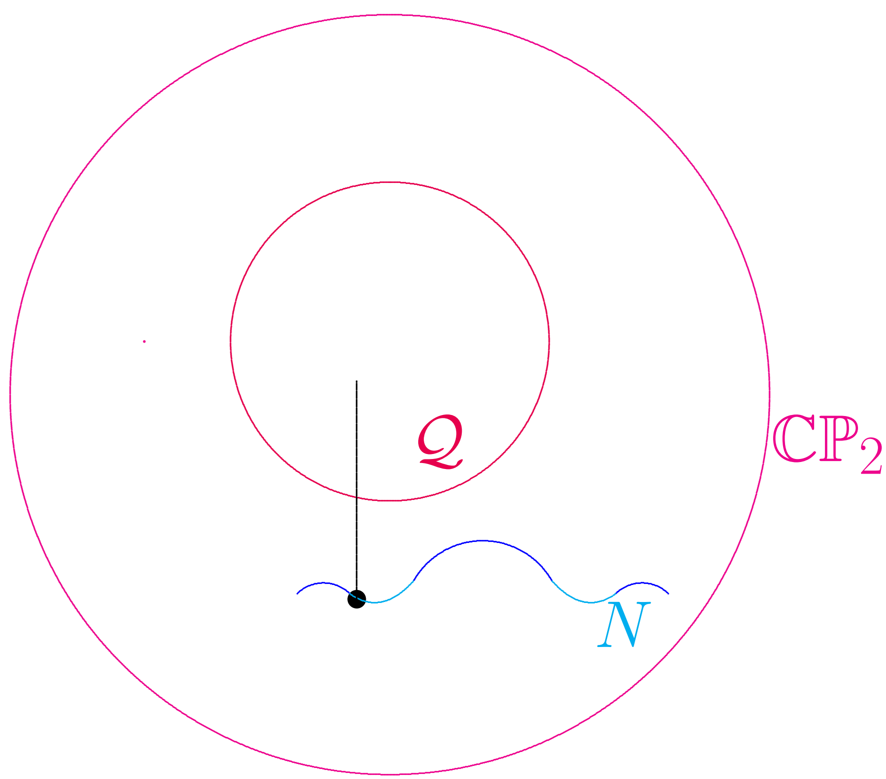

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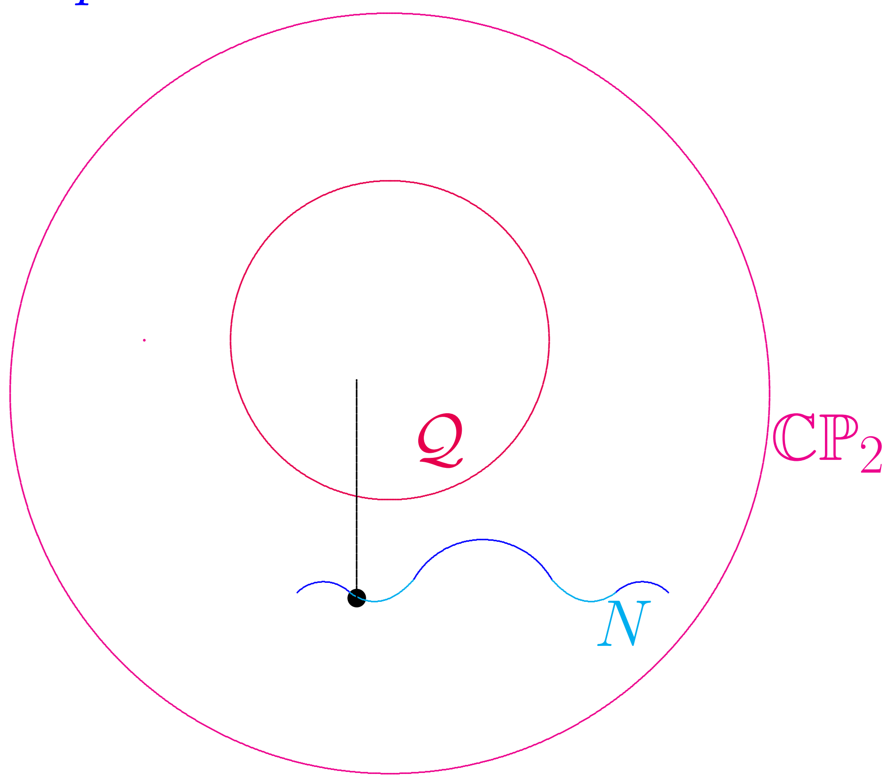

