Do three of the following problems. 33 points each.

1. Let $M$ and $N$ be smooth, non-empty manifolds, and let $F : M \to N$ be a smooth submersion. If $M$ is compact and $N$ is connected, show that $F$ is onto, and that $N$ is compact.

As a corollary of the inverse function theorem, any submersion is an open map. In particular, $F(M) \subset N$ is open. On the other hand, since $F$ is continuous and $M$ is compact, $F(M)$ is compact, too; and since $N$ is Hausdorff, it follows that $F(M) \subset N$ is closed. Thus $F(M)$ is a non-empty open and closed subset of $N$. Since $N$ is connected, it follows that $N = F(M)$. In particular, $F$ is surjective. Since we have already shown that $F(M)$ is compact, it in particular follows that $N$ is compact, as desired.

2. Let $\mathbb{R}^2$ be equipped with its usual $(x, y)$ coordinates, and let $\varphi$ be the 1-form on $\mathbb{R}^2 - \{0\}$ given by

$$
\varphi = \frac{y \, dx - x \, dy}{x^2 + 4y^2}.
$$

Let $\gamma : [0, 1] \to \mathbb{R}^2$ be the smooth curve defined by

$$
\gamma(t) = (2 \cos(2\pi t), \sin(2\pi t))\, .
$$

Compute $d\varphi$ and $\int_\gamma \varphi$. Is $\varphi$ closed? Is it exact?
Let us first notice that
\[ d\varphi = d \left( \frac{y \, dx - x \, dy}{x^2 + 4y^2} \right) = d \left( \frac{1}{x^2 + 4y^2} \right) \wedge (y \, dx - x \, dy) + \frac{1}{x^2 + 4y^2} \, d(y \, dx - x \, dy) \]
\[ = - \frac{d(x^2 + 4y^2)}{(x^2 + 4y^2)^2} \wedge (y \, dx - x \, dy) - \frac{2 \, dx \wedge dy}{x^2 + 4y^2} \]
\[ = - \frac{(2x \, dx + 8y \, dy) \wedge (y \, dx - x \, dy)}{(x^2 + 4y^2)^2} - \frac{2 \, dx \wedge dy}{x^2 + 4y^2} \]
\[ = \frac{2 \, dx \wedge dy}{x^2 + 4y^2} - \frac{2 \, dx \wedge dy}{x^2 + 4y^2} \]
\[ = 0. \]

This shows that \( \varphi \) is closed. On the other hand,
\[
\int_\gamma \varphi = \int_0^1 \gamma^* \varphi = \int_0^1 \sin(2\pi t) \, d \left( 2 \cos(2\pi t) - 2 \cos(2\pi t) \sin(2\pi t) \frac{d}{(2 \cos(2\pi t))^2 + 4(\sin(2\pi t))^2} \right) = \int_0^1 \frac{-4\pi \sin^2(2\pi t) - 4\pi \cos^2(2\pi t)}{4} \, dt = -\pi \int_0^1 dt = -\pi \neq 0.
\]

If \( \varphi \) were exact, there would be a function \( f \) with \( \varphi = df \), implying that
\[
\int_\gamma \varphi = \int_0^1 d(f \circ \gamma) = f(\gamma(1)) - f(\gamma(0)) = 0
\]
because \( \gamma(0) = \gamma(1) = (2, 0) \). This contradiction shows that, although \( \varphi \) is closed, it is not exact.
3. (a) Let $N$ be a smooth $n$-manifold, and let $F : M \to N$ be a smooth map from another manifold to $N$. If $\omega$ is any $n$-form on $N$, show that $F^* \omega$ is closed.

Since $N$ is $n$-dimensional, $\mathcal{A}^{n+1}(N) = 0$. But we are given that $\omega \in \mathcal{A}^n(N)$, and it follows that $d\omega \in \mathcal{A}^{n+1}(N)$. Hence $d\omega = 0$, and it follows that

$$d(F^* \omega) = F^* d\omega = F^* 0 = 0$$

because exterior differentiation commutes with pull-backs. This proves that $F^* \omega$ is closed.

