Mid-Term Solutions

Geometry/Topology II

Spring 2009

Do four of the following problems. 25 points each.

1. Recall that a smooth n-manifold N has been defined to be orientable iff it admits a coordinate atlas for which every coordinate change

$$(x^1, \ldots, x^n) \longmapsto (y^1, \ldots, y^n)$$

has positive Jacobian determinant:

$$\det \left[\frac{\partial y^j}{\partial x^k} \right] > 0.$$

Prove that N is orientable (by this definition) iff there exists a smooth n-form $\omega \in \Omega^n(N)$ such that $\omega \neq 0$ everywhere. Then show that this happens iff the rank-1 vector bundle $\Lambda^n \to N$ is trivial.

A non-zero n-form ω becomes

$$\omega = f \; dx^1 \wedge \cdots \wedge dx^n$$

in any coordinate chart for some $f \neq 0$; moreover, we can change the sign of f by, for instance, replacing x^1 with $-x^1$. Now if $(y^1, \ldots, y^n)$ is another coordinate chart, and if

$$\omega = h \; dy^1 \wedge \cdots \wedge dy^n$$

then, on the overlap region,

$$\det \left[\frac{\partial y^j}{\partial x^k} \right] = \frac{f}{h}$$

so any two charts in which ω has positive coefficient function are orientation-compatible. The collection of all such charts is therefore an oriented atlas for M.

Conversely, suppose that $\{(x^1_\alpha, \dots x^n_\alpha) : U_\alpha \to \mathbb{R}^n\}$ is an oriented smooth atlas for M. Let $\{f_\alpha\}$ be a smooth partition of unity subordinate to the cover U_α; thus, each smooth function $f_\alpha : M \to$ is non-negative, is supported in U_α, and only finitely many f_α's are non-zero on some neighborhood of any point. We may therefore set

$$\omega = \sum_\alpha f_\alpha dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha.$$

Because

$$\det \left[\frac{\partial x^j_\alpha}{\partial x^k_\beta} \right] > 0 \quad \forall \alpha, \beta,$$

each summand is then non-negative in each chart of our oriented atlas, and since at least one summand is actually positive at any given point, we will then have $\omega \neq 0$ everywhere on M.

On the other hand, a rank-1 vector bundle $E \to M$ is trivial iff it has a nowhere-zero section. Indeed, if $\sigma : M \to E$ is a non-zero section, then the map $M \times \mathbb{R} \to E$ given by $(p, t) \mapsto t\sigma(p)$ is an isomorphism between the trivial bundle and E; conversely, since $p \mapsto (p, 1)$ is a non-zero section of the trivial bundle $M \times \mathbb{R} \to M$, any bundle isomorphic to this trivial bundle has a non-zero section. Applying this the rank-1 bundle $\Lambda^n \to M$ thus shows that Λ^n is trivial iff M carries a nowhere zero n-form; and the previous argument then shows that this is in turn equivalent to the orientability of the n-manifold M.

2. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function, and let $\Sigma := f^{-1}(0)$. If $df \neq 0$ at each point $p \in \Sigma$, prove that Σ is a smooth orientable n-manifold.

To show that Σ is a smooth manifold, one uses the inverse function theorem. Namely, near any point $p \in \Sigma$, there must some j such that $\partial f / \partial x^j \neq 0$. One then considers the map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by $(f, x^1, \dots, \widehat{x^j}, \dots, x^{n+1})$, where the "hat" means "omit this entry." The Jacobian determinant of this map is then $(-1)^{j+1} \partial f / \partial x^j \neq 0$, so the inverse function theorem guarantees that this a local diffeomorphism near p in $\mathbb{R}^{n+1}$, and sends a neighborhood of

p in Σ to a relatively open set in $\{0\} \times \mathbb{R}^n$. This provides the charts necessary to make Σ a smooth n-manifold.