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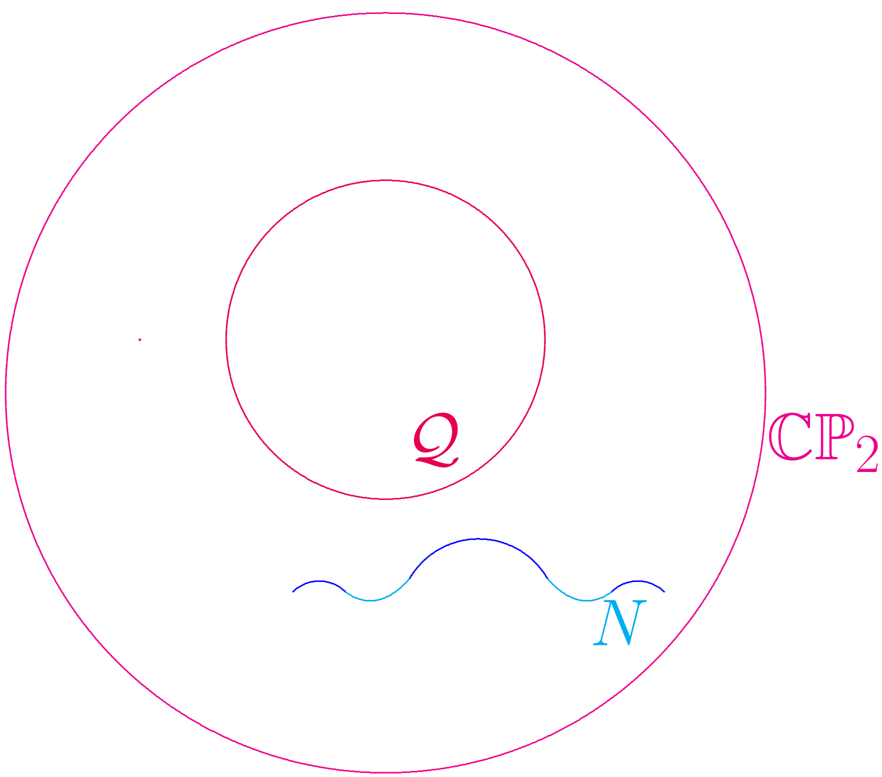
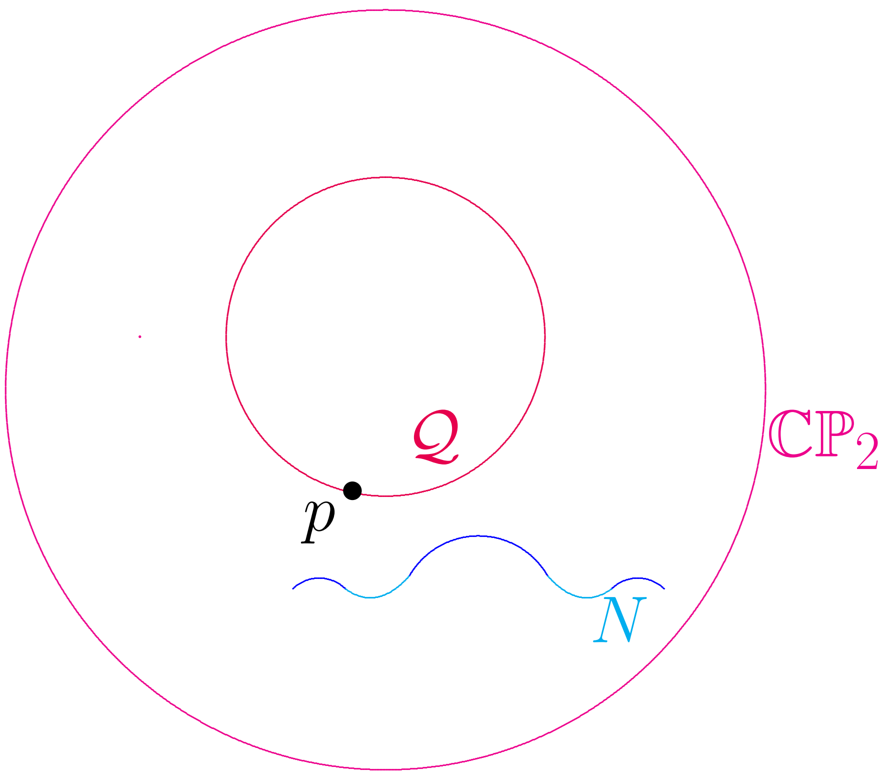
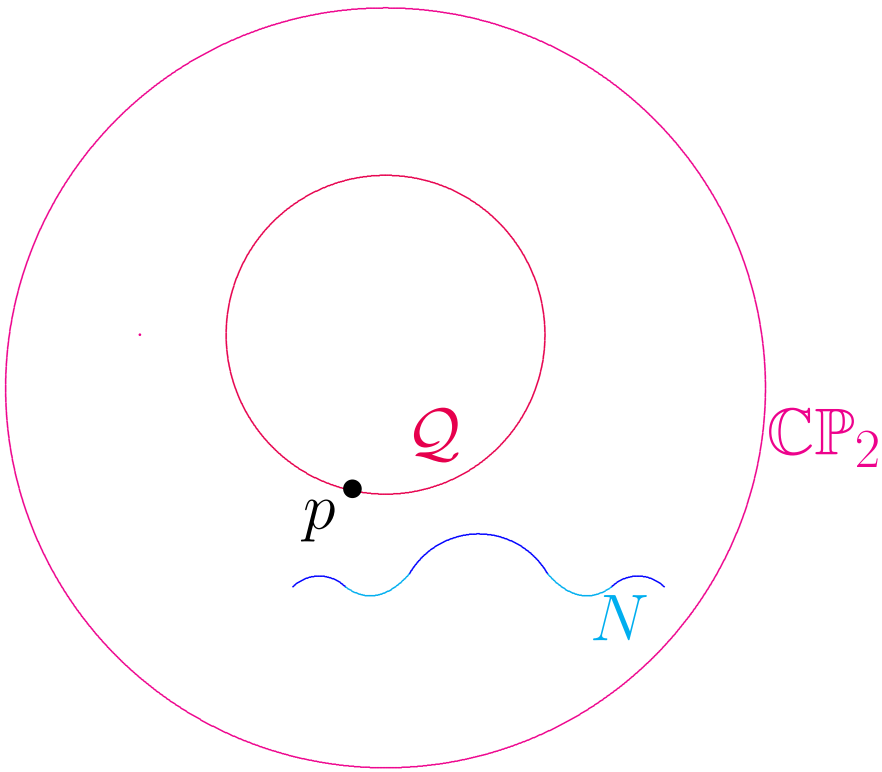


Theorem. *Let* $N \subset \mathbb{CP}_2$ *be any docile surface, and let* $p \in \mathcal{Q}$



Theorem. *Let* $N \subset \mathbb{CP}_2$ *be any docile surface, and let* $p \in \mathcal{Q}$ *be any point of the reference conic.*



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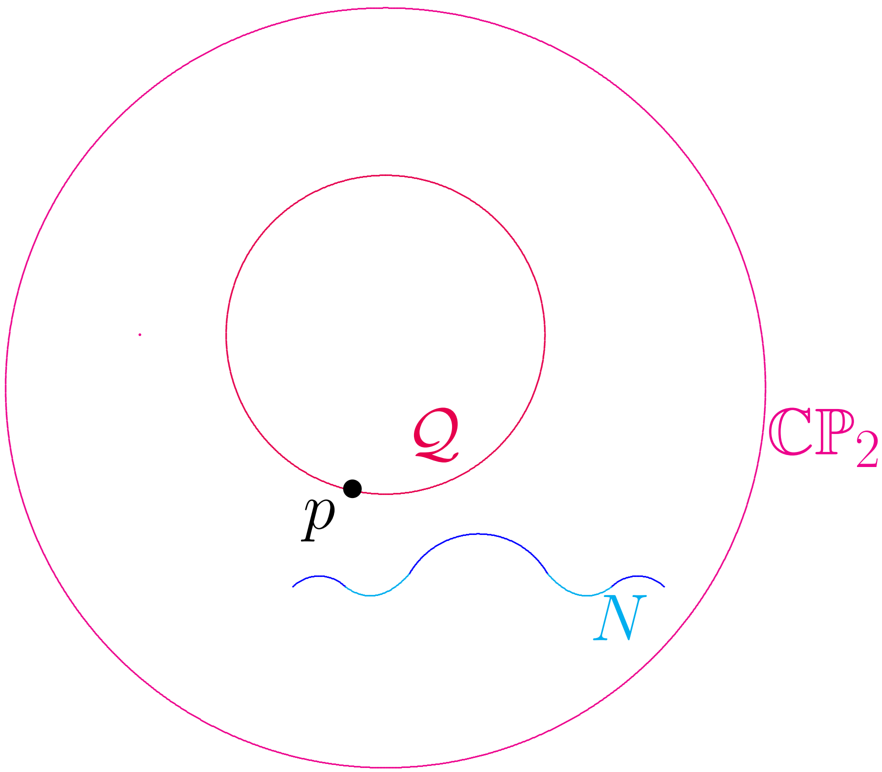

