Math 531 - Midterm Exam Solutions

Problem 1. Submersions are open maps, hence, F is open. Now let $C \subseteq M$ be a closed set. Since M is compact, C is also compact. So $F(C)$ is compact. So $F(C)$ is closed. Hence, F is an open map, and F is also a closed map. Therefore, $F(M) \subseteq N$ the image of F is both open and closed. Since M is non-empty $F(M)$ is non-empty and since N is connected $F(M) = N$, so F is onto. Since M is compact, so is $F(M)$. So N is compact.

Problem 2. Consider the diffeomorphism $F: \underset{(u,v)}{\mathbb{R}^2} \to \underset{(x,y)}{\mathbb{R}^2}$

$F(u,v) = (2u, v)$. And consider $\tilde{\gamma}: [0,1] \to \mathbb{R}^2$

$\tilde{\gamma}(t) = (\cos(2\pi t), \sin(2\pi t))$. Then $(F \circ \tilde{\gamma}) = \gamma$, and

$$F^* \varphi = \frac{v\, d(2u) - 2u\, dv}{4u^2 + 4v^2} = \frac{1}{2} \frac{v\,du - u\,dv}{u^2+v^2}$$

Now let us switch to polar coordinates in the (u,v)-plane,

$$\left.\begin{aligned} u &= r\cos\theta \\ v &= r\sin\theta \end{aligned}\right\} \quad \begin{aligned} &u^2+v^2 = r^2 \\ &r > 0 \\ &-\pi < \theta < \pi \end{aligned} \quad \left.\begin{aligned} du &= \cos\theta\, dr - r\sin\theta\, d\theta \\ dv &= \sin\theta\, dr + r\cos\theta\, d\theta \end{aligned}\right\}$$

(Note: polar coordinates are valid on the open set $U \subseteq \mathbb{R}^2$, $U = \mathbb{R}^2 \setminus \{\text{origin, negative } x\text{-axis}\}$)

$$F^* \varphi = \frac{1}{2} \frac{v\,du - u\,dv}{u^2+v^2} = \frac{1}{2} \frac{1}{r^2} \Big(-r^2 \sin^2\theta\, d\theta$$

$$r\sin\theta\cos\theta\, dr - r\cos\theta\sin\theta\, dr - r^2\cos^2\theta\, d\theta \Big)$$

$$= -\frac{1}{2}\, d\theta$$

Now, $\int_\gamma \varphi = \int_{F\circ\tilde\gamma} \varphi = \int_{\tilde\gamma} F^*\varphi$ It is now clear that the integral is $-\frac{1}{2}(2\pi) = -\pi$. But to be perfectly rigorous, we must introduce another chart $\tilde{U} \overset{\circ}{\subseteq} \mathbb{R}^2$, $\tilde{U} = \mathbb{R}^2 \setminus \{\text{origin, positive x-axis}\}$

And coordinates $\left.\begin{array}{l} x = \tilde{r}\cos\tilde\theta \\ y = \tilde{r}\sin\tilde\theta \end{array}\right\}$ $\begin{array}{l} \tilde{r} > 0 \\ 0 < \tilde\theta < 2\pi \end{array}$

Clearly on $U \cap \tilde{U} = \mathbb{R}^2 \setminus \{x\text{-axis}\}$ $r = \tilde{r}$, $\theta = \tilde\theta$ and so $d\theta = d\tilde\theta$. Now $\int_{\tilde\gamma} F^*\varphi = \int_{\tilde\gamma_1} F^*\varphi + \int_{\tilde\gamma_2} F^*\varphi$

where $\tilde\gamma_1 : [0,\frac{1}{2}] \to \mathbb{R}^2$ $\tilde\gamma_1(t) = (\cos(2\pi(t-\frac{1}{4})), \sin(2\pi(t-\frac{1}{4})))$

and $\tilde\gamma_2 : [\frac{1}{2}, 1] \to \mathbb{R}^2$ $\tilde\gamma_2(t) = (\cos(2\pi(t-\frac{1}{4})), \sin(2\pi(t-\frac{1}{4})))$

