MAT 568: Differential Geometry

Homework # 2

Due Thursday, 10/29/15

1. Let (M, g) be a connected Riemannian manifold, let $\varpi : \widetilde{M} \to M$ be its universal cover, and let $\tilde{g} = \varpi^* g$ be the pull-back metric on $\widetilde{M}$. Prove that $(\widetilde{M}, \tilde{g})$ is complete iff (M, g) is complete.

2. A Riemannian manifold n-manifold (M, g) is said to be *flat* if its curvature tensor is identically zero. Show that a Riemannian manifold is flat if and only if every point $p \in M$ has a neighborhood which is isometric to an open set in Euclidean space $\mathbb{R}^n$.

Hint: Consider the exponential map.

3. Define $(n+1)$-dimensional Minkowski space $\mathbb{R}^{n,1}$ to be $\mathbb{R}^{n+1} = \{(x^0, x^1, \ldots x^n)\}$ equipped with the pseudo-Riemannian metric

$$h = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2.$$

Define *hyperbolic n-space* $\mathcal{H}^n$ to be the connected component $x^0 > 0$ of the hyperboloid of two sheets given by

$$-(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = -1,$$

equipped with the Riemannian metric $g := j^*h$, where $j : \mathcal{H}^n \hookrightarrow \mathbb{R}^{n,1}$ is the inclusion map. Then define *stereographic projection* coordinates $(u^1, \ldots, u^n)$ on $\mathcal{H}^n$ by requiring that $(x^0, x^1, \ldots x^n)$, $(-1, 0, \ldots, 0)$, and $(0, u^1, \ldots, u^n)$ be collinear in $\mathbb{R}^{n+1}$.

(a) Show that the coordinates $(u^1, \ldots, u^n)$ provide a diffeomorphism between $\mathcal{H}^n$ and the unit ball $\varrho < 1$ in $\mathbb{R}^n$, where

$$\varrho := \sqrt{\sum_{k=1}^{n} (u^k)^2}$$

is the Euclidean radius in $\mathbb{R}^n$, in such a manner that g becomes

$$g = \left(\frac{2}{1-\varrho^2}\right)^2 \sum_{k=1}^{n} du^k \otimes du^k.$$

This is called the *Poincaré ball model* of hyperbolic space. Notice that g just the Euclidean metric times a function. In particular, two vectors are orthogonal with respect to the Poincaré-ball metric iff they are orthogonal with respect to the Euclidean metric.

(b) By means of a symmetry argument, prove that every straight line through the origin $(u^1, \ldots, u^n) = (0, \ldots, 0)$ is a geodesic in the Poincaré model of $\mathcal{H}^n$.

(c) By means of another symmetry argument, show that any parallel vector field along such a geodesic is tangent to a Euclidean plane though the origin. If Y is a Jacobi field along a unit speed geodesic ray $\gamma(t)$, $t \geq 0$, where $\gamma(0)$ is the origin, $Y(0) = 0$ and $\langle Y, \gamma'(t)\rangle \equiv 0$, then prove that Y is a parallel vector field times $\varrho/(1-\varrho^2)$.

(d) If Π is any 2-dimensional subspace of the tangent space of the unit ball at the origin, use Jacobi's equation and part (c) to show that the associated sectional curvature of the Poincaré metric is given by

$$K(\Pi) = -1.$$

(e) By applying a composition of translations and an inversion $\vec{v} \mapsto \vec{v}/\|\vec{v}\|^2$, show that the Poincaré ball model is isometric to the upper half-space model

$$\left(\left\{(y^1, \ldots, y^n) \in \mathbb{R}^n \;\middle|\; y^0 > 0\right\}, \frac{(dy^1)^2 + \cdots + (dy^n)^2}{(y^n)^2}\right)$$

of hyperbolic n-space. Next, use this to give a proof of the fact that the isometry group of $(\mathcal{H}^n, g)$ acts transitively— i.e. that any point of $\mathcal{H}^n$ can be carried to any other point by means of an isometry. Then use this, in conjunction with part (d), to show that $\mathcal{H}^n$ has constant sectional curvature -1, and that its curvature tensor is therefore explicitly given by

$$R(W, X, Y, Z) = g(X, Y)g(W, Z) - g(W, Y)g(X, Z).$$

4. Let $f(y)$ be a smooth positive function on some open interval $I \subset \mathbb{R}$, and consider the Riemannian metric

$$g = [f(y)]^2(dx^2 + dy^2)$$

on $(\mathbb{R} \times I) \subset \mathbb{R}^2$.