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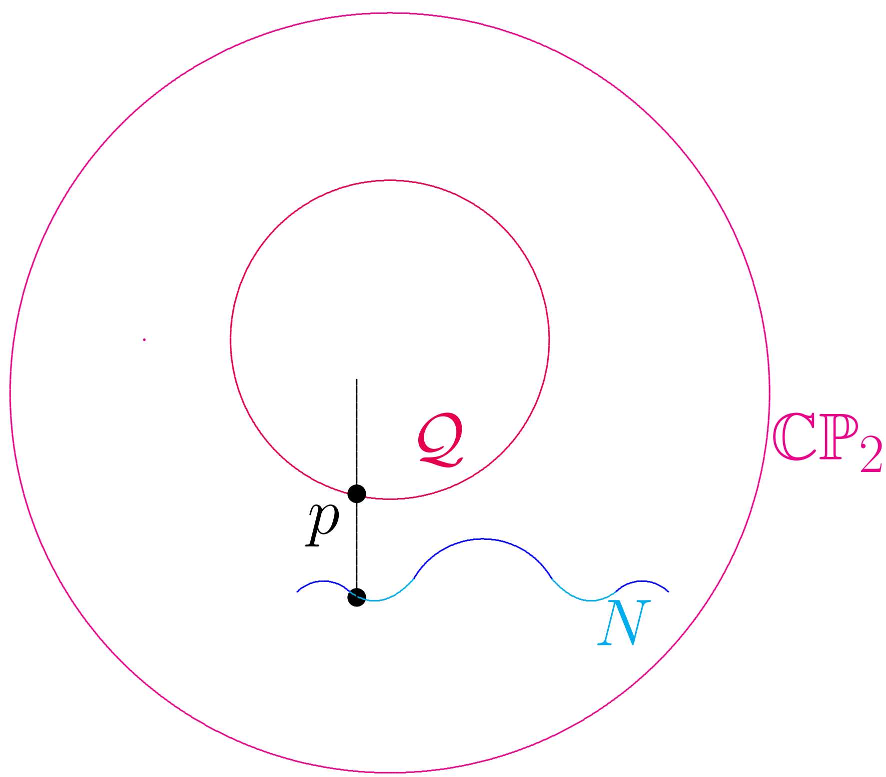

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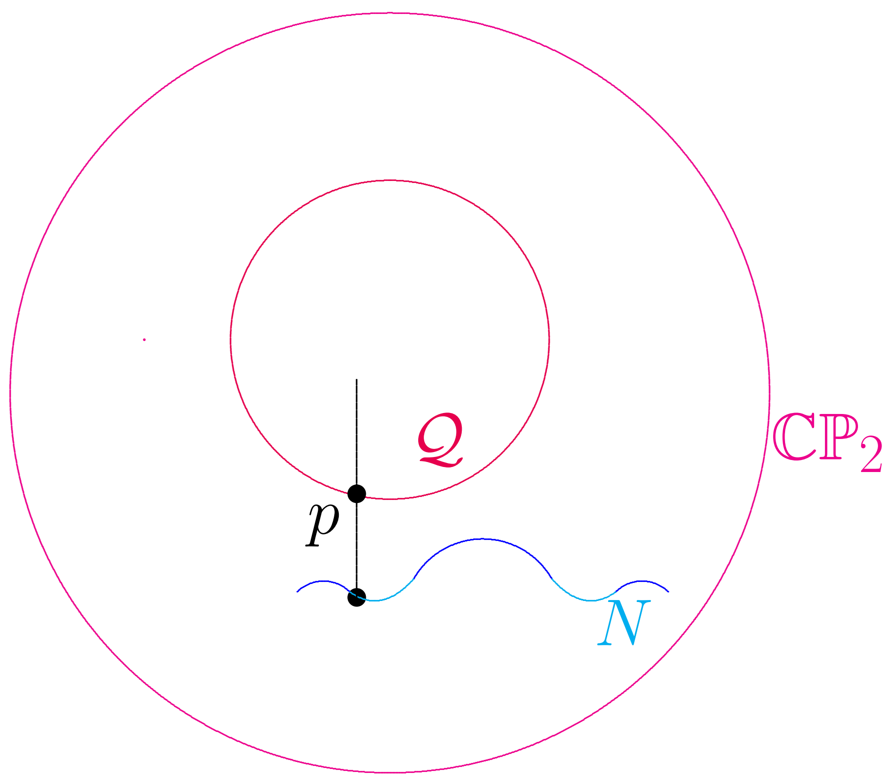

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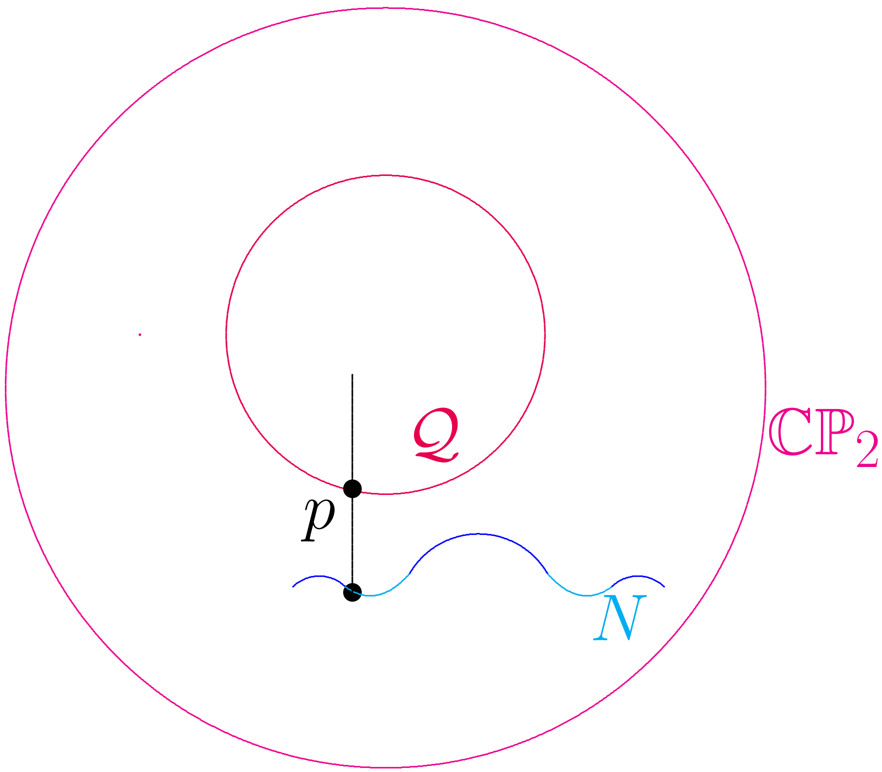