It remains to show that Σ is orientable. While this can be done in various ways, a particularly nice method is to invoke Problem 1, and show that Σ admits an n-form which is everywhere non-zero. One such form can be gotten by considering the vector field

$$X = \operatorname{grad} f = \sum_{j=1}^{n+1} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^j}$$

on $\mathbb{R}^{n+1}$, and then considering the n-form

$$\begin{aligned} \omega &= X \lrcorner\, dx^1 \wedge \cdots \wedge dx^{n+1} \\ &= \frac{\partial f}{\partial x^1} dx^2 \wedge \cdots \wedge dx^{n+1} + \cdots + (-1)^n \frac{\partial f}{\partial x^{n+1}} dx^1 \wedge \cdots \wedge dx^n \end{aligned}$$

on $\mathbb{R}^{n+1}$. The pull-back of ω to $\Sigma \hookrightarrow \mathbb{R}^{n+1}$ is then non-zero everywhere, because, for example,

$$df \wedge \omega = \left[\sum_{j=1}^{n+1} \left(\frac{\partial f}{\partial x^j} \right)^2 \right] dx^1 \wedge \cdots \wedge dx^{n+1}$$

is non-zero in a neighborhood of Σ.

3. Suppose that $\varpi : E \to M$ is a smooth rank-1 vector bundle over a manifold M, and let $\mathbf{0}_M \subset E$ denote the image of the zero section of E. If $E - \mathbf{0}_M$ is connected, prove that every section $M \to E$ must have a zero. Then use this to prove that the Möbius band is a non-orientable 2-manifold.

Suppose that $\varpi : E \to M$ is a smooth rank-1 vector bundle. If there were a section $\sigma : M \to E$ with $\sigma \neq 0$, we could define a smooth map $F : M \times \mathbb{R} \to E$ by

$$F(p, t) = t\sigma(x)$$

and inspection then reveals that this map would be an isomorphism of vector bundles. In particular, $E - \mathbf{0}_M \approx M \times (\mathbb{R} - \{0\})$ could not be connected.

By contraposition, this shows that if $E - \mathbf{0}_M$ is connected, then any section σ of E must have a zero.

Now let us show the Möbius band B is not-orientable. If not, by Problem 1, there would be a non-zero 2-form on B. Thus, if $\Lambda^2 \to B$ is the bundle of 2-forms, and if $E \to S^1$ denotes its restriction to the middle S^1, then the rank-1 bundle $E \to S^1$ would have to admit a a non-zero section, and $E - \mathbf{0}_{S^1}$ would have to be disconnected. But $E \to S^1$ is bundle-isomorphic to B itself, thought of as a bundle over the circle! (For example, contraction of 2-forms with the vector field $\partial/\partial\theta$ on S^1 identifies E with B^*, and B^* can then be identified with B by using the obvious inner product on fibers.) Thus, if B were orientable as a 2-manifold, removing the middle S^1 would have to disconnect B. But in fact $B - \mathbf{0}_{S^1}$ deform-retracts to a circle, and so in particular is connected. This contradiction proves that B is not orientable.

4. Consider the vector fields X and Y on $\mathbb{R}^3$ defined by

$$\begin{aligned} X &= x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} \\ Y &= x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1} \end{aligned}$$

(a) Explicitly find the flows generated by X and Y.

The flow generated by X is explicitly given by

$$\phi_t(x^1, x^2, x^3) = (e^t x^1, e^t x^2, e^t x^3)$$

while the flow of Y is given by

$$\psi_t(x^1, x^2, x^3) = (x^1\cos t - x^2 \sin t, x^1 \sin t + x^2 \cos t, x^3).$$

(b) Compute $[X, Y]$.

$$\begin{aligned} [X,Y] &= XY - YX \\ &= X(x^1)\frac{\partial}{\partial x^2} - X(x^2)\frac{\partial}{\partial x^1} - \left[Y(x^1)\frac{\partial}{\partial x^1} + Y(x^2)\frac{\partial}{\partial x^2} + Y(x^3)\frac{\partial}{\partial x^3}\right] \\ &= x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1} - \left[-x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}\right] \\ &= 0. \end{aligned}$$

(c) Explain why your answers to (a) and (b) are related, and consistent.

$[X, Y] = 0 \iff \phi_t$ commutes with ψ_s $\forall s, t$. But indeed,

$$\psi_s \phi_t(x^1, x^2, x^3) = (e^t x^1 \cos s - e^t x^2 \sin s, e^t x^1 \sin s + e^t x^2 \cos s, e^t x^3)$$

while

$$\phi_t \psi_s(x^1, x^2, x^3) = (e^t(x^1 \cos s - x^2 \sin s), e^t(x^1 \sin s + x^2 \cos s), e^t x^3),$$

so they do commute, as predicted.