And the image of $\tilde\gamma_1 \subset U$, and the image of $\tilde\gamma_2 \subset \tilde{U}$, so

$$\int_{\gamma} F^*\varphi = \int_{\tilde\gamma_1} F^*\varphi + \int_{\tilde\gamma_2} F^*\varphi = \frac{-1}{2}\left(\int_{\tilde\gamma_1} d\theta + \int_{\tilde\gamma_2} d\tilde\theta\right) = -\frac{1}{2}\Big(\theta(\tilde\gamma_1(\tfrac{1}{2})) - \theta(\tilde\gamma(0))$$

$$+ \tilde\gamma_2(1) - \tilde\gamma_2(\tfrac{1}{2})\Big) = \frac{-1}{2}\left(\frac{\pi}{2} - \frac{-\pi}{2} + \frac{3\pi}{2} - \frac{\pi}{2}\right) = \frac{-1}{2}(2\pi) = -\pi$$

Further, φ is closed since $F^* d\varphi\big|_U = d(F^*\varphi)\big|_U = d(-\frac{1}{2}d\theta)\big|_U = 0$ and similarly on $\tilde{U}$. But F^* is an isomorphism, since F is a diffeo, so $F^*d\varphi = 0 \implies d\varphi = 0$. However, φ is not exact, since if φ were exact then, there would be a function $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ s.t. $\varphi = df$. But then $\int_\gamma \varphi = \int_\gamma df = f(\gamma(1)) - f(\gamma(0)) = 0.$
$\neq -\pi.$

Problem 3. a) $d(F^*\omega) = F^*(d\omega) = F^*(0) = 0$

The first equality follows because d commutes with pullback under a smooth map, the second equality follows since $\omega \in \mathcal{A}^n(N)$ and $\dim N = n$, so $\mathcal{A}^{n+1}(N) = \{0\}$ so $d\omega \in \mathcal{A}^{n+1}(N)$ is automatically 0.

b. The point of the problem is that if we let

$F: \mathbb{R}^3 \setminus \{(0,0,0)\} \longrightarrow S^2$ be given by $F(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z)$ and let $\omega =$ the area form on S^2 induced from $\mathbb{R}^3$, then $F^*\omega = \psi$. Let's expand on this and make the argument rigorous. Let $f: \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = x^2+y^2+z^2-1$. Then $df|_{(x,y,z)} = 2x\,dx + 2y\,dy + 2z\,dz$, so $df|_{(x,y,z)} = 0$ only at the origin. In particular, 0 is a regular value and so $f^{-1}(0) = S^2 \subseteq \mathbb{R}^3$ is an embedded submanifold, and $T_pS^2 \subseteq T_p\mathbb{R}^3$ is equal to $\ker(df|_p)$. Define $\omega(X,Y) =$ signed area of the parallelogram spanned by $X, Y \in T_pS^2 \subseteq T_p\mathbb{R}^3$, or formally, if $X = x^1\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y} + x^3\frac{\partial}{\partial z}$ and $Y = y^1\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + y^3\frac{\partial}{\partial z}$ and $p = (a^1, a^2, a^3)$, then $df_p(X_p) = a^i x^j \delta_{ij} = 0$, $df_p(Y_p) = a^i y^j \delta_{ij} = 0$

$$\omega_p(X_p, Y_p) = \begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ a^1 & a^2 & a^3 \end{vmatrix}$$

(this is actually the volume of the parallelopiped with base the parallelogram spanned by X, Y and height the unit vector (a^1, a^2, a^3) which is orthogonal to X, Y).

ω so defined is clearly a 2-form on T_pS^2 and is smooth in p.
Now we need to check that $F^*\omega = \psi$. We can do this directly, but it is quicker to use a trick.