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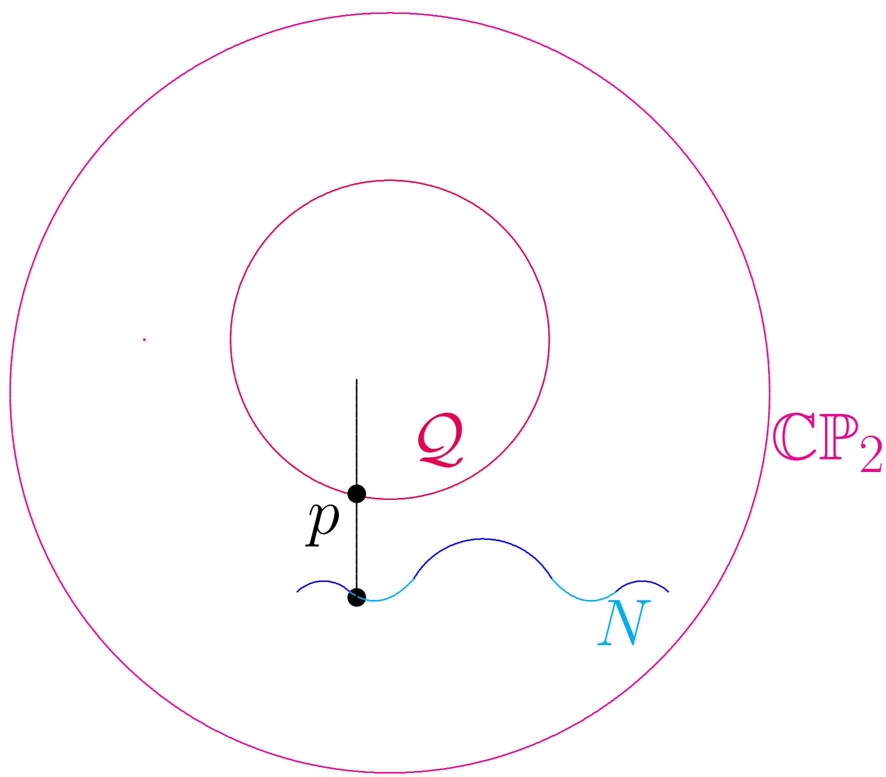

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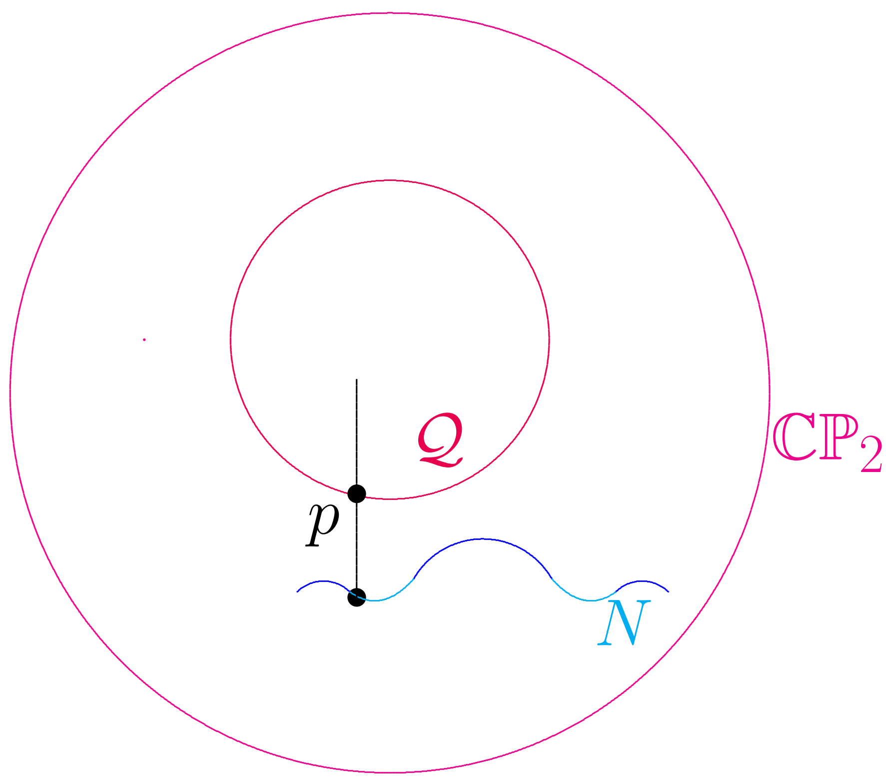

