# Einstein Metrics, Harmonic Forms, and Symplectic Four-Manifolds

Claude LeBrun
Stony Brook University

**Abstract**

If $M$ is the underlying smooth oriented 4-manifold of a Del Pezzo surface, we consider the set of Riemannian metrics $h$ on $M$ such that $W^+(\omega, \omega) > 0$, where $W^+$ is the self-dual Weyl curvature of $h$, and $\omega$ is a non-trivial self-dual harmonic 2-form on $(M, h)$. While this open region in the space of Riemannian metrics contains all the known Einstein metrics on $M$, we show that it contains no others. Consequently, it contributes exactly one connected component to the moduli space of Einstein metrics on $M$.

## 1 Introduction

Given a smooth compact 4-manifold $M$, one would like to completely understand the moduli space $\mathscr{E}(M)$ of the Einstein metrics it carries. Recall that an *Einstein metric* [4] means a Riemannian metric $h$ which has constant Ricci curvature, in the sense that it solves the Einstein equation

$$r = \lambda h$$

where $r$ is the Ricci tensor of $h$ and $\lambda$ is a real number, called the *Einstein constant* of $h$. The *moduli space* $\mathscr{E}(M)$ of Einstein metrics on $M$ is by definition the quotient of the set of Einstein metrics by the action of the group $\mathcal{D}\!\mathit{iff}(M) \times \mathbb{R}^\times$ of self-diffeomorphisms and constant rescalings. For simplicity, we may give $\mathscr{E}(M)$ the quotient topology induced by the $C^\infty$-topology on the space of smooth metric tensors; however, it is worth noting that, for reasons of elliptic regularity [13], this coincides [1] with the metric topology induced by the Gromov-Hausdorff distance between unit-volume Einstein metrics.

While our understanding of this problem is rather limited for general 4-manifolds, there are specific cases where our knowledge is quite complete.

In particular, if $M$ is the 4-torus, or $K3$, or a compact real or complex-hyperbolic 4-manifold, the Einstein moduli space $\mathscr{E}(M)$ is known to be *connected* [4, 5, 20]. This should be contrasted with the pattern that predominates in higher dimensions, where Einstein moduli spaces are typically disconnected, and indeed often have infinitely many connected components [7, 8, 34].

This article will explore related uniqueness questions for Einstein metrics on the small but important class of smooth compact 4-manifolds that arise as *Del Pezzo surfaces*. These 4-manifolds are characterized [10] by two properties: they admit Einstein metrics with $\lambda > 0$, and they also admit symplectic structures. Up to diffeomorphism, there are exactly ten such manifolds, namely $S^2 \times S^2$ and the nine connected sums $\mathbb{CP}_2 \# m\overline{\mathbb{CP}}_2$, $m = 0, 1, \ldots, 8$. The known Einstein metrics on these spaces all have $\lambda > 0$, and our objective here is to completely characterize these known Einstein metrics by a curvature condition. To this end, notice that, with their standard orientations, each of these 4-manifolds $M$ has $b_+(M) = 1$. This is equivalent to saying that, for any Riemann metric $h$ on any of these compact oriented 4-manifolds, there is, up to an overall multiplicative constant, a unique non-trivial self-dual harmonic 2-form $\omega$. We can therefore consider those Riemannian metrics $h$ on $M$ which satisfy the curvature inequality

$$W^+(\omega, \omega) > 0 \tag{1}$$

at every point of $M$, where $W^+$ is the self-dual Weyl tensor of $h$. Note that, whenever $b_+(M) = 1$, this condition only depends on the metric $h$, since the harmonic form $\omega$ is then uniquely determined up to a non-zero multiplicative constant. Our first main result is the following:

**Theorem A.** *Let $(M, h)$ be a compact oriented Einstein 4-manifold with $b_+ = 1$, and suppose that condition (1) holds at every point of $M$. Then $M$ is diffeomorphic to a Del Pezzo surface, in such a way that $h$ becomes*

- *a Kähler-Einstein metric with $\lambda > 0$; or*
- *a constant multiple of the Page metric on $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$; or*
- *a constant multiple of the Chen-LeBrun-Weber metric on $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$.*

*Conversely, every metric on this list satisfies (1) at every point.*

The proof of this result is given in §2 below, and proceeds by proving that the given Einstein metric must be conformally Kähler. Our method makes strong use of the fact that the second Bianchi identity implies that the self-dual Weyl curvature $W^+$ of an oriented 4-dimensional Einstein manifold is *harmonic* as a bundle-valued 2-form. In fact, the proof does not really require the assumption that $b_+(M) = 1$; it suffices to assume that there is a harmonic self-dual 2-form $\omega$ on $(M, h)$ such that (1) holds at every point.