5. (a) Let φ be any smooth differential form on an open subset of $\mathbb{R}^n$, and let Y denote the vector field $\partial/\partial x^1$. Working directly from the definition of the Lie derivative, show that

$$\mathcal{L}_Y \varphi = Y \lrcorner\, d\varphi + d(Y \lrcorner\, \varphi). \tag{1}$$

Any k-form φ on $\mathbb{R}^n$ is a finite sum of terms

$$f(x^1, \dots, x^n) dx^I$$

and terms

$$h(x^1, \dots, x^n) dx^1 \wedge dx^J$$

where I and J are multi-indices with $I, J \subset \{2, \dots, n\}$, $|I| = n$, $|J| = n - 1$. Since both sides are linear in φ, we merely need check (1) for forms of this simple type.

Now the flow of $Y = \partial/\partial x^1$ is explicitly given by

$$\psi_t(x^1, x^2, \dots, x^n) = (x^1 + t, x^2, \dots, x^n),$$

so

$$\mathcal{L}_Y(f\ dx^I) = \frac{\partial f}{\partial x^1} dx^1 \wedge dx^I,$$

whereas

$$\begin{aligned} Y \lrcorner\, d(f dx^I) + d(Y \lrcorner\, f dx^I) &= Y \lrcorner \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \right) + d(0) \\ &= \frac{\partial f}{\partial x^1} dx^1 \wedge dx^I, \end{aligned}$$

so (1) holds for these cases. On the other hand,

$$\mathcal{L}_Y(h\ dx^1 \wedge dx^J) = \frac{\partial h}{\partial x^1} dx^1 \wedge dx^J,$$

while

$$\begin{aligned} Y \lrcorner\ d(h\ dx^1 \wedge dx^J) + d(Y \lrcorner\ \ h\ \ dx^1 \wedge dx^J) \\ &= \frac{\partial}{\partial x^1} \lrcorner \left(\sum_{j=2}^{n} \frac{\partial h}{\partial x^i} dx^j \wedge dx^1 \wedge dx^J \right) + d(h\ dx^J) \\ &= -\sum_{j=2}^{n} \frac{\partial h}{\partial x^j} dx^j \wedge dx^J + \sum_{i=1}^{n} \frac{\partial h}{\partial x^i} dx^i \wedge dx^J \\ &= \frac{\partial h}{\partial x^1} dx^1 \wedge dx^J \end{aligned}$$

and (1) therefore holds for terms of both types, as claimed.

(b) Show that (1) holds if $Y \equiv 0$ and φ any differential form.

If $Y \equiv 0$, then its flow is the identity, and $\mathcal{L}_Y \varphi = 0$ for any φ. But in this case, the right-hand side of (1) vanishes, too, and the formula therefore holds.

(c) If Y is any smooth vector field and φ is any smooth differential form on a smooth manifold M, use (a) and (b) to show that (1) holds on all of M by first showing that it holds on a dense open subset of M.

Let $U \subset M$ be the set where $Y \neq 0$, and let $V \subset M$ be the *interior* of the set where $Y = 0$. Thus $M = \overline{U} \cup V$. However, every point of U has a neighborhood on which we can find a coordinate system in which $Y = \partial/\partial x^1$; and every point of V has a neighborhood on which $Y \equiv 0$. Hence (1) holds on U by part (a), and (1) holds on V by part (b). Thus, the difference of the two sides of (1) vanishes on $U \cup V$. By continuity, it therefore also vanishes on $\overline{U \cup V} = M$, and we have therefore shown that (1) holds everywhere.

6. Let Y be the vector field on $\mathbb{R}^3$ appearing in problem 5 (Misprint! Should have read problem 4!), and let

$$\varphi = x^1 dx^2 \wedge dx^3.$$

Compute the Lie derivative $\mathcal{L}_Y\varphi$ in two ways:

(a) from the definition; and

Using the flow found in Problem 4,

$$\begin{aligned}
\psi_t^*\varphi &= (x^1\cos t - x^2\sin t)d(x^1\sin t + x^2\cos t)\wedge dx^3 \\
&= (x^1\cos t - x^2\sin t)(\sin t dx^1 + \cos t dx^2)\wedge dx^3 \\
&= (\frac{1}{2}x^1\sin 2t - x^2\sin^2 t)dx^1\wedge dx^3 + (x^1\cos^2 t - \frac{1}{2}x^2\sin 2t)dx^2\wedge dx^3
\end{aligned}$$