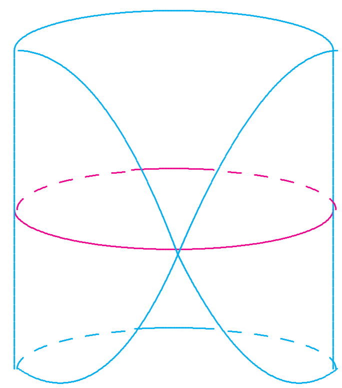

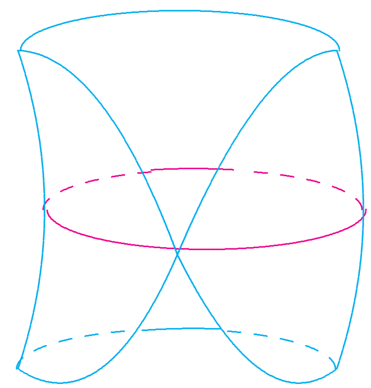

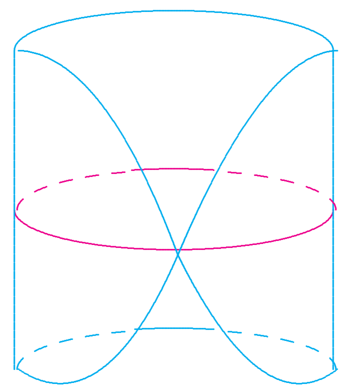

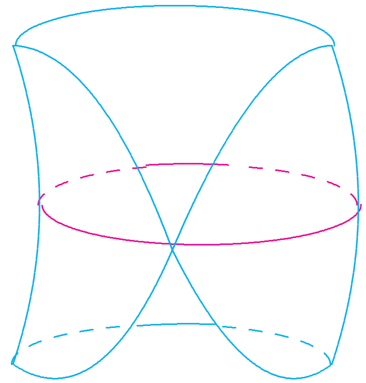

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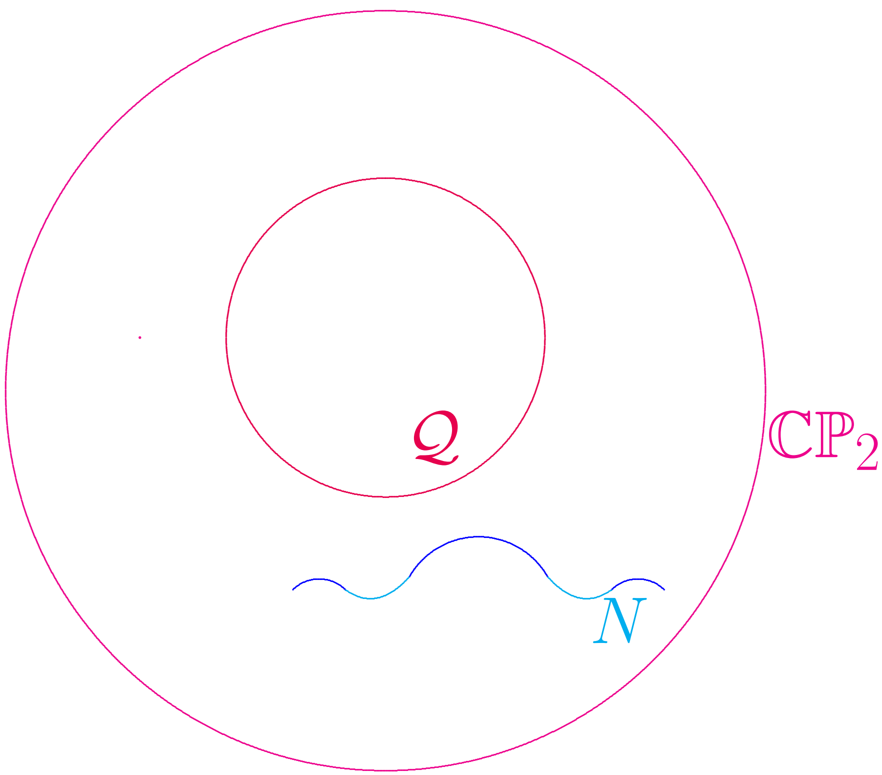

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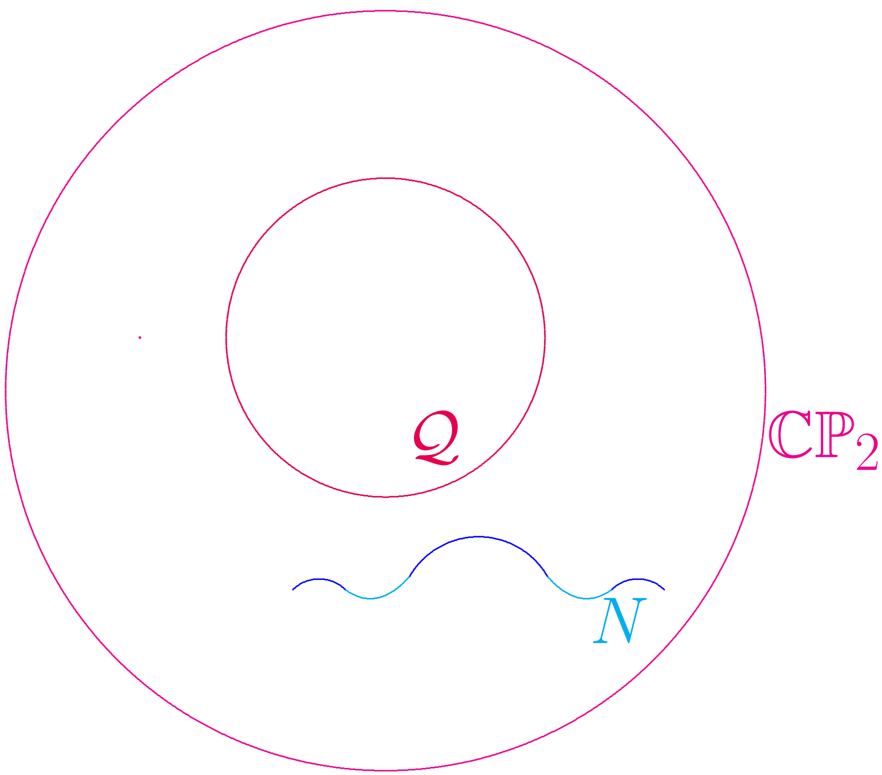

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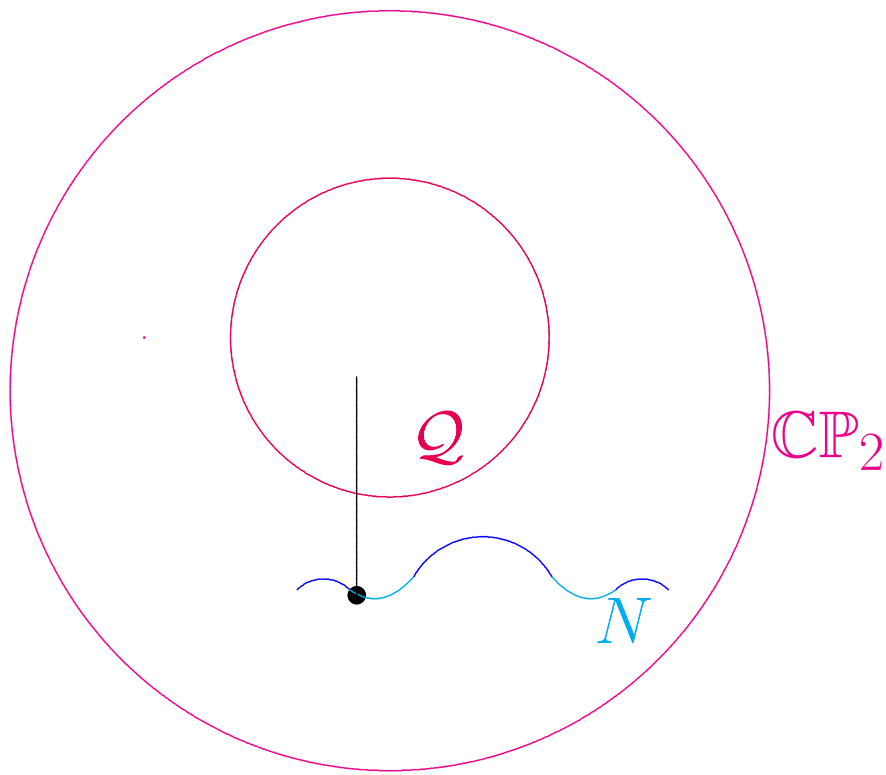

