Zoll Manifolds,

Complex Surfaces, &

Holomorphic Disks, IV

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Autumn School on Holomorphic Disks
Schloss Rauischholzhausen, November 16, 2018

Joint work with

Lionel Mason
Oxford University

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Zoll Manifolds and Complex Surfaces
J. Diff. Geom. 347 (2002) 453–535.

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Zoll Metrics, Branched Covers,
and Holomorphic Disks,
Comm. An. Geom. 18 (2010) 475–502.

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Definition. *Zoll projective structure $[\nabla]$ on M is the projective equivalence class of some torsion-free affine connection ∇ for which the image of each maximally-extended geodesic is a simple closed curve.*

Theorem A. *Let* $[\nabla]$ *be Zoll projective structure on a compact surface* M^2. *If*

$$\pi_1(M) \neq 0,$$

there is a diffeomorphism

$$\Phi : M \xrightarrow{\approx} \mathbb{RP}^2$$

such that $[\nabla] = [\Phi^* \triangledown]$, *where* $\triangledown$ *is the Levi-Civita connection of the standard, constant curvature Riemannian metric* h *on* $\mathbb{RP}^2$.

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Similarly for Zoll metrics.

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Notice that only remaining case is $M = S^2 \dots$

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Now for a more geometric reformulation. . .

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$\longleftrightarrow$ $\{N^2 \subset \mathbb{CP}_2$ *that are "twisted Lagrangian"*$\}$

Proposition. *Let* M^2 *be any surface, and let* $\mathcal{Z}^4 = \mathbb{P}T_{\mathbb{C}}M$ *be its projectivized complexified tangent bundle.*

Then any affine connection ∇ *on* M *determines a rank-2 sub-bundle* $\mathbf{D} \subset T_{\mathbb{C}}\mathcal{Z}$ *with*

$$[C^1(\mathbf{D}), C^1(\mathbf{D})] \subset C^0(\mathbf{D})$$

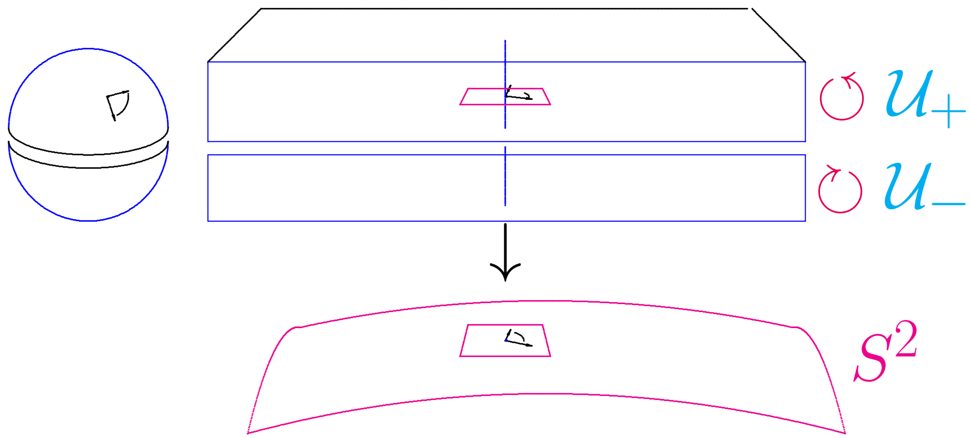
and

$$\dim \mathbf{D}_z \cap \overline{\mathbf{D}}_z = \begin{cases} 0 & \text{if } z \notin \mathbb{P}TM, \\ 1 & \text{if } z \in \mathbb{P}TM. \end{cases}$$

Moreover, two connections ∇ *and* $\hat{\nabla}$ *give rise to the same* $\mathbf{D}$ *iff they are projectively equivalent.*

Blow down $\mathbb{P}TM$:

• If $M \approx S^2$, $\mathbb{P}TM$ divides $\mathcal{Z}^4$ into two components $\mathcal{U}_\pm$:



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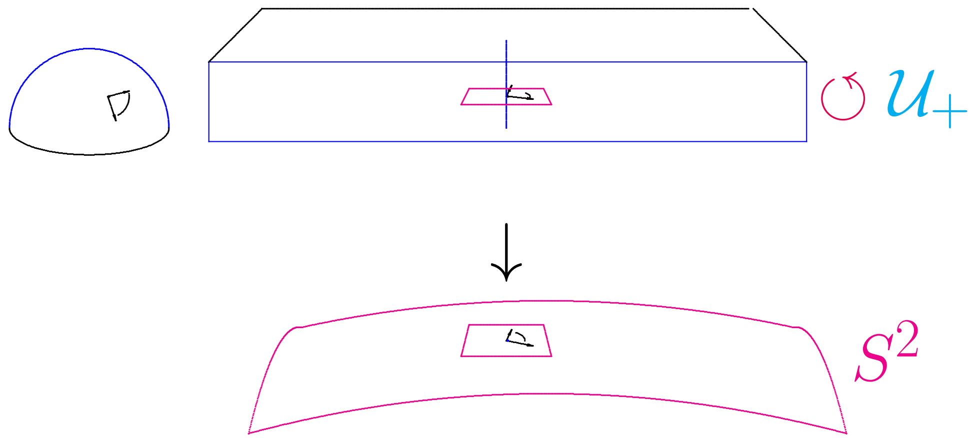

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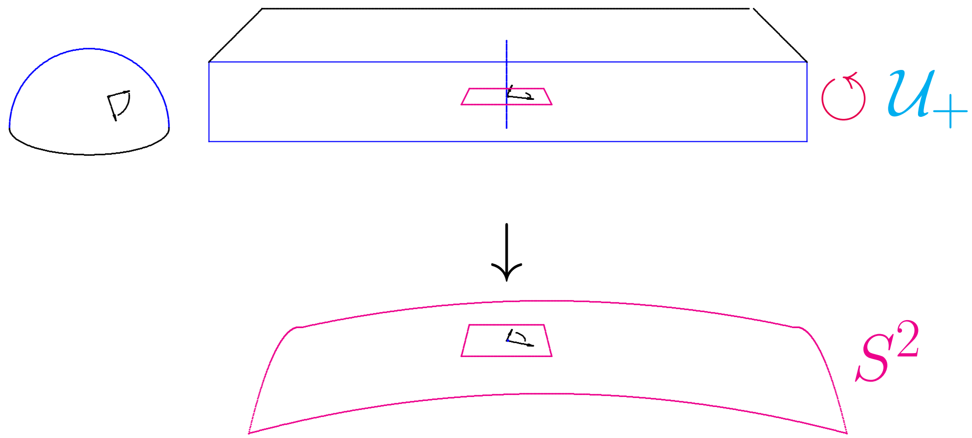

Let $\mathcal{Z}_+ = \mathcal{U}_+ \cup \mathbb{P}TM$, and let $\mathcal{N}$ be obtained from $\mathcal{Z}_+$ by collapsing $\partial\mathcal{Z}_+ = \mathbb{P}TM$ to N.

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$[\nabla]$	J	Integrability Theorem
C^{14}	C^4	Newlander-Nirenberg (1957)
C^{10}	C^2	Malgrange (1968)
C^3	Lipschitz	Hill-Taylor (2002)

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- $\pi_1(\mathcal{N}) = 0$, $b_2(\mathcal{N}) = 1$.
- $N^2 \subset \mathcal{N}^4$ totally real.
- If $M \approx S^2$, family of holomorphic disks D^2 with $\partial D^2 \subset N$. Interiors foliate $\mathcal{N} - N$.

