Mid-Term Solutions

Geometry/Topology II

Spring 2012

Do three of the following problems. 33 points each.

1. Let M and N be smooth, non-empty manifolds, and let $F : M \to N$ be a smooth submersion. If M is compact and N is connected, show that F is onto, and that N is compact.

As a corollary of the inverse function theorem, any submersion is an open map. In particular, $F(M) \subset N$ is open. On the other hand, since F is continuous and M is compact, $F(M)$ is compact, too; and since N is Hausdorff, it follows that $F(M) \subset N$ is closed. Thus $F(M)$ is a non-empty open and closed subset of N. Since N is connected, it follows that $N = F(M)$. In particular, F is surjective. Since we have already shown that $F(M)$ is compact, it in particular follows that N is compact, as desired.

2. Let $\mathbb{R}^2$ be equipped with its usual (x, y) coordinates, and let φ be the 1-form on $\mathbb{R}^2 - \{0\}$ given by

$$\varphi = \frac{y\ dx - x\ dy}{x^2 + 4y^2} \ .$$

Let $\gamma : [0, 1] \to \mathbb{R}^2$ be the smooth curve defined by

$$\gamma(t) = (2\cos(2\pi t), \sin(2\pi t)) \ .$$

Compute $d\varphi$ and $\int_\gamma \varphi$. Is φ closed? Is it exact?

Let us first notice that

$$\begin{aligned}
d\varphi &= d\left(\frac{y\ dx - x\ dy}{x^2+4y^2}\right) \\
&= d\left(\frac{1}{x^2+4y^2}\right) \wedge (y\ dx - x\ dy) + \frac{1}{x^2+4y^2}\ d(y\ dx - x\ dy) \\
&= -\frac{d(x^2+4y^2)}{(x^2+4y^2)^2} \wedge (y\ dx - x\ dy) - \frac{2\ dx \wedge dy}{x^2+4y^2} \\
&= -\frac{(2x\ dx + 8y\ dy) \wedge (y\ dx - x\ dy)}{(x^2+4y^2)^2} - \frac{2\ dx \wedge dy}{x^2+4y^2} \\
&= \frac{(2x^2+8y^2)\ dx \wedge dy}{(x^2+4y^2)^2} - \frac{2\ dx \wedge dy}{x^2+4y^2} \\
&= \frac{2\ dx \wedge dy}{x^2+4y^2} - \frac{2\ dx \wedge dy}{x^2+4y^2} \\
&= 0.
\end{aligned}$$

This shows that φ is closed. On the other hand,

$$\begin{aligned}
\int_\gamma \varphi &= \int_0^1 \gamma^*\varphi \\
&= \int_0^1 \frac{\sin(2\pi t)\ d\ 2\cos(2\pi t) - 2\cos(2\pi t)\ d\sin(2\pi t)}{(2\cos(2\pi t))^2 + 4(\sin(2\pi t))^2} \\
&= \int_0^1 \frac{(-4\pi\sin^2(2\pi t) - 4\pi\cos^2(2\pi t))\ dt}{4} \\
&= -\pi \int_0^1 dt = -\pi \neq 0\ .
\end{aligned}$$

If φ were exact, there would be a function f with $\varphi = df$, implying that

$$\int_\gamma \varphi = \int_0^1 d(f \circ \gamma) = f(\gamma(1)) - f(\gamma(0)) = 0$$

because $\gamma(0) = \gamma(1) = (2,0)$. This contradiction shows that, although φ is closed, it is not exact.

3. (a) Let N be a smooth n-manifold, and let $F : M \to N$ be a smooth map from another manifold to N. If ω is any n-form on N, show that $F^*\omega$ is closed.

Since N is n-dimensional, $\mathcal{A}^{n+1}(N) = 0$. But we are given that $\omega \in \mathcal{A}^n(N)$, and it follows that $d\omega \in \mathcal{A}^{n+1}(N)$. Hence $d\omega = 0$, and it follows that

$$d(F^*\omega) = F^*d\omega = F^*0 = 0$$

because exterior differentiation commutes with pull-backs. This proves that $F^*\omega$ is closed.

