Mid-Term Solutions

Geometry/Topology II

Spring 2009

Do four of the following problems. 25 points each.

1. Recall that a smooth $n$-manifold $N$ has been defined to be orientable iff it admits a coordinate atlas for which every coordinate change

$$(x^1, \ldots, x^n) \mapsto (y^1, \ldots, y^n)$$

has positive Jacobian determinant:

$$\det \left[ \frac{\partial y^j}{\partial x^k} \right] > 0.$$

Prove that $N$ is orientable (by this definition) iff there exists a smooth $n$-form $\omega \in \Omega^n(N)$ such that $\omega \neq 0$ everywhere. Then show that this happens iff the rank-1 vector bundle $\Lambda^n \to N$ is trivial.

A non-zero $n$-form $\omega$ becomes

$$\omega = f \, dx^1 \wedge \cdots \wedge dx^n$$

in any coordinate chart for some $f \neq 0$; moreover, we can change the sign of $f$ by, for instance, replacing $x^1$ with $-x^1$. Now if $(y^1, \ldots, y^n)$ is another coordinate chart, and if

$$\omega = h \, dy^1 \wedge \cdots \wedge dy^n$$

then, on the overlap region,

$$\det \left[ \frac{\partial y^j}{\partial x^k} \right] = \frac{f}{h}.$$
so any two charts in which \( \omega \) has positive coefficient function are orientation-compatible. The collection of all such charts is therefore an oriented atlas for \( M \).

Conversely, suppose that \( \{(x^1_\alpha, \ldots x^n_\alpha) : U_\alpha \to \mathbb{R}^n\} \) is an oriented smooth atlas for \( M \). Let \( \{f_\alpha\} \) be a smooth partition of unity subordinate to the cover \( U_\alpha \); thus, each smooth function \( f_\alpha : M \to \mathbb{R} \) is non-negative, is supported in \( U_\alpha \), and only finitely many \( f_\alpha \)'s are non-zero on some neighborhood of any point. We may therefore set

\[
\omega = \sum_\alpha f_\alpha dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha.
\]

Because

\[
\det \left[ \frac{\partial x^j_\alpha}{\partial x^k_\beta} \right] > 0 \ \forall \alpha, \beta,
\]

each summand is then non-negative in each chart of our oriented atlas, and since at least one summand is actually positive at any given point, we will then have \( \omega \neq 0 \) everywhere on \( M \).

On the other hand, a rank-1 vector bundle \( E \to M \) is trivial iff it has a nowhere-zero section. Indeed, if \( \sigma : M \to E \) is a non-zero section, then the map \( M \times \mathbb{R} \to E \) given by \((p, t) \mapsto t\sigma(p)\) is an isomorphism between the trivial bundle and \( E \); conversely, since \( p \mapsto (p, 1) \) is a non-zero section of the trivial bundle \( M \times \mathbb{R} \to M \), any bundle isomorphic to this trivial bundle has a non-zero section. Applying this the rank-1 bundle \( \Lambda^n \to M \) thus shows that \( \Lambda^n \) is trivial iff \( M \) carries a nowhere zero \( n \)-form; and the previous argument then shows that this is in turn equivalent to the orientability of the \( n \)-manifold \( M \).

2. Let \( f : \mathbb{R}^{n+1} \to \mathbb{R} \) be a smooth function, and let \( \Sigma := f^{-1}(0) \). If \( df \neq 0 \) at each point \( p \in \Sigma \), prove that \( \Sigma \) is a smooth orientable \( n \)-manifold.

To show that \( \Sigma \) is a smooth manifold, one uses the inverse function theorem. Namely, near any point \( p \in \Sigma \), there must some \( j \) such that \( \partial f / \partial x^j \neq 0 \). One then considers the map \( \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) given by \((f, x^1, \ldots, \hat{x^j}, \ldots, x^{n+1})\), where the “hat” means “omit this entry.” The Jacobian determinant of this map is then \((-1)^{j+1} \partial f / \partial x^j \neq 0 \), so the inverse function theorem guarantees that this a local diffeomorphism near \( p \) in \( \mathbb{R}^{n+1} \), and sends a neighborhood of
p in Σ to a relatively open set in \( \{ 0 \} \times \mathbb{R}^n \). This provides the charts necessary to make Σ a smooth \( n \)-manifold.

It remains to show that Σ is orientable. While this can be done in various ways, a particularly nice method is to invoke Problem 1, and show that Σ admits an \( n \)-form which is everywhere non-zero. One such form can be gotten by considering the vector field

\[
X = \text{grad } f = \sum_{j=1}^{n+1} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^j}
\]
on \( \mathbb{R}^{n+1} \), and then considering the \( n \)-form

\[
\omega = X \lrcorner \, dx^1 \wedge \cdots \wedge dx^{n+1} = \frac{\partial f}{\partial x^1} dx^2 \wedge \cdots \wedge dx^{n+1} + \cdots + (-1)^n \frac{\partial f}{\partial x^{n+1}} dx^1 \wedge \cdots \wedge dx^n
\]
on \( \mathbb{R}^{n+1} \). The pull-back of \( \omega \) to \( \Sigma \hookrightarrow \mathbb{R}^{n+1} \) is then non-zero everywhere, because, for example,

\[
df \wedge \omega = \left[ \sum_{j=1}^{n+1} \left( \frac{\partial f}{\partial x^j} \right)^2 \right] dx^1 \wedge \cdots \wedge dx^{n+1}
\]
is non-zero in a neighborhood of Σ.