(b) Use part (a), with $N = S^2$, to show that the 2-form

$$\psi = \frac{x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

on $\mathbb{R}^3 - \{0\}$ is closed, without resorting to brute-force calculation.

If $X$ denotes the radially directed vector field $X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$, then $\psi = \mu(X, \_, \_)$, where

$$\mu = \frac{dx \wedge dy \wedge dz}{(x^2 + y^2 + z^2)^{3/2}}.$$ 

It follows that $\psi(X, Y) = \mu(X, X, Y) = 0$ for any $Y$, since $\mu$ is alternating.

Now if $F : \mathbb{R}^3 - \{0\} \to S^2$

$$(x, y, z) \mapsto \frac{(x, y, z)}{\|(x, y, z)\|}$$

is the radial projection, then $X$ spans the kernel of $F_*$ at each point. Thus, if $j : S^2 \hookrightarrow \mathbb{R}^3 - \{0\}$ is the inclusion of the standard unit 2-sphere, and if we let $\omega$ denote the 2-form on $S^2$ given by $\omega := j^* \psi$, then $\psi$ must agree with $F^* \omega$ at every point of the unit sphere. However, $\psi$ is also invariant under $(x, y, z) \mapsto (\lambda x, \lambda y, \lambda z)$ for any constant $\lambda > 0$, since this substitution multiplies both the numerator and denominator by $\lambda^3$; meanwhile, for any $\lambda > 0$, this formula defines a diffeomorphism $\Phi_\lambda : \mathbb{R}^3 - \{0\} \to \mathbb{R}^3 - \{0\}$ such that $F \circ \Phi_\lambda = F$, and it follows that $\Phi_\lambda^*(F^* \omega) = F^* \omega$ for any $\lambda > 0$. Since $\Phi_\lambda$ sends the unit 2-sphere to the 2-sphere of radius $\lambda$, it follows that $\psi$ and $F^* \omega$ also agree along the sphere of radius $\lambda$, for any $\lambda > 0$. Hence $\psi = F^* \omega$, and part (a) therefore implies that $\psi$ is closed.
4. (a) Let $M$ be a smooth non-empty compact manifold, and suppose that $\beta$ is a smooth 1-form which is non-zero at every point of $M$. Show that $\beta$ is not exact.

Suppose that $\beta = df$ for some smooth function $f : M \to \mathbb{R}$. Then $f$ has a maximum at some point $p$, and $df = 0$ at $p$, since otherwise $f$ would have to increase in some direction along some curve through $p$. But since $\beta = df$ by assumption, this would show that $\beta = 0$ at $p$, which contradicts our hypothesis. Thus such a 1-form $\beta$ can never be exact.

(b) Let $M$ be a smooth, non-empty compact manifold which admits a smooth submersion $F : M \to S^1$. Use part (a) of problems 3 and 4 to show that $H^1(M) \neq 0$.

If $F : M \to S^1$ is a submersion, then $F^* : T_q^* S^1 \to T_p^* M$ is an injection, for all $q \in S^1$ and $p \in F^{-1}(\{q\})$. Thus, if we choose a 1-form $\phi$ on $S^1$ which is nowhere zero, its pull-back $\beta := F^* \phi$ will also be non-zero everywhere. If $M$ is compact, then, by part (a) above, $\beta = F^* \phi$ cannot be exact. On the other hand, by part (a) of problem 3, $F^* \phi$ is closed:

$$d(F^* \phi) = F^* d\phi = F^* 0 = 0$$

Thus, if we can find a nowhere-zero 1-form $\phi$ on $S^1$, we have produced a closed 1-form on $M$ with is exact, and so proved that

$$H^1(M) = \left\{ \text{closed 1-forms on } M \right\} \neq 0.$$

Now there is an obvious choice of such a $\phi$, and its usual name illustrates how important it is that $M$ has been assumed to be compact. Indeed, the famous form in question is usually misleadingly denoted by “$d\theta$.” Here the the angle function $\theta$ is not actually defined on $S^1$, but rather only on its universal cover $\mathbb{R}$. Notice that, by part (a), this form is certainly not exact on $S^1$. However, its pull-back to $\mathbb{R}$ is exact — “upstairs,” it simply becomes the differential of the standard coordinate. Notice that the covering map $\mathbb{R} \to S^1$ is a submersion, but this does not contradict our conclusion, because $\mathbb{R}$ is non-compact.