While the inequality (1) may have a somewhat unfamiliar flavor, it is interestingly related to the positivity of scalar curvature. Indeed, any harmonic self-dual 2-form $\omega$ satisfies the Weitzenböck formula

$$\frac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 + \frac{s}{3}|\omega|^2 = 2W^+(\omega, \omega),$$

and it therefore follows that any metric of scalar curvature $s > 0$ must at least satisfy (1) at *some* points of $M$. Thus, while Theorem A does not provide a complete classification of $\lambda > 0$ Einstein metrics on Del Pezzo surfaces, it does represent a step in that direction.

However, our method requires (1) to hold *everywhere*, rather than just at certain points. This is a strong condition, because it guarantees that the closed self-dual 2-form $\omega$ is nowhere zero, and therefore implies that $(M, \omega)$ is a symplectic manifold. If $h$ is a Riemannian metric on a smooth compact oriented 4-manifold $M$ with $b_+(M) \neq 0$, we will thus say that $h$ is of *symplectic type* if there is a self-dual harmonic 2-form on $(M, h)$ such that $\omega \neq 0$ at every point of $M$. This is actually a conformally invariant condition; if $h$ is of symplectic type, and if $u$ is a smooth positive function, then $uh$ is also of symplectic type. For this reason, it is also natural to say that the conformal class $[h]$ is of symplectic type if there is a self-dual harmonic 2-form on $(M, [h])$ which is everywhere non-zero. This is an *open condition* on $[h]$, in the sense that the set of conformal classes of symplectic type is automatically open [22] in the $C^2$ topology.

Condition (1) is also conformally invariant. Namely, if we replace $h$ with $uh$ for some positive function $u$, then $W^+(\omega, \omega)$ is replaced with $u^{-3}W^+(\omega, \omega)$, thereby leaving the the sign of $W^+(\omega, \omega)$ unaltered at any given point. We will henceforth say that the conformal class $[h]$ is of *positive symplectic type* if, for some choice of $h$-compatible self-dual harmonic 2-form $\omega$, condition (1) holds everywhere on $M$. This obviously implies that $\omega \neq 0$ everywhere, so positive symplectic type implies symplectic type. The condition of positive symplectic type is once again open in the $C^2$ topology.

With these concepts in place, we are now ready to formulate our other main result, which is a direct consequence of Theorem A:

**Theorem B.** *Let $M$ be the underlying smooth compact 4-manifold of a Del Pezzo surface. Let $\mathscr{E}(M)$ denote the the moduli space of Einstein metrics*

*$h$ on $M$, and let $\mathscr{E}^+_\omega(M) \subset \mathscr{E}(M)$ be the open subset arising from Einstein metrics $h$ for which the corresponding conformal classes $[h]$ are of positive symplectic type. Then $\mathscr{E}^+_\omega(M)$ is connected. Moreover, if $b_2(M) \leq 5$, then $\mathscr{E}^+_\omega(M)$ exactly consists of a single point.*

*Proof.* A Del Pezzo surface is by definition a compact complex surface $(M^4, J)$ whose first Chern class is a Kähler class. As complex manifolds, the Del Pezzo surfaces are exactly $\mathbb{CP}_1 \times \mathbb{CP}_1$ and the blow-ups of $\mathbb{CP}_2$ at $m$ distinct points, $0 \leq m \leq 8$, such that no three points are on a line, no six are on a conic, and no eight are on a nodal cubic with one of the given points at the node [11, 24]. When $b_2 \leq 5$, there is consequently, up to biholomorphism, only one Del Pezzo complex structure for each diffeotype, since we can simultaneously move up to four generically located points in the projective plane to standard positions via a suitable projective linear transformation. For larger values of $b_2$, the choice of complex structure instead essentially depends on $2b_2 - 10$ complex parameters; however, the various possibilities still form a single connected family, since the set of prohibited configurations of $m = b_2 - 1$ points in $\mathbb{CP}_2$ is a finite union of complex hypersurfaces, and so has real codimension 2.

Given a Del Pezzo surface $(M, J)$ with fixed complex structure, there is always a $\lambda > 0$ Einstein metric $h$ which can be written as $h = s^{-2}g$ for a $J$-compatible extremal Kähler metric $g$ with scalar curvature $s > 0$. In most cases, one can simply take $h = g$, so that $h$ is a Kähler-Einstein metric. By a result of Tian [26, 32], a Del Pezzo surface $(M, J)$ admits a $J$-compatible Kähler-Einstein metric iff it has reductive automorphism group. This excludes only two Del Pezzo surfaces, namely the ones diffeomorphic to $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$ and $\mathbb{CP}_2\#2\overline{\mathbb{CP}}_2$; these two do not admit Kähler-Einstein metrics, but they nonetheless admit conformally Kähler, $\lambda > 0$ Einstein metrics, known as the Page and Chen-LeBrun-Weber metrics [10, 12, 27], respectively. In [23], it was then shown that any conformally Kähler, Einstein metric on a compact complex surface is either Kähler-Einstein, or else is isometric to a constant multiple of one of these two special metrics. In particular, up to complex automorphisms and constant rescalings, there is exactly one conformally Kähler, Einstein metric for each Del Pezzo complex structure. Theorem A therefore tells us that the moduli space $\mathscr{E}^+_\omega(M)$ can be identified with the moduli space of Del Pezzo complex structures. Since we have seen that the latter is always pathwise connected, and moreover consists of a single point when $b_2(M) \leq 5$, the claim follows. □