Lemma (Yau)**.**

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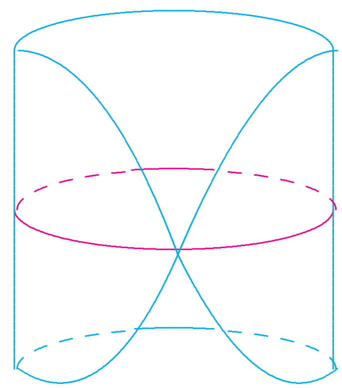
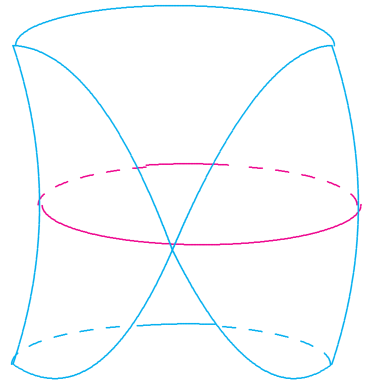
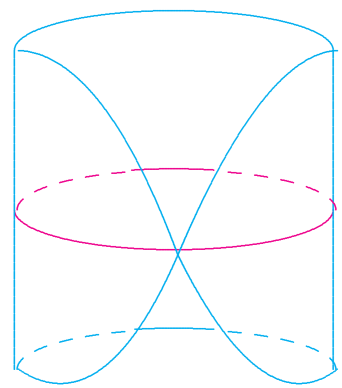
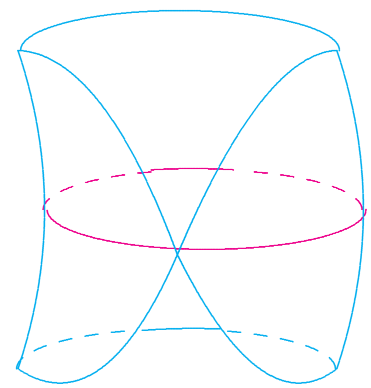
Moreover, $\pm\omega$ *determines the metric* g *up to an overall multiplicative constant.*

If embedding $N \hookrightarrow \mathbb{CP}_2$ perturbed, disks survive. . .









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Back in 2002, we took a low-tech route...

Theorem. *Let* N *be any embedding of* $\mathbb{RP}^2$ *into* $\mathbb{CP}_2$ *which is* C^{2k+5} *close to the standard one. Let* $\{\ell_x \mid x \in M\}$ *be the family of circles in* N *which bound constructed holomorphic disks in* $\mathbb{CP}_2$*. For each* $y \in N$*, set*

$$\mathfrak{C}_y = \{x \in M \mid y \in \ell_x\}.$$

Then there is a unique C^k *Zoll projective structure* $[\nabla]$ *on* $M \approx S^2$ *for which every* $\mathfrak{C}_y$ *is a geodesic.*

If embedding $N \hookrightarrow \mathbb{CP}_2$ perturbed, disks survive. . .

By 2010, we were using better tools. . .

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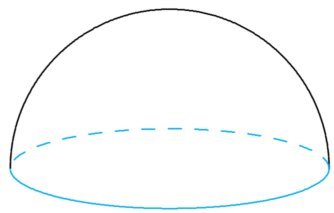

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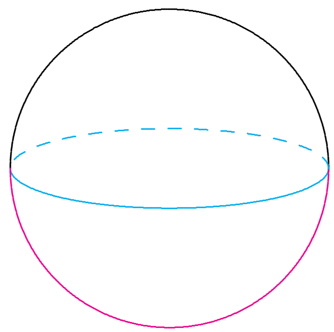

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$$\begin{array}{ccccc} E & = & N & \cup_{\nu} & \overline{N} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{CP}_1 & = & D & \cup_{\partial D} & \overline{D} \end{array}$$

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Equals $+1$ in our case:

$$E \cong \mathcal{O}(1).$$

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$$h^1(CP_1, \mathcal{O}(1)) = 0$$
$$h^0(CP_1, \mathcal{O}(1)) = 2$$

cf. Kodaira's Theorem
on deformation of complex submanifolds

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(Forsternic, Gromov, et al.)
Perturbation of holomorphic disks.

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Perturbation of holomorphic disks.

Our disks Fredholm regular, & index 1 $\Longrightarrow$ moduli space of disks is smooth 2-manifold.