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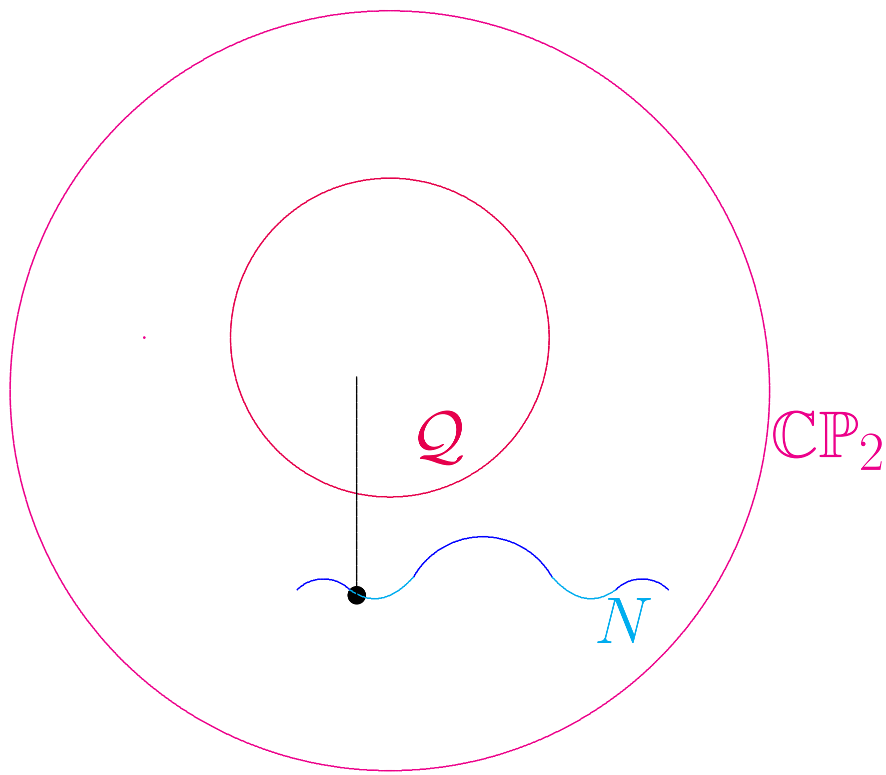

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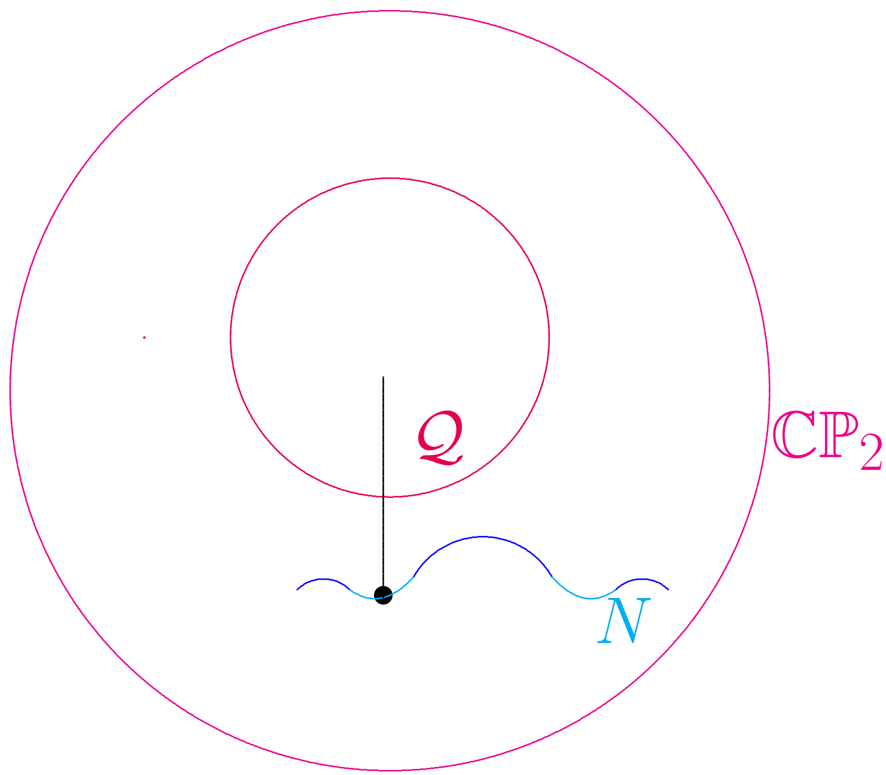