We now conclude this introduction with a consequence of Theorem B:

**Corollary 1.** *For any Del Pezzo surface $M$, $\mathscr{E}^+_\omega(M)$ is exactly a connected component of $\mathscr{E}(M)$.*

Indeed, it suffices to prove that the path-connected space $\mathscr{E}^+_\omega(M)$ is both open and closed in $\mathscr{E}(M)$. Since $\mathscr{E}(M)$ has the quotient topology, the fact that it is open follows from the fact that the set of metrics with positive symplectic conformal class is open and invariant under the action of $\mathscr{D}\!\mathit{iff}(M) \times \mathbb{R}^\times$. On the other hand, it is also closed, because, except in cases where $\mathscr{E}^+_\omega(M)$ is now known to be a single point, the Einstein metrics in question are all Kähler, and requiring that a Riemannian metric carry a parallel almost-complex structure is a closed condition.

## 2 Harmonic Self-Dual Weyl Curvature

Recall that we say that a conformal class $[h]$ on an a compact oriented 4-manifold $M$ is of *symplectic type* if there is a harmonic self-dual 2-form $\omega$ on $(M,h)$ such that $\omega \neq 0$ everywhere on $M$. This is indeed a conformally invariant condition, because the Hodge star operator is conformally invariant; moreover, it is an open condition [22] with respect to the $C^2$ topology. Since any self-dual 2-form $\omega$ satisfies

$$\omega \wedge \omega = \omega \wedge \star\omega = |\omega|^2_h \; d\mu_h,$$

it follows that an appropriate $\omega$ is actually a symplectic form on $M$ if $[h]$ is of symplectic type. Assuming this, the conformally related metric $g \in [h]$ given by $g = 2^{-1/2}|\omega|_h h$ is then an *almost-Kähler metric*, in the sense that $g$ is related to the symplectic form $\omega$ via $g = \omega(\cdot, J\cdot)$ for a unique almost-complex structure $J$ on $M$. For our purposes, the important point is that, in dimension 4, the almost-Kähler condition is equivalent to saying that $\omega$ is harmonic and self-dual with respect to $g$, and that $|\omega|^2_g \equiv 2$.

While our primary aim here is to learn something about Einstein metrics, we will more generally focus on oriented Riemannian 4-manifolds $(M,h)$ with *harmonic self-dual Weyl curvature*, in the sense that $\delta W^+ := -\nabla \cdot W^+ = 0$. When $h$ is Einstein, this property holds, as a consequence of the second Bianchi identity. However, we will see in due course that $\delta W^+ = 0$ is in general much weaker than the Einstein condition.

When $[h]$ is of symplectic type, it will prove profitable to study this equation from the point of view of the conformally related almost-Kähler metric $g$. This is quite tractable, because the divergence-free condition on a section of $\odot^2_0\Lambda^+$ is conformally invariant, albeit [28] *with an unexpected* *conformal weight*. In practice, this means that if $h = f^2 g$ has the property that $\delta W^+ = 0$, then $g$ will instead have the property that $\delta(fW^+) = 0$. For us, the important point is that this then implies a Weitzenböck formula

$$0 = \nabla^*\nabla(fW^+) + \frac{s}{2} fW^+ - 6fW^+ \circ W^+ + 2f|W^+|^2 I \tag{2}$$

for $fW^+$, considered as a section of $\mathrm{End}(\Lambda^+)$; cf. [12, 15, 28].

To exploit this effectively, we will need the following identity:

**Lemma 1.** *Any 4-dimensional almost-Kähler manifold satisfies*

$$\langle W^+, \nabla^*\nabla(\omega \otimes \omega)\rangle = [W^+(\omega, \omega)]^2 + 4|W^+(\omega)|^2 - sW^+(\omega, \omega)$$

*at every point.*

*Proof.* First notice that the oriented Riemannian 4-manifold $(M, g)$ satisfies

$$\Lambda^+ \otimes \mathbb{C} = \mathbb{C}\omega \oplus K \oplus \overline{K},$$

where $K = \Lambda^{2,0}_J$ is the canonical line bundle of the almost-complex manifold $(M, J)$. Locally choosing a unit section $\varphi$ of $K$, we thus have

$$\nabla\omega = \alpha \otimes \varphi + \bar{\alpha} \otimes \bar{\varphi}$$

for a unique 1-form $\alpha \in \Lambda^{1,0}_J$, since $\nabla_{[a}\omega_{bc]} = 0$ and $\omega^{bc}\nabla_a \omega_{bc} = 0$. If