Thus

$$\frac{d}{dt}\psi_t^*\varphi = (x^1\cos 2t - 2x^2\sin t\cos t)dx^1\wedge dx^3 - (2x^1\cos t\sin t + x^2\cos 2t)dx^2\wedge dx^3$$

and hence

$$\mathcal{L}_Y\varphi = \frac{d}{dt}\psi_t^*\varphi|_{t=0} = x^1dx^1\wedge dx^3 - x^2dx^2\wedge dx^3$$

(b) by means of (1).

$$\begin{aligned}
\mathcal{L}_Y\varphi &= Y \lrcorner\, d\varphi + d(Y \lrcorner\, \varphi) \\
&= \left(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\right) \lrcorner\, dx^1\wedge dx^2\wedge dx^3 + d\left[\left(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\right) \lrcorner\, x^1dx^2\wedge dx^3\right] \\
&= -x^1dx^1\wedge dx^3 - x^2dx^2\wedge dx^3 + d\left[(x^1)^2dx^3\right] \\
&= -x^1dx^1\wedge dx^3 - x^2dx^2\wedge dx^3 + 2x^1dx^1\wedge dx^3 \\
&= x^1dx^1\wedge dx^3 - x^2dx^2\wedge dx^3
\end{aligned}$$

Then verify that your two answers agree.

My apologies for the confusion caused by the misprint!

7. Let M be a smooth n-manifold, and let $\omega \in \Omega^n(M)$ be a smooth n-form on M. Let $p \in M$ be any point such that $\omega \neq 0$ at p. Prove that there is a coordinate system $(x^1, \ldots, x^n)$ on a neighborhood U of p in which

$$\omega = dx^1\wedge dx^2\wedge\cdots\wedge dx^n.$$

In an arbitrary coordinate system in which p corresponds to the origin,

$$\omega = f(y^1, y^2, \ldots, y^n) dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n.$$

Setting

$$x^1 = \int_0^{y^1} f(t, y^2, \ldots, y^n) dt,$$

we then have

$$dx^1 = f \ dy^1 + \text{ terms involving } dy^2, \ldots, dy^n$$

and so

$$\omega = dx^1 \wedge dy^2 \wedge \cdots \wedge dy^n.$$

Setting $y^j = x^j$ for $j = 2, \ldots, n$, we thus have

$$\omega = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

Moreover, the inverse function theorem then guarantees that $(x^1, \ldots, x^n)$ is a coordinates system in some neighborhood of p, since

$$\det \left[\frac{\partial x^j}{\partial y^k} \right] = f \neq 0$$

at the origin.

8. Let $\psi \in \Omega^{n-1}(M)$ be a smooth $(n-1)$-form on a smooth n-manifold M.

(a) Let $(y^1, \ldots, y^n) : U \to \mathbb{R}^n$ be any smooth coordinate system on M. Prove that there is a unique smooth vector field X on U such that

$$\psi = X \lrcorner (dy^1 \wedge \cdots \wedge dy^n).$$

The general $(n-1)$-form on U is given by

$$\psi = \sum_{j=1}^{n} f_j \ dy^1 \wedge \cdots \wedge \widehat{dy^j} \wedge \cdots \wedge dy^n$$

where the “hat” means “omit this term.” Demanding that

$$X = \sum_j X^j \frac{\partial}{\partial y^j}$$

satisfy

$$\psi = X \lrcorner (dy^1 \wedge \cdots \wedge dy^n).$$

is thus equivalent to requiring that

$$X^j = (-1)^j f_j$$

(b) Then use this to prove that, near any $p \in M$ where $\psi \neq 0$, one can choose coordinates $(x^1, \ldots, x^n)$ for which

$$\psi = f \ dx^2 \wedge \cdots \wedge dx^n$$

for some smooth positive function $f(x^1, \ldots, x^n)$.

At any such p, and for any coordinate system about p, one has $X|_p \neq 0$. Now choose a new coordinate system about p in which $X = \partial/\partial x^1$. Since

$$dy^1 \wedge \cdots \wedge dy^n = f \ dx^1 \wedge \cdots \wedge dx^n$$

in these coordinates, for some $f \neq 0$ near p, it therefore follows that

$$\begin{aligned} \psi &= X \lrcorner (dy^1 \wedge \cdots \wedge dy^n) \\ &= \frac{\partial}{\partial x^1} \lrcorner (f \ dx^1 \wedge \cdots \wedge dx^n) \\ &= f \ dx^2 \wedge \cdots \wedge dx^n, \end{aligned}$$

as claimed