Theorem. *Let* $N \hookrightarrow \mathbb{CP}_2$ *be a totally real embedding of* $\mathbb{RP}^2$ *which arises from a* $C^{k,\alpha}$ *projective structure* $[\nabla]$ *on* $M \approx S^2$, $k \geq 3$, $\alpha \in (0,1)$. *Then there is a* $C^{k+1,\alpha}$ *Riemannian metric* g *on* M *whose Levi-Civita connection* ∇ *belongs to the projective class* $[\nabla]$ *iff (modulo* $PSL(3,\mathbb{C})$*) the surface* N *avoids the conic* $\mathcal{Q}$ *defined by*

$$ {z_1}^2 + {z_2}^2 + {z_3}^2 = 0, $$

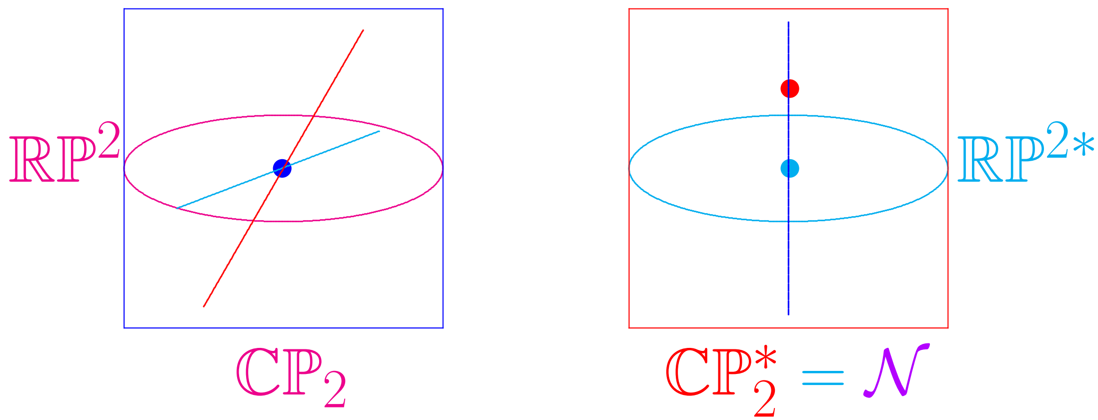
and is Lagrangian with respect to the sign-ambiguous symplectic structure $\pm\omega$ *on* $\mathbb{CP}_2 - \mathcal{Q}$ *defined by*

$$ \pm \Im m \left(\frac{z_1\, dz_2 \wedge dz_3 + z_2\, dz_3 \wedge dz_1 + z_3\, dz_1 \wedge dz_2}{({z_1}^2 + {z_2}^2 + {z_3}^2)^{3/2}} \right). $$

Definition. *A compact connected smoothly embedded 2-manifold* $N \subset\subset \mathbb{CP}_2$ *will be called a* docile surface *relative to* $\mathcal{Q}$ *if*

- N *is a totally real submanifold of* $\mathbb{CP}_2$*;*
- N *is disjoint from the conic* $\mathcal{Q}$*; and*
- N *is transverse to each tangent projective line of the conic* $\mathcal{Q}$*.*

Theorem (LM 2010). *Let* $N \subset \mathbb{CP}_2$ *be any docile surface, and let* M *denote the moduli space of all holomorphic disks in* $(\mathbb{CP}_2, N)$ *which represent the generator of* $H_2(\mathbb{CP}_2, N) \cong \mathbb{Z}$. *Then* M *is diffeomorphic to* S^2. *The interiors of these disks foliate* $\mathbb{CP}_2 - N$, *and the intersection pattern of their boundaries defines a unique Zoll projective structure* $[\nabla]$ *on* M. *Moreover, the reference conic* $\mathcal{Q}$ *induces a specific conformal structure* $[g]$ *on* M, *and there is a unique* $\triangledown \in [\nabla]$ *which is a Weyl connection for the conformal class* $[g]$.



$\mathbb{RP}^2$
$\mathbb{CP}_2$
$\mathbb{RP}^{2*}$
$\mathbb{CP}_2^* = \mathcal{N}$

End, Part IV