$$\circledast : \Lambda^+ \times \Lambda^+ \to \odot^2_0 \Lambda^+$$

denotes the symmetric trace-free product, we therefore have

$$(\nabla_e \omega) \circledast (\nabla^e \omega) = 2|\alpha|^2 \varphi \circledast \bar{\varphi} = -\frac{1}{4}|\nabla\omega|^2 \omega \circledast \omega$$

and we thus deduce that

$$\begin{aligned}
\langle W^+, \nabla^*\nabla(\omega \otimes \omega)\rangle &= 2W^+(\omega, \nabla^*\nabla\omega) - 2W^+(\nabla_e\omega, \nabla^e\omega) \\
&= 2W^+(\omega, \nabla^*\nabla\omega) + \frac{1}{2}|\nabla\omega|^2 W^+(\omega, \omega) \\
&= 2W^+(\omega, 2W^+(\omega) - \frac{s}{3}\omega) + \left[W^+(\omega, \omega) - \frac{s}{3}\right] W^+(\omega, \omega) \\
&= -\frac{2}{3} sW^+(\omega, \omega) + 4|W^+(\omega)|^2 + \left[W^+(\omega, \omega) - \frac{s}{3}\right] W^+(\omega, \omega) \\
&= [W^+(\omega, \omega)]^2 + 4|W^+(\omega)|^2 - sW^+(\omega, \omega)
\end{aligned}$$

where we have used the Weitzenböck formula

$$0 = \nabla^*\nabla\omega - 2W^+(\omega) + \frac{s}{3}\omega$$

for the harmonic self-dual 2-form $\omega$, as well as the associated key identity

$$\frac{1}{2}|\nabla\omega|^2 = W^+(\omega,\omega) - \frac{s}{3}$$

resulting from the fact that $|\omega|^2 \equiv 2$. □

Plugging this into our Weitzenböck formula (2) and integrating by parts, we thus see that whenever a compact almost-Kähler 4-manifold $(M, g, \omega)$ satisfies $\delta(fW^+) = 0$, we then automatically have

$$\begin{aligned}
0 &= \int_M \Big\langle \Big(\nabla^*\nabla fW^+ + \frac{s}{2}fW^+ - 6fW^+\circ W^+ + 2f|W^+|^2 I\Big), \omega\otimes\omega\Big\rangle d\mu \\
&= \int_M \Big[\langle W^+, \nabla^*\nabla(\omega\otimes\omega)\rangle + \frac{s}{2}W^+(\omega,\omega) - 6|W^+(\omega)|^2 + 2|W^+|^2|\omega|^2\Big] f\ d\mu \\
&= \int_M \Big[\Big([W^+(\omega,\omega)]^2 + 4|W^+(\omega)|^2 - sW^+(\omega,\omega)\Big) \\
&\qquad\qquad + \frac{s}{2}W^+(\omega,\omega) - 6|W^+(\omega)|^2 + 4|W^+|^2\Big] f\ d\mu \\
&= \int_M \Big[[W^+(\omega,\omega)]^2 - \frac{s}{2}W^+(\omega,\omega) - 2|W^+(\omega)|^2 + 4|W^+|^2\Big] f\ d\mu\ .
\end{aligned}$$

In other words, letting $W^+(\omega)^\perp$ denote the component of $W^+(\omega)$ perpendicular to $\omega$, any compact almost-Kähler manifold $(M, g, \omega)$ with $\delta(fW^+) = 0$ satisfies the identity

$$\int_M sW^+(\omega,\omega) f\ d\mu = 8\int_M \left(|W^+|^2 - \frac{1}{2}|W^+(\omega)^\perp|^2\right) f\ d\mu\ . \qquad (3)$$

To proceed further, we will now need another algebraic observation:

**Lemma 2.** *Any 4-dimensional almost-Kähler manifold satisfies*

$$|W^+|^2 - \frac{1}{2}|W^+(\omega)^\perp|^2 \geq \frac{3}{8}\left[W^+(\omega,\omega)\right]^2$$

*at every point, and equality can only hold at points where* $W^+(\omega)^\perp = 0$.

*Proof.* If $A = [A_{jk}]$ is any symmetric trace-free $3 \times 3$ matrix, the fact that $A_{33} = -(A_{11} + A_{22})$ implies that

$$\sum_{jk} A_{jk}^2 \geq 2A_{21}^2 + A_{11}^2 + A_{22}^2 + A_{33}^2 = 2A_{21}^2 + \frac{3}{2}A_{11}^2 + 2(\frac{A_{11}}{2} + A_{22})^2$$

and we therefore conclude that

$$|A|^2 \geq 2A_{21}^2 + \frac{3}{2}A_{11}^2.$$

If we now let $A$ represent $W^+ : \Lambda^+ \to \Lambda^+$ with respect to an orthogonal basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$ for $\Lambda^+$ such that $\omega = \sqrt{2}\varepsilon_1$ and $W^+(\omega)^\perp \propto \varepsilon_2$, this inequality becomes

$$|W^+|^2 \geq |W^+(\omega)^\perp|^2 + \frac{3}{8}\left[W^+(\omega,\omega)\right]^2$$

which not only proves the desired inequality, but shows that it is actually strict whenever $\omega$ is not an eigenvector of $W^+$. □