3. Suppose that \( \varpi : E \to M \) is a smooth rank-1 vector bundle over a manifold \( M \), and let \( 0_M \subset E \) denote the image of the zero section of \( E \). If \( E \setminus 0_M \) is connected, prove that every section \( \sigma : M \to E \) must have a zero. Then use this to prove that the Möbius band is a non-orientable 2-manifold.

Suppose that \( \varpi : E \to M \) is a smooth rank-1 vector bundle. If there were a section \( \sigma : M \to E \) with \( \sigma \neq 0 \), we could define a smooth map \( F : M \times \mathbb{R} \to E \) by

\[
F(p, t) = t\sigma(x)
\]
and inspection then reveals that this map would be an isomorphism of vector bundles. In particular, \( E \setminus 0_M \approx M \times (\mathbb{R} \setminus \{0\}) \) could not be connected.

By contraposition, this shows that if \( E \setminus 0_M \) is connected, then any section \( \sigma \) of \( E \) must have a zero.
Now let us show the Möbius band $B$ is not-orientable. If not, by Problem 1, there would be a non-zero 2-form on $B$. Thus, if $\Lambda^2 \to B$ is the bundle of 2-forms, and if $E \to S^1$ denotes its restriction to the middle $S^1$, then the rank-1 bundle $E \to S^1$ would have to admit a non-zero section, and $E - 0_{S^1}$ would have to be disconnected. But $E \to S^1$ is bundle-isomorphic to $B$ itself, thought of as a bundle over the circle! (For example, contraction of 2-forms with the vector field $\partial/\partial \theta$ on $S^1$ identifies $E$ with $B^*$, and $B^*$ can then be identified with $B$ by using the obvious inner product on fibers.) Thus, if $B$ were orientable as a 2-manifold, removing the middle $S^1$ would have to disconnect $B$. But in fact $B - 0_{S^1}$ deform-retracts to a circle, and so in particular is connected. This contradiction proves that $B$ is not orientable.

4. Consider the vector fields $X$ and $Y$ on $\mathbb{R}^3$ defined by

\[
X = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3}, \\
Y = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}.
\]

(a) Explicitly find the flows generated by $X$ and $Y$.

The flow generated by $X$ is explicitly given by

\[
\phi_t(x^1, x^2, x^3) = (e^t x^1, e^t x^2, e^t x^3)
\]

while the flow of $Y$ is given by

\[
\psi_t(x^1, x^2, x^3) = (x^1 \cos t - x^2 \sin t, x^1 \sin t + x^2 \cos t, x^3).
\]

(b) Compute $[X, Y]$.

\[
[X, Y] = XY - YX = X(x^1) \frac{\partial}{\partial x^2} - X(x^2) \frac{\partial}{\partial x^1} - \left[ Y(x^1) \frac{\partial}{\partial x^1} + Y(x^2) \frac{\partial}{\partial x^2} + Y(x^3) \frac{\partial}{\partial x^3} \right]
\]

\[
= x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} - \left[ -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} \right]
\]

\[
= 0.
\]
(c) Explain why your answers to (a) and (b) are related, and consistent.

\[ [X,Y] = 0 \iff \phi_t \text{ commutes with } \psi_s \ \forall s,t. \]

But indeed,

\[ \psi_s \phi_t(x^1, x^2, x^3) = (e^t x^1 \cos s - e^t x^2 \sin s, e^t x^1 \sin s + e^t x^2 \cos s, e^t x^3) \]

while

\[ \phi_t \psi_s(x^1, x^2, x^3) = (e^t(x^1 \cos s - x^2 \sin s), e^t(x^1 \sin s + x^2 \cos s), e^t x^3), \]

so they do commute, as predicted.

5. (a) Let \( \varphi \) be any smooth differential form on an open subset of \( \mathbb{R}^n \), and let \( Y \) denote the vector field \( \partial / \partial x^1 \). Working directly from the definition of the Lie derivative, show that

\[ \mathcal{L}_Y \varphi = Y \lrcorner d\varphi + d(Y \lrcorner \varphi). \]

Any \( k \)-form \( \varphi \) on \( \mathbb{R}^n \) is a finite sum of terms

\[ f(x^1, \ldots, x^n)dx^I \]

and terms

\[ h(x^1, \ldots, x^n)dx^1 \wedge dx^I \]

where \( I \) and \( J \) are multi-indices with \( I, J \subset \{2, \ldots, n\}; |I| = n, |J| = n-1 \). Since both sides are linear in \( \varphi \), we merely need check (1) for forms of this simple type.

Now the flow of \( Y = \partial / \partial x^1 \) is explicitly given by

\[ \psi_t(x^1, x^2, \ldots, x^n) = (x^1 + t, x^2, \ldots, x^n), \]

so

\[ \mathcal{L}_Y(f dx^I) = \frac{\partial f}{\partial x^1} dx^1 \wedge dx^I, \]

whereas

\[ Y \lrcorner d(fd^I) + d(Y \lrcorner fd^I) = Y \lrcorner \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \right) + d(0) \]

\[ = \frac{\partial f}{\partial x^1} dx^1 \wedge dx^I, \]
so (1) holds for these cases. On the other hand,

$$\mathcal{L}_Y(h \, dx^1 \wedge dx^J) = \frac{\partial h}{\partial x^1} \, dx^1 \wedge dx^J,$$

while

$$Y \cdot d(h \, dx^1 \wedge dx^J) + d(Y \cdot h \, dx^1 \wedge dx^J)$$

$$= \frac{\partial}{\partial x^1} \cdot \left( \sum_{j=2}^{n} \frac{\partial h}{\partial x^j} \, dx^j \wedge