(a) Argue by symmetry that the curves $x =$ constant are geodesics, and that $\partial/\partial x$ is a Jacobi field along any such geodesic. Then use Jacobi's equation to prove that the Gauss curvature of this metric is given by

$$K = -\frac{(\log f)''}{f^2}.$$

(b) Use the fact that $\partial/\partial x$ is a Killing field to show that $[f(y)]^2\ dx/dt$ is constant along any unit-speed geodesic $(x(t), y(t))$. Now write down the equation stating that $(dx/dt, dy/dt)$ has unit length, and use the "conservation law" we have just found to eliminate t. Conclude that the graph $y(x)$ swept out by a non-vertical geodesic satisfies the equation

$$\frac{dy}{dx} = \pm\sqrt{[aF(y))]^2 - 1}$$

for some constant a. Then express x as an integral in y.

(c) Apply parts (a) and (b) to the special case of $f(y) = 1/y$, thereby once again showing that the upper-half-space model of the hyperbolic plane has Gauss curvature -1, and that that its non-vertical geodesics are the semi-circles orthogonal to the boundary line.

(d) Use a symmetry argument to show that any geodesic in the upper-half-space model of $\mathcal{H}^n$ is contained in a vertical half-plane. Conclude that its geodesics are exactly the semi-circles orthogonal to the boundary hyperplane.

(e) Show that an inversion $\vec{v} \mapsto \vec{v}/\|\vec{v}\|^2$ of $\mathbb{R}^n - \{0\}$ sends every $(n-1)$-dimensional hypersphere[1] to another hypersphere. By taking intersections, deduce that it also takes every circle to a circle. Use this and the isometry you constructed in 3(e) to show that the geodesics of the Poincaré ball metric are exactly the intersections of the ball with Euclidean circles (or line segments) that meet the boundary sphere orthogonally.

[1] Here a *hypersphere* basically means the boundary of some ball of arbitrary Euclidean radius, centered at an arbitrary point in Euclidean space. To keep the above assertion as simple and clean as possible, though, let us also agree that Euclidean hyper-planes are to be considered as hyperspheres here, since they can be constructed as limits of ordinary hyperspheres, by letting the radius and center both go to infinity.

5. Given a smooth function $r = F(z) > 0$, there is a surface of revolution $\Sigma \subset \mathbb{R}^3$ parameterized by

$$(x, y, z) = (F(z)\cos\theta, F(z)\sin\theta, z).$$

(a) Let g be the metric induced on Σ by pulling back the Euclidean metric from $\mathbb{R}^3$. Show that this metric is given by

$$g = \left[1 + \left(\frac{dF}{dz}\right)^2\right] dz^2 + [F(z)]^2 d\theta^2.$$

Then show, by symmetry, that the curves $\theta =$ constant are geodesics of g, and that, along any such geodesic, the vector field $\partial/\partial\theta$ is a Jacobi field.

(b) Set $r = F(z)$ and $v = dr/dz = dF/dz$. Show that Jacobi's equation for the above Jacobi field can be written as

$$\frac{1}{\sqrt{1+v^2}}\frac{d}{dz}\frac{v}{\sqrt{1+v^2}} + Kr = 0,$$

where K is the Gauss curvature. Then show that this can then be transformed into the differential equation

$$\frac{d}{dr}\frac{1}{1+v^2} = 2Kr$$

by considering v as a function of r.

(c) By solving this equation when $K = 0$, describe the most general surface of revolution with vanishing Gauss curvature.

(d) Solve[2] the above equation when $K = +1$, while imposing the "initial condition" that $v = 0$ when $r = 1$. What is the resulting surface of revolution? What would happen if we did not impose the given initial condition?

(e) By solving the above equation, find $z = F^{-1}(r)$ if we instead set $K = -1$ and impose the "initial condition" that $v \to 0$ as $r \to 0$.

The resulting profile curve $r = F(z)$ is called a "tractrix," a Latin term meaning *something that drags*. Show that this curve is characterized by the requirement that, *along its own tangent line*, the distance from the curve

[2] In practice, you will actually be solving for $z = F^{-1}(r)$.

to the z-axis is always 1; thus, it is the curve that would be swept out by a small, heavy object dragged along by a rigid rod attached to a "tractor" moving along the positive z-axis of the rz-plane .

The corresponding surface of revolution Σ is called a *pseudosphere* (or tracticoid). Make a sketch of this surface Σ. By construction, it is a smooth surface of Gauss curvature -1 in $\mathbb{R}^3$. Show, however, that this smooth surface Σ is geodesically incomplete.