Combining (3) with Lemma 2 now yields the global inequality

$$\int_M sW^+(\omega,\omega)f\ d\mu \geq 3\int_M \left[W^+(\omega,\omega)\right]^2 f\ d\mu, \tag{4}$$

with equality only if $W^+(\omega)^\perp \equiv 0$. It thus follows that

$$0 \geq \int_M W^+(\omega,\omega)\left(W^+(\omega,\omega) - \frac{s}{3}\right)\ f\ d\mu.$$

However, since $\frac{1}{2}|\nabla\omega|^2 = W^+(\omega,\omega) - \frac{s}{3}$ for any almost-Kähler 4-manifold, this proves the following:

**Proposition 1.** *Let $(M^4, g, \omega)$ be a compact almost-Kähler manifold, and suppose that, for some positive function $f$, the conformally related metric $h = f^2 g$ has harmonic self-dual Weyl curvature. Then $(M, g, \omega)$ satisfies the inequality*

$$0 \geq \int_M W^+(\omega,\omega)|\nabla\omega|^2 f\ d\mu. \tag{5}$$

This has an interesting immediate consequence:

**Proposition 2.** *Let $(M^4, g, \omega)$ be a compact connected almost-Kähler manifold with $W^+(\omega,\omega) \geq 0$, and suppose that the conformally related metric $h = f^2 g$ satisfies $\delta W^+ = 0$. Then either $g$ is a Kähler metric with scalar curvature $s = c/f$ for some constant $c > 0$, or else $g$ satisfies $W^+ \equiv 0$, and so is an anti-self-dual metric.*

*Proof.* Recall that $f > 0$ by convention, and that $W^+(\omega, \omega) \geq 0$ by assumption. Thus (5) implies that

$$\int_M W^+(\omega, \omega)|\nabla\omega|^2 f \, d\mu = 0,$$

so that $\nabla\omega = 0$ wherever $W^+(\omega, \omega) \neq 0$. If $U \subset M$ is the open subset where $W^+(\omega, \omega) \neq 0$, the restriction of $g$ to $U$ is therefore Kähler. On the other hand, by hypothesis, $g$ satisfies $\delta(fW^+) = 0$. However, for any Kähler manifold of real dimension 4, $W^+$ is the trace-free part of $(s/4)\omega \otimes \omega$, where the scalar curvature $s$ satisfies $s = 3W^+(\omega, \omega)$. It follows that $d[fW^+(\omega, \omega)] = 0$ on $U$. By continuity, we therefore have $d[fW^+(\omega, \omega)] = 0$ on the closure $\overline{U}$ of $U$, too. On the other hand, $fW^+(\omega, \omega) \equiv 0$ on $M - \overline{U}$, so we also have $d[fW^+(\omega, \omega)] = 0$ on the open set $M - \overline{U}$. Hence $d[fW^+(\omega, \omega)] = 0$ on all of $M$. Since $M$ is connected, it follows that $fW^+(\omega, \omega) = c/3$ for some non-negative constant $c \geq 0$. If $c > 0$, $M = U$, and $(M, g)$ is a Kähler manifold, with $s = 3W^+(\omega, \omega) = c/f$. On the other hand, if $c = 0$, we have $W^+(\omega, \omega) \equiv 0$, and therefore have equality in (4). However, this implies that $W^+(\omega)^\perp \equiv 0$, and (3) therefore implies that $W^+ \equiv 0$, as claimed. $\square$

As a special case, we therefore obtain the following key result:

**Theorem 1.** *Let $(M, h)$ be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$. If the conformal class $[h]$ is of positive symplectic type, then $h = s^{-2}g$ for a unique Kähler metric $g$ of scalar curvature $s > 0$. Conversely, if $g$ is any Kähler metric of positive scalar curvature, the conformally related metric $h = s^{-2}g$ satisfies $\delta W^+ = 0$.*