First, note that $\psi = \frac{1}{\rho^3}(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy) \in \mathcal{A}^2(\mathbb{R}^3)$ is rotationally invariant (we used $\rho = (x^2+y^2+z^2)^{1/2}$), i.e. for any $T \in SO(3)$, $T^*\psi = \psi$. Indeed, for a rotation $T_{z,\theta}$ by angle θ around the z-axis $T^*_{z,\theta}\psi = T^*_{z,\theta}\frac{1}{\rho^3}\left(\underbrace{(x\,dy - y\,dx)}_{r^2 d\theta}\wedge dz + z\,dx\wedge dy\right)$ $= \frac{1}{\rho^3}T^*_{z,\theta}(r^2 d\theta\wedge dz + z\,dx\wedge dy) = \frac{1}{\rho^3}(r^2 d\theta\wedge dz + z\,dx\wedge dy)$, where we used $r = (x^2+y^2)^{1/2}$, and the fact that ρ, r, z are fixed by $T_{z,\theta}$, and $T^*_{z,\theta}d\theta = d\theta$ and $T^*_{z,\theta}(dx\wedge dy) = dx\wedge dy$ since the latter is the area form on $\mathbb{R}^2$, which is preserved by a rotation around the z-axis. Since ψ is unchanged by a cyclic permutation of the axis ψ is invariant under rotations around the x and y axes as well, and hence, rotationally invariant as claimed.

Now, F commutes with rotations, and ω is obviously rotation for $t>0$ invariant, so we can check that $F^*_{(t,0,0)}\omega_{(1,0,0)} = \frac{1}{t^2}dy\wedge dz$ to show that $F^*\omega = \psi$ everywhere. Compute:

$$\left(\frac{1}{t^2}dy\wedge dz\right)(X,Y) = \frac{1}{t^2}(X^2Y^3 - X^3Y^2)$$

$$F^*_{(1,0,0)}\omega_{(1,0,0)}(X,Y) = \omega_{(1,0,0)}(F_{*(t,0,0)}X, F_{*(t,0,0)}Y) = \omega_{(1,0,0)}\left(\frac{X^2}{t}\frac{\partial}{\partial y} + \frac{X^3}{t}\frac{\partial}{\partial z}, \frac{Y^2}{t}\frac{\partial}{\partial y} + \frac{Y^3}{t}\frac{\partial}{\partial z}\right) = \frac{1}{t^2}\begin{vmatrix} 0 & X^2 & X^3 \\ 0 & Y^2 & Y^3 \\ 1 & 0 & 0 \end{vmatrix} = \frac{1}{t^2}(X^2Y^3 - X^3Y^2).$$

Now, since $\psi = F^*\omega$, $\omega \in \mathcal{A}(S^2)$, by part a $d\psi = d(F^*\omega)$ $= F^*(d\omega) = F^*(0) = 0.$

Problem 4.a. Suppose β is exact so $\exists f \in C^\infty(M)$ s.t. $\beta = df$.
Since M is compact, f attains its maximum on M, at say $p \in M$. By elementary calculus $\forall X \in T_pM$ $X(f)|_p = 0$
But $0 = X(f)|_p = df|_p(X) = \beta_p(X)$. So $\beta_p = 0$, contradicting the non-vanishing of β.

b. As we saw in Problem 2, $\exists d\Theta \in \mathcal{A}^1(S^1)$ a non-vanishing 1-form on S^1. (Note, there is no $\Theta \in C^\infty(S^1)$ s.t. the 1-form in question is actually $d\Theta$, but $\Theta = \mathrm{Tan}^{-1}(y/x) \in C^\infty(\mathcal{U})$, $\mathcal{U} = \{(x,y) \in S^1 \mid x \neq 0\}$, and $\tilde{\Theta} = \frac{\pi}{2} - \mathrm{Tan}^{-1}(x/y) \in C^\infty(\tilde{\mathcal{U}})$, $\tilde{\mathcal{U}} = \{(x,y) \in S^1 \mid y \neq 0\}$, and $\mathcal{U} \cap \tilde{\mathcal{U}} = Q_1 \amalg Q_2 \amalg Q_3 \amalg Q_4$ and $\Theta - \tilde{\Theta}|_{Q_i} = C_i$, where C_i is a constant, and Q_i are the 4 quadrants excluding the axes).

Let $\beta \in \mathcal{A}^1(M)$ be given by $F^* d\Theta$, then β is non-vanishing. Indeed, if $p \in M$ and $\beta_p = 0$, then $\forall X \in T_pM$ $\beta_p(X) = 0$, but $0 = \beta_p(X) = F^*_p(d\Theta|_{F(p)})(X) = d\Theta|_{F(p)}(F_{*p}X)$. But F is a submersion, so F_{*p} is onto $T_{F(p)}S^1$, so $\exists X \in T_pM$ s.t. $F_{*p}X = \frac{\partial}{\partial\theta}\Big|_{F(p)}$ and so $d\Theta|_{F(p)}(F_{*p}X) = d\Theta|_{F(p)}\left(\frac{\partial}{\partial\theta}\Big|_{F(p)}\right) = 1$, contradiction.