(b) Use part (a), with $N = S^2$, to show that the 2-form

$$\psi = \frac{x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

on $\mathbb{R}^3 - \{0\}$ is closed, without resorting to brute-force calculation.

If X denotes the radially directed vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, then $\psi = \mu(X, \_\_, \_\_)$, where

$$\mu = \frac{dx \wedge dy \wedge dz}{(x^2 + y^2 + z^2)^{3/2}}.$$

It follows that $\psi(X, Y) = \mu(X, X, Y) = 0$ for any Y, since μ is alternating.

Now if

$$\begin{aligned} F : \mathbb{R}^3 - \{0\} &\longrightarrow S^2 \\ (x, y, z) &\longmapsto \frac{(x, y, z)}{\|(x, y, z)\|} \end{aligned}$$

is the radial projection, then X spans the kernel of F_* at each point. Thus, if $j : S^2 \hookrightarrow \mathbb{R}^3 - \{0\}$ is the inclusion of the standard unit 2-sphere, and if we let ω denote the 2-form on S^2 given by $\omega := j^*\psi$, then ψ must agree with $F^*\omega$ at every point of the unit sphere. However, ψ is also invariant under $(x, y, z) \mapsto (\lambda x, \lambda y, \lambda z)$ for any constant $\lambda > 0$, since this substitution multiplies both the numerator and denominator by λ^3; meanwhile, for any $\lambda > 0$, this formula defines a diffeomorphism $\Phi_\lambda : \mathbb{R}^3 - \{0\} \to \mathbb{R}^3 - \{0\}$ such that $F \circ \Phi_\lambda = F$, and it follows that $\Phi_\lambda^*(F^*\omega) = F^*\omega$ for any $\lambda > 0$. Since Φ_λ sends the unit 2-sphere to the 2-sphere of radius λ, it follows that ψ and $F^*\omega$ also agree along the sphere of radius λ, for any $\lambda > 0$. Hence $\psi = F^*\omega$, and part (a) therefore implies that ψ is closed.

4. (a) Let M be a smooth non-empty compact manifold, and suppose that β is a smooth 1-form which is non-zero at every point of M. Show that β is not exact.

Suppose that $\beta = df$ for some smooth function $f : M \to \mathbb{R}$. Then f has a maximum at some point p, and $df = 0$ at p, since otherwise f would have to increase in some direction along some curve through p. But since $\beta = df$ by assumption, this would show that $\beta = 0$ at p, which contradicts our hypothesis. Thus such a 1-form β can never be exact.

(b) Let M be a smooth, non-empty compact manifold which admits a smooth submersion $F : M \to S^1$. Use part (a) of problems 3 and 4 to show that $H^1(M) \neq 0$.

If $F : M \to S^1$ is a submersion, then $F^* : T^*_q S^1 \to T^*_p M$ is an injection, for all $q \in S^1$ and $p \in F^{-1}(\{q\})$. Thus, if we choose a 1-form ϕ on S^1 which is nowhere zero, its pull-back $\beta := F^*\phi$ will also be non-zero everywhere. If M is compact, then, by part (a) above, $\beta = F^*\phi$ cannot be exact. On the other hand, by part (a) of problem 3, $F^*\phi$ is closed:

$$d(F^*\phi) = F^*d\phi = F^*0 = 0$$

Thus, if we can find a nowhere-zero 1-form ϕ on S^1, we have produced a closed 1-form on M with is exact, and so proved that

$$H^1(M) = \frac{\{\text{ closed 1-forms on } M\ \}}{\{\text{ exact 1-forms on } M\ \}} \neq 0.$$

Now there is an obvious choice of such a ϕ, and its usual name illustrates how important it is that M has been assumed to be compact. Indeed, the famous form in question is usually misleadingly denoted by "$d\theta$." Here the the angle function θ is not actually defined on S^1, but rather only on its universal cover $\mathbb{R}$. Notice that, by part (a), this form is certainly *not* exact on S^1. However, its pull-back to $\mathbb{R}$ *is* exact — "upstairs," it simply becomes the differential of the standard coordinate. Notice that the covering map $\mathbb{R} \to S^1$ is a submersion, but this does not contradict our conclusion, because $\mathbb{R}$ is non-compact.

Alternatively, you might cite the fact that any orientable n-manifold N admits a nowhere-zero n-form ϕ; we proved this by a partition-of-unity argument. The fact that S^1 admits a non-zero 1-form ϕ is thus precisely equivalent to the fact that the circle is orientable.

5. Let N be a smooth n-manifold, and let $L \subset N$ and $M \subset N$ be smooth submanifolds, of dimensions ℓ and m, respectively. We say that L and M are *transverse* submanifolds if, at every $p \in L \cap M$, the tangent spaces T_pL and T_pM together span T_pN:

$$T_pN = T_pL + T_pM \quad \forall p \in L \cap M.$$

If L and M are transverse and are not disjoint, show that their intersection $L \cap M$ is a submanifold. What is its dimension?

(**Hint:** in a neighborhood of p, $L \cap M$ us characterized by the vanishing of a finite collection of smooth real functions with independent differentials.)

Set $k_1 = n - \ell$ and $k_2 = n - m$; these numbers are called the *codimensions* of L and M, respectively. Near a point p of $L \cap M$, we can find coordinates $(x^1, \dots, x^\ell, u^1, \dots, u^{k_1})$ on an open set $\mathcal{U} \subset N$, $p \in \mathcal{U}$, such that $L \cap \mathcal{U}$ is given by the equations $u^1 = \cdots = u^{k_1} = 0$; similarly, we can find a coordinate system $(y^1, \dots, y^m, v^1, \dots, v^{k_2})$ on an open set $\mathcal{V} \subset N$, $p \in \mathcal{V}$, such that $M \cap \mathcal{V}$ is given by the equations $v^1 = \cdots = v^{k_2} = 0$. Now consider the neighborhood of $p \in N$ defined by $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$, together with the smooth map $F : \mathcal{W} \to \mathbb{R}^{k_1+k_2}$ defined by $(u^1, \dots, u^{k_1}, v^1, \dots, v^{k_2})$. Since L and M are transverse, $T_pN \cong (T_pL \oplus T_pM)/(T_pL \cap T_pM)$, and it follows that the tautological projection

$$T_pN \to (T_pN/T_pL) \oplus (T_pN/T_pM)$$

is onto. Hence $du^1, \cdots, du^{k_1}, dv^1, \cdots, dv^{k_2}$ are linearly independent at p, and the derivative of $F : \mathcal{W} \to \mathbb{R}^{k_1+k_2}$ at p is therefore surjective. Thus, as a corollary of the inverse function theorem, F restricts to some smaller neighborhood $\mathcal{W}'$ of p as a submersion, and the equations $u^1 = \cdots = u^{k_1} = v^1 = \cdots = v^{k_2} = 0$ define a submanifold of dimension

$$n - (k_1 + k_2) = n - (n - \ell) - (n - m) = \ell + m - n$$

of the open subset $\mathcal{W}' \subset N$. However, this submanifold exactly consists of points of $\mathcal{W}'$ which solve the equations $u^1 = \cdots = u^{k_1} = 0$ and also solve the equations $v^1 = \cdots = v^{k_2} = 0$; in other words, it is exactly the subset $(L \cap \mathcal{W}') \cap (M \cap \mathcal{W}') = (L \cap M) \cap \mathcal{W}'$. We have therefore shown that $L \cap M$ is a submanifold near each of its points p, and is therefore a smooth submanifold. Moreover, our argument shows that the dimension of $L \cap M$ is $\ell + m - n$.