*Proof.* To say that $[h]$ is of positive symplectic type means that there is a self-dual harmonic 2-form $\omega$ on $(M, h)$ such that $W^+(\omega, \omega) > 0$ at every point of $M$. Rescaling $h$ to make $\omega$ have constant norm $\sqrt{2}$ results in an almost-Kähler metric $\hat{g}$ such that $h = \hat{f}^2\hat{g}$ for some positive function $\hat{f}$. If $h$ satisfies $\delta W^+ = 0$, the almost-Kähler metric $\hat{g}$ then satisfies $W^+(\omega, \omega) > 0$ and $\delta(\hat{f}W^+) = 0$, so Proposition 2 then tells us that $\hat{g}$ is actually Kähler, with scalar curvature $\hat{s} = c/\hat{f}$ for some positive constant $c$. In particular, $M$ admits a Kähler metric with positive scalar curvature, and Yau's vanishing theorem [35] for the geometric genus therefore implies that $b_+(M) = 1$. Thus the choice of $\omega$ is in fact unique up to an overall multiplicative constant, and the choice of $\hat{g}$ is therefore determined up to constant rescalings. But if, for a positive constant $a$, we replace $\hat{g}$ with $g = a^2\hat{g}$, we must also replace $\hat{f}$ with $f = a^{-1}\hat{f}$; and note that the scalar curvature of $g$ is then $s = a^{-2}\hat{s}$. Since $\hat{f} = c\hat{s}^{-1}$, we then have $h = \hat{f}^2\hat{g} = c^2\hat{s}^{-2}\hat{g} = c^2(a^2 s)^{-2}(a^{-2}g) =$

$(ca^{-3})^2s^{-2}g$. This shows that setting $a = \sqrt[3]{c}$ results in a Kähler metric $g$ such that $h = s^{-2}g$, and moreover shows that this choice yields the only Kähler metric with this property.

Conversely [12], if $g$ is a Kähler metric with $s > 0$, $s^{-1}W^+$ is parallel, so that, in particular, we have $\delta(s^{-1}W^+) = 0$. Thus $h = s^{-2}g$ satisfies $\delta W^+ = 0$, as promised. □

Theorem A is now a straightforward consequence. Indeed, since the second Bianchi identity implies that any Einstein metric on an oriented 4-manifold satisfies $\delta W^+ = 0$, Theorem 2 tells us that every Einstein metric $h$ of positive symplectic type must be conformally Kähler. Moreover, since the conformal class $[h]$ contains a representative $g$ with $s > 0$, the constant scalar curvature $4\lambda$ of $h$ must [33] be positive, too. Theorem A therefore follows from the known classification [23] of conformally Kähler, Einstein metrics on compact 4-manifolds.

## 3 Almost-Kähler Manifolds Revisited

As an added bonus, the results of §2 also have interesting consequences in the narrower context of almost-Kähler geometry; for related work, see [6]. Our main such application is the following:

**Theorem 2.** *Let $(M, g, \omega)$ be a compact almost-Kähler 4-manifold with non-negative scalar curvature and harmonic self-dual Weyl tensor:*

$$s \geq 0, \qquad \delta W^+ = 0.$$

*Then $(M, g, \omega)$ is a constant-scalar-curvature Kähler manifold.*

*Proof.* For any almost-Kähler manifold,

$$W^+(\omega, \omega) = \frac{s}{3} + \frac{1}{2}|\nabla\omega|^2$$

so that the hypothesis $s \geq 0$ implies $W^+(\omega, \omega) \geq 0$. Proposition 2, with $f = 1$, therefore tells us that $(M, g)$ is Kähler, with scalar curvature $s = c/f = c$ for some positive constant $c$, or else that $W^+ \equiv 0$. In the latter case, we then have $0 = 3W^+(\omega, \omega) \geq s \geq 0$, so $s \equiv 0$, and hence $|\nabla\omega|^2 = 2W^+(\omega, \omega) - 2s/3 = 0$. Thus $(M, g)$ is constant-scalar-curvature Kähler, even in the exceptional case. □

Conversely, any constant-scalar-curvature Kähler manifold of real dimension 4 satisfies $\delta W^+ = 0$, independent of the sign of $s$. While the study of "cscK" (constant-scalar-curvature Kähler) metrics on compact complex surfaces is an active area of ongoing research, many existence results are already available [3, 14, 17, 19, 29, 31]. However, we should emphasize that the non-negativity of the scalar curvature plays a crucial role in Theorem 2. For example, there exist many compact almost-Kähler manifolds with $W^+ \equiv 0$ which are not Kähler. Indeed, such examples can be obtained [16] by deforming scalar-flat Kähler metrics through anti-self-dual conformal classes, and then conformally rescaling to make $|\omega| \equiv \sqrt{2}$. Examples of this type automatically have $s \leq 0$, with $s < 0$ on an open dense subset.

Since any Einstein 4-manifold satisfies $\delta W^+ = 0$, Theorem 2 provides a new proof of Sekigawa's breakthrough result [30] on the Goldberg conjecture:

**Corollary 2** (Sekigawa)**.** *Every compact almost-Kähler Einstein 4-manifold with non-negative Einstein constant is Kähler-Einstein.*

This fact provided a useful guidepost en route to the present results.