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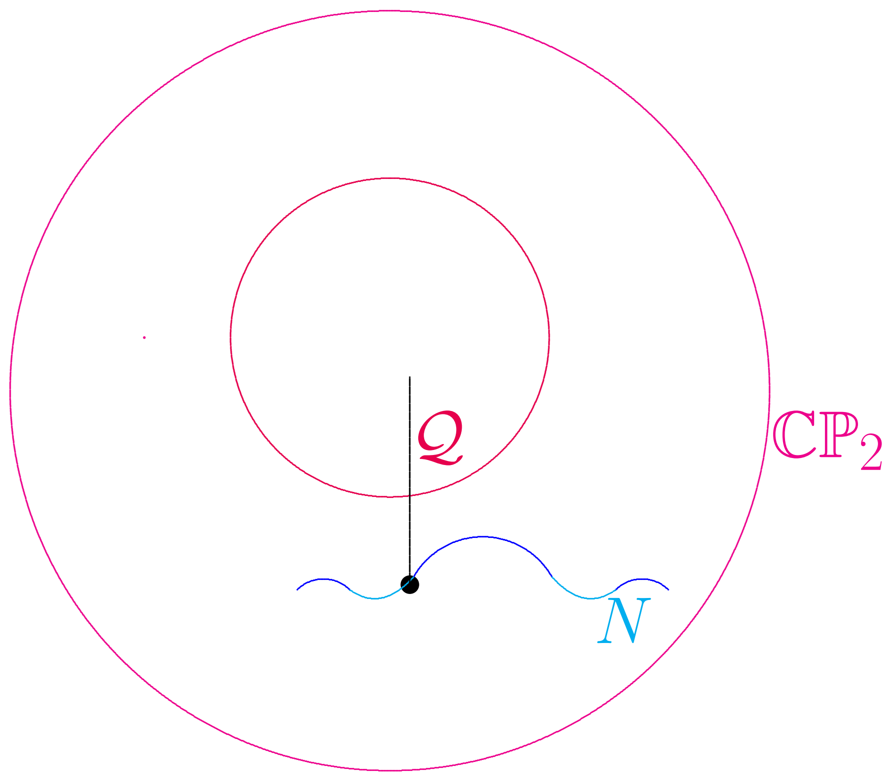

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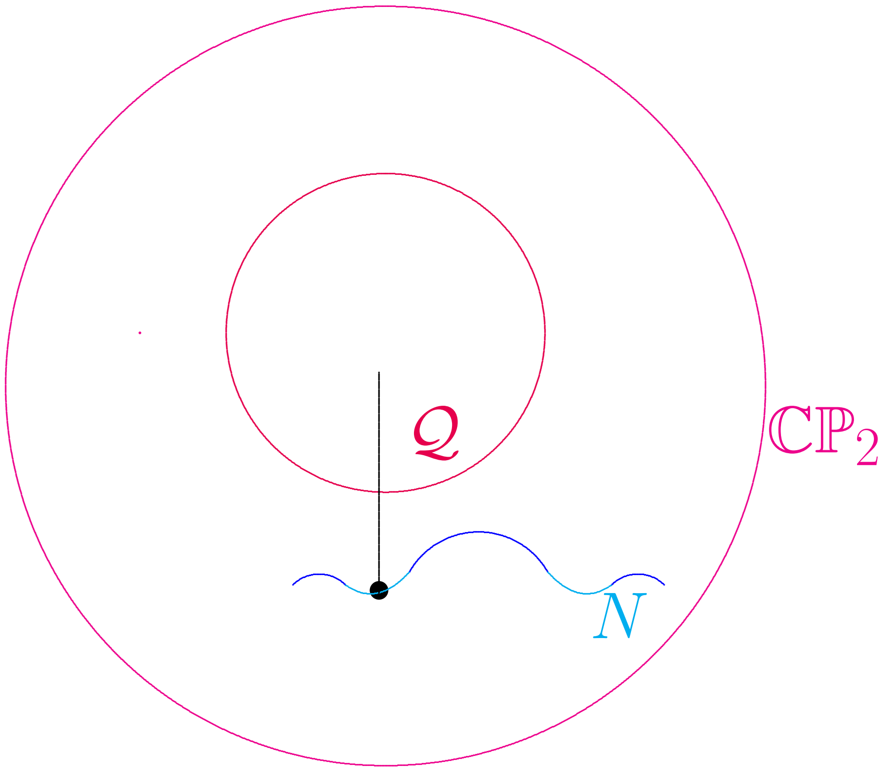

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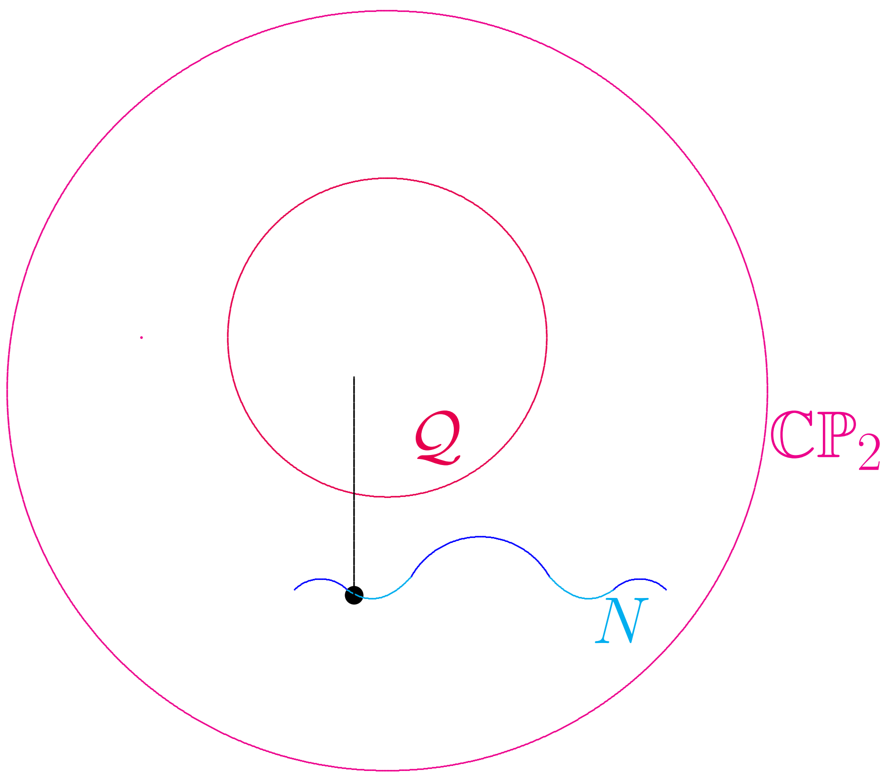