But if β is non-vanishing, by part a it is not exact.
On the other hand by 3a, β is closed ($d\beta = dF^*d\Theta = F^*d(d\Theta) = 0$ since $d(d\Theta) \in \mathcal{A}^2(S^1) = \{0\}$). Hence, $[\beta]$ is a nonzero element of the space of closed 1-forms modulo the subspace of exact 1-forms, so $[\beta] \neq 0$ in $H^1_{deRham}(M)$.

Problem 5. Let $q \in L \cap M \subset N$. $\exists\, \mathcal{U} \overset{\circ}{\subset} N$ a coordinate neighborhood around q, and coordinates $(x^1, \dots, x^n) : \mathcal{U} \to \mathbb{R}^n$ s.t. $(x^1, \dots, x^m)$ are coordinates on $M \cap \mathcal{U}$ ($n = \dim N$, $m = \dim M$ and $\ell = \dim L$). Also $\exists\, \tilde{\mathcal{U}} \overset{\circ}{\subset} N$, a coordinate neighborhood of q, and coordinates $(\tilde{x}^1, \dots, \tilde{x}^n) : \tilde{\mathcal{U}} \to \mathbb{R}^n$ s.t. $(\tilde{x}^1, \dots, \tilde{x}^\ell)$ are coordinates on $L \cap \tilde{\mathcal{U}}$. Let $V = \mathcal{U} \cap \tilde{\mathcal{U}}$, and restrict the x^i and $\tilde{x}^j$ to V. Then $M \cap V =$

$\{p \in V \mid x^{m+1}(p) = x^{m+2}(p) = \dots = x^n(p) = 0\}$ and $L \cap V =$

$\{p \in V \mid \tilde{x}^{\ell+1}(p) = \tilde{x}^{\ell+2}(p) = \dots = \tilde{x}^n(p) = 0\}$. Now consider

$F : V \to \mathbb{R}^{2n-m-\ell}$, $F(p) = (x^{m+1}(p), \dots, x^n(p), \tilde{x}^{\ell+1}(p), \dots, \tilde{x}^n(p))$.

Then $L \cap M \cap V = F^{-1}(0, \dots, 0)$. Claim: $(0, 0, \dots, 0)$ is a regular value of F. Indeed, $\forall q \in L \cap M \cap V$, $\xi \in T_q N$

$(DF_q)\xi = (dx^{m+1}|_q(\xi), \dots, dx^n|_q(\xi), d\tilde{x}^{\ell+1}|_q(\xi), \dots, d\tilde{x}^n|_q(\xi))$

If $(DF_q)\xi = 0$, then $dx^i|_q(\xi) = 0$ $m+1 \le i \le n$ and $d\tilde{x}^j|_q(\xi) = 0$ $\ell+1 \le j \le n$, so $\xi \in T_q M$ and $\xi \in T_q L$.

And obviously if $\xi \in T_q M \cap T_q L$ then $(DF_q)\xi = 0$.

So $\ker DF_q = T_q M \cap T_q L$. Now, $T_q M + T_q L = T_q N$

So $m + \ell - \dim(T_q M \cap T_q L) = n$. So $\dim \ker DF_q = -n + m + \ell$. DF_q is a map from an n-dimensional space to $\mathbb{R}^{(n-m)+(n-\ell)}$, and since $T_q M + T_q L = T_q N$, $m + \ell \geq n$. So

$\dim \operatorname{Image} DF_q = n - (-n + m + \ell) = 2n - m - \ell$. So DF_q is onto. So $(0, \dots, 0) \in \mathbb{R}^{2n-m-\ell}$ is a regular value, so $F^{-1}(0, \dots, 0) = L \cap M \cap V$ is a manifold of dimension $\dim \ker DF_q = m + \ell - n$. It follows that $L \cap M$ is a submanifold.