The proof of Theorem 2 in fact still works if we merely impose the ostensibly weaker hypothesis that $s + tW^+(\omega, \omega) \geq 0$ for some constant $t \geq 0$, since any such hypothesis will imply that $W^+(\omega, \omega) \geq 0$, with $s = 0$ if equality holds. In particular, one reaches exactly the same conclusion if we merely assume that the so-called star-scalar curvature

$$s^* = s + |\nabla\omega|^2 = \frac{s}{3} + 2W^+(\omega, \omega)$$

is non-negative:

**Proposition 3.** *Let $(M, g, \omega)$ be a compact almost-Kähler 4-manifold with non-negative star-scalar curvature and harmonic self-dual Weyl tensor:*

$$s^* \geq 0, \qquad \delta W^+ = 0.$$

*Then $(M, g, \omega)$ is a constant-scalar-curvature Kähler manifold.*

Kirchberg [18] has elsewhere investigated almost-Kähler 4-manifolds with harmonic Weyl tensor and positive star-scalar curvature. Since the hypothesis $\delta W = 0$ is equivalent to $\delta W^+ = \delta W^- = 0$, and is therefore stronger than the hypothesis $\delta W^+ = 0$ of Proposition 3, we can recover many of Kirchberg's results from our own. For example, we can deduce the following clarification of [18, Corollary 3.13]:

**Corollary 3.** *Let $(M, g, \omega)$ be a compact almost-Kähler 4-manifold with non-negative scalar curvature and harmonic Weyl tensor:*

$$s \geq 0, \qquad \delta W^{+} = \delta W^{-} = 0.$$

*Then $(M^4, g, J)$ is either a Kähler-Einstein manifold with $\lambda \geq 0$, or else is locally symmetric, with universal cover $(\tilde{M}, \tilde{g})$ isometric to the Riemannian product of two constant-curvature surfaces, where one factor is a 2-sphere.*

*Proof.* Theorem 2 tells us that $(M, g, J)$ is Kähler and has constant scalar curvature. But since the entire Weyl tensor is actually assumed to be harmonic, the second Bianchi identity also tells us that

$$\nabla_{[c} r_{d]b} = \nabla_a W^a{}_{bcd} + \frac{1}{6} g_{b[c} \nabla_{d]} s = 0,$$

so that the covariant derivative $\nabla r$ of the Ricci tensor must therefore be completely symmetric. Decomposing $\otimes^3 \Lambda^1_{\mathbb{C}}$ into $\otimes^3 (\Lambda^{1,0} \oplus \Lambda^{0,1})$, we thus have

$$\nabla_\kappa r_{\mu\bar{\nu}} = \nabla_{\bar{\nu}} r_{\mu\kappa} = 0 \qquad \text{and} \qquad \nabla_{\bar{\kappa}} r_{\mu\bar{\nu}} = \nabla_\mu r_{\bar{\kappa}\bar{\nu}} = 0.$$

The Ricci tensor of our Kähler manifold is therefore parallel, and the primitive part $\mathring{\rho} \in \Lambda^{-}$ of its Ricci form must therefore be parallel, too. If $\mathring{\rho} = 0$, $(M, g, J)$ is Kähler-Einstein. Otherwise, the holonomy of $(M, g)$ fixes both $\omega$ and $\mathring{\rho}$, and so must be contained in $U(1) \times U(1) \subset U(2) \subset SO(4)$. In the latter case, the de Rham splitting theorem [4] then implies that the universal cover $(\tilde{M}, \tilde{g})$ of $(M, g)$ is a Riemannian product $(M_1, g_1) \times (M_2, g_2)$ of two complete, constant-curvature Riemann surfaces; and if $g$ is not Einstein, and therefore not flat, the assumption that $s \geq 0$ then forces at least one factor $(M_j, g_j)$ to have positive Gauss curvature. □

The Narasimhan-Seshadri theorem [25] provides a complete existence theory for the non-Einstein metrics of Corollary 3. Indeed, a compact complex manifold $(M^4, J)$ admits such a locally-product Kähler metric iff it is a geometrically ruled complex surfaces that arises as the projectivization of a polystable rank-2 holomorphic vector bundles over a compact complex curve. For related results, see [2, 9, 21].

**Acknowledgments.** The author would like to thank Tedi Draghici for subsequently pointing out some of his own related work, and the anonymous referee for suggesting ways to streamline and clarify the exposition. This work was supported in part by NSF grant DMS-1205953.

## References


[1] M. T. Anderson, *The $L^2$ structure of moduli spaces of Einstein metrics on 4-manifolds*, Geom. Funct. Anal., 2 (1992), pp. 29–89.

[2] V. Apostolov and C. Tønnesen-Friedman, *A remark on Kähler metrics of constant scalar curvature on ruled complex surfaces*, Bull. London Math. Soc., 38 (2006), pp. 494–500.

[3] C. Arezzo and F. Pacard, *Blowing up and desingularizing constant scalar curvature Kähler manifolds*, Acta Math., 196 (2006), pp. 179–228.

[4] A. L. Besse, *Einstein manifolds*, vol. 10 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1987.

[5] G. Besson, G. Courtois, and S. Gallot, *Entropies et rigidités des espaces localement symétriques de courbure strictement négative*, Geom. and Func. An., 5 (1995), pp. 731–799.

[6] D. E. Blair and T. Draghici, *Remarks on Weyl curvature in almost Kähler geometry*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 52(100) (2009), pp. 241–249.