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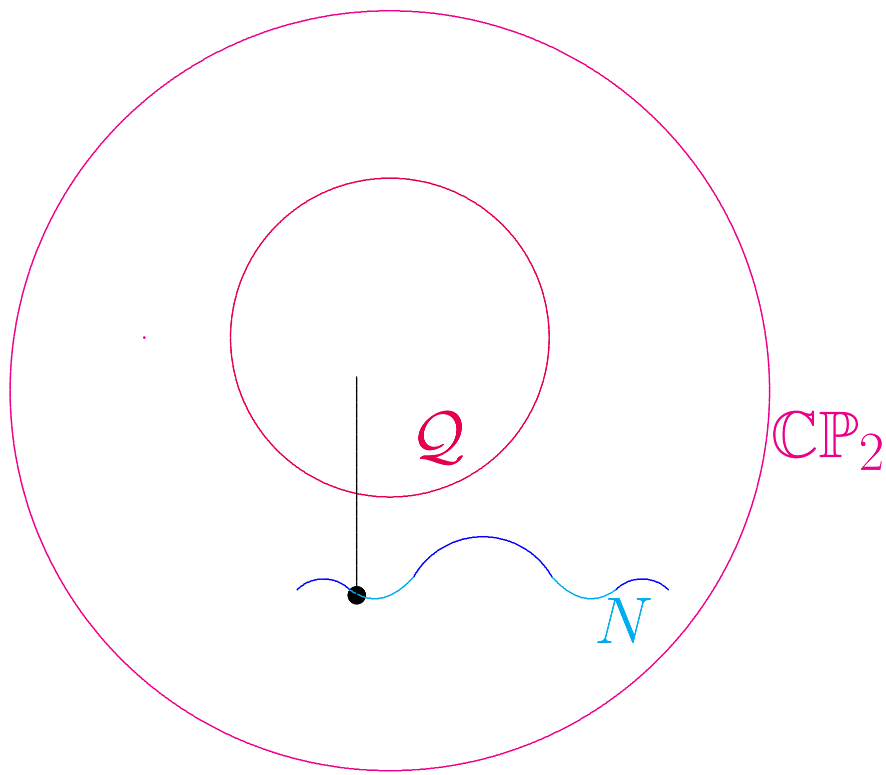

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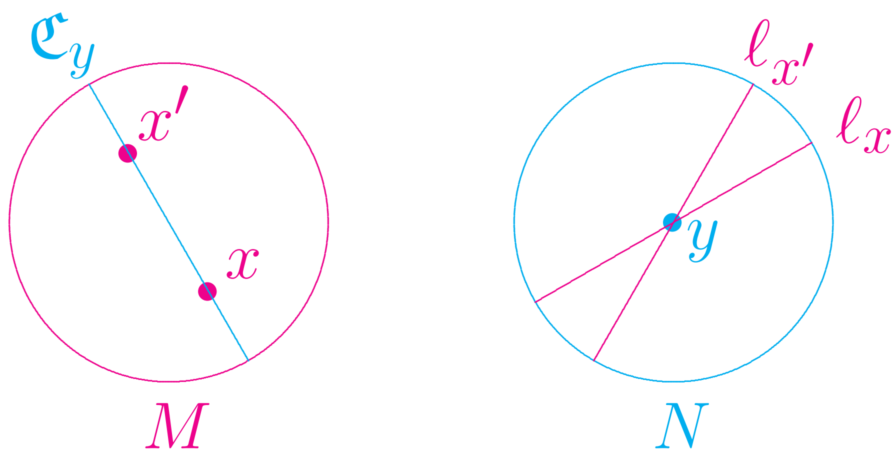

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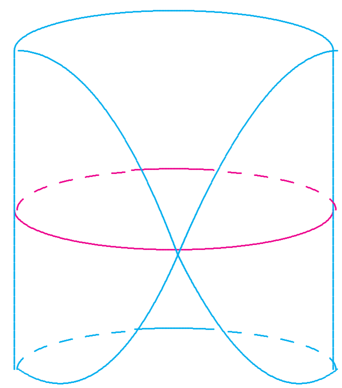

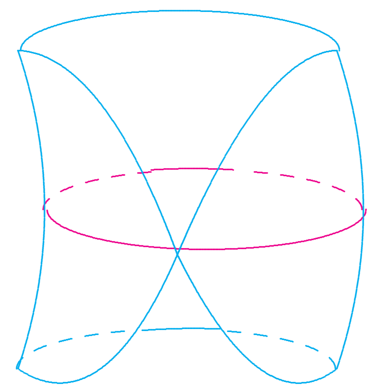

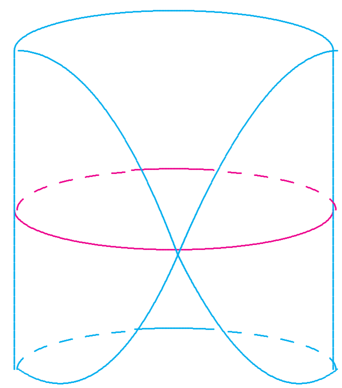

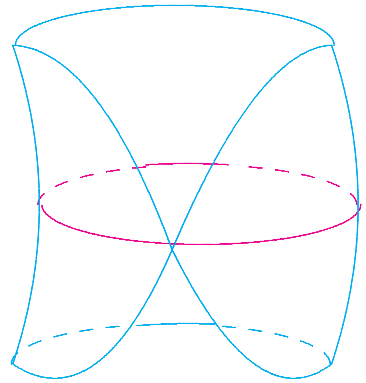

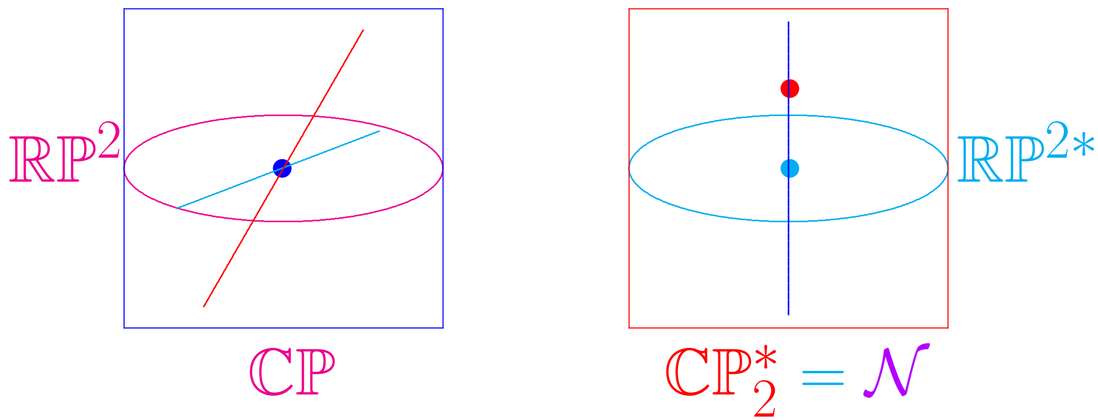
$\mathbb{RP}^2$
$\mathbb{CP}$
$\mathbb{RP}^{2*}$
$\mathbb{CP}^*_2 = \mathcal{N}$

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End, Part VII