Alternatively, you might cite the fact that any orientable $n$-manifold $N$ admits a nowhere-zero $n$-form $\phi$; we proved this by a partition-of-unity argument. The fact that $S^1$ admits a non-zero 1-form $\phi$ is thus precisely equivalent to the fact that the circle is orientable.
Let $N$ be a smooth $n$-manifold, and let $L \subset N$ and $M \subset N$ be smooth submanifolds, of dimensions $\ell$ and $m$, respectively. We say that $L$ and $M$ are transverse submanifolds if, at every $p \in L \cap M$, the tangent spaces $T_p L$ and $T_p M$ together span $T_p N$:

$$T_p N = T_p L + T_p M \quad \forall p \in L \cap M.$$

If $L$ and $M$ are transverse and are not disjoint, show that their intersection $L \cap M$ is a submanifold. What is its dimension?

(Hint: in a neighborhood of $p$, $L \cap M$ is characterized by the vanishing of a finite collection of smooth real functions with independent differentials.)

Set $k_1 = n - \ell$ and $k_2 = n - m$; these numbers are called the codimensions of $L$ and $M$, respectively. Near a point $p$ of $L \cap M$, we can find coordinates $(x^1, \ldots, x^\ell, u^1, \ldots, u^{k_1})$ on an open set $U \subset N$, $p \in U$, such that $L \cap U$ is given by the equations $u^1 = \cdots = u^{k_1} = 0$; similarly, we can find a coordinate system $(y^1, \ldots, y^m, v^1, \ldots, v^{k_2})$ on an open set $V \subset N$, $p \in V$, such that $M \cap V$ is given by the equations $v^1 = \cdots = v^{k_2} = 0$. Now consider the neighborhood of $p \in N$ defined by $W = U \cap V$, together with the smooth map $F : W \to \mathbb{R}^{k_1 + k_2}$ defined by $(u^1, \ldots, u^{k_1}, v^1, \ldots, v^{k_2})$. Since $L$ and $M$ are transverse, $T_p N \cong (T_p L \oplus T_p M)/(T_p L \cap T_p M)$, and it follows that the tautological projection

$$T_p N \to (T_p N/T_p L) \oplus (T_p N/T_p M)$$

is onto. Hence $du^1, \ldots, du^{k_1}, dv^1, \ldots, dv^{k_2}$ are linearly independent at $p$, and the derivative of $F : W \to \mathbb{R}^{k_1 + k_2}$ at $p$ is therefore surjective. Thus, as a corollary of the inverse function theorem, $F$ restricts to some smaller neighborhood $W'$ of $p$ as a submersion, and the equations $u^1 = \cdots = u^{k_1} = v^1 = \cdots = v^{k_2} = 0$ define a submanifold of dimension $n - (k_1 + k_2) = n - (n - \ell) - (n - m) = \ell + m - n$ of the open subset $W' \subset N$. However, this submanifold exactly consists of points of $W'$ which solve the equations $u^1 = \cdots = u^{k_1} = 0$ and also solve the equations $v^1 = \cdots = v^{k_2} = 0$; in other words, it is exactly the subset $(L \cap W') \cap (M \cap W') = (L \cap M) \cap W'$. We have therefore shown that $L \cap M$ is a submanifold near each of its points $p$, and is therefore a smooth submanifold. Moreover, our argument shows that the dimension of $L \cap M$ is $\ell + m - n$. 

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