[7] C. Böhm, *Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces*, Invent. Math., 134 (1998), pp. 145–176.

[8] C. P. Boyer, K. Galicki, and J. Kollár, *Einstein metrics on spheres*, Ann. of Math. (2), 162 (2005), pp. 557–580.

[9] D. Burns and P. De Bartolomeis, *Stability of vector bundles and extremal metrics*, Invent. Math., 92 (1988), pp. 403–407.

[10] X. X. Chen, C. LeBrun, and B. Weber, *On conformally Kähler, Einstein manifolds*, J. Amer. Math. Soc., 21 (2008), pp. 1137–1168.

[11] M. Demazure, *Surfaces de del Pezzo, II, III, IV, V*, in Séminaire sur les Singularités des Surfaces, vol. 777 of Lecture Notes in Mathematics, Berlin, 1980, Springer, pp. 21–69.

[12] A. Derdziński, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compositio Math., 49 (1983), pp. 405–433.

[13] D. M. DeTurck and J. L. Kazdan, *Some regularity theorems in Riemannian geometry*, Ann. Sci. École Norm. Sup. (4), 14 (1981), pp. 249–260.

[14] S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom., 62 (2002), pp. 289–349.

[15] M. J. Gursky, *Four-manifolds with $\delta W^+ = 0$ and Einstein constants of the sphere*, Math. Ann., 318 (2000), pp. 417–431.

[16] I. Kim, *Almost-Kähler Anti-Self-Dual Metrics*, PhD thesis, State University of New York at Stony Brook, 2014.

[17] J. Kim, C. LeBrun, and M. Pontecorvo, *Scalar-flat Kähler surfaces of all genera*, J. Reine Angew. Math., 486 (1997), pp. 69–95.

[18] K.-D. Kirchberg, *Some integrability conditions for almost Kähler manifolds*, J. Geom. Phys., 49 (2004), pp. 101–115.

[19] C. LeBrun, *Scalar-flat Kähler metrics on blown-up ruled surfaces*, J. Reine Angew. Math., 420 (1991), pp. 161–177.

[20] ——, *Einstein metrics and Mostow rigidity*, Math. Res. Lett., 2 (1995), pp. 1–8.

[21] ——, *Polarized 4-manifolds, extremal Kähler metrics, and Seiberg-Witten theory*, Math. Res. Lett., 2 (1995), pp. 653–662.

[22] ——, *Yamabe constants and the perturbed Seiberg-Witten equations*, Comm. An. Geom., 5 (1997), pp. 535–553.

[23] ——, *On Einstein, Hermitian 4-manifolds*, J. Differential Geom., 90 (2012), pp. 277–302.

[24] Y. I. Manin, *Cubic Forms: Algebra, Geometry, Arithmetic*, North-Holland Publishing Co., Amsterdam, 1974. Translated from the Russian by M. Hazewinkel.

[25] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2), 82 (1965), pp. 540–567.

[26] Y. Odaka, C. Spotti, and S. Sun, *Compact moduli spaces of Del Pezzo surfaces and Kähler-Einstein metrics.* e-print arXiv:1210.0858 [math.DG], 2012.

[27] D. Page, *A compact rotating gravitational instanton*, Phys. Lett., 79B (1979), pp. 235–238.

[28] R. Penrose and W. Rindler, *Spinors and space-time. Vol. 2*, Cambridge University Press, Cambridge, 1986. Spinor and twistor methods in space-time geometry.

[29] Y. Rollin and M. Singer, *Construction of Kähler surfaces with constant scalar curvature*, J. Eur. Math. Soc. (JEMS), 11 (2009), pp. 979–997.

[30] K. Sekigawa, *On some* 4*-dimensional compact Einstein almost Kähler manifolds*, Math. Ann., 271 (1985), pp. 333–337.

[31] Y. Shu, *Compact complex surfaces and constant scalar curvature Kähler metrics*, Geom. Dedicata, 138 (2009), pp. 151–172.

[32] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math., 101 (1990), pp. 101–172.

[33] N. Trudinger, *Remarks concerning the conformal deformation of metrics to constant scalar curvature*, Ann. Scuola Norm. Sup. Pisa, 22 (1968), pp. 265–274.

[34] C. van Coevering, *Sasaki-Einstein 5-manifolds associated to toric 3-Sasaki manifolds*, New York J. Math., 18 (2012), pp. 555–608.

[35] S. T. Yau, *On the curvature of compact Hermitian manifolds*, Invent. Math., 25 (1974), pp. 213–239.

Department of Mathematics, State University of New York, Stony Brook, NY 11794-3651 USA

e-mail: claude@math.sunyb.edu

Keywords: Einstein metric, Del Pezzo surface, Weyl curvature, moduli space, harmonic form, Kähler, almost-Kähler, symplectic.

MSC classification: 53C25 (Primary), 14J26, 32J15, 53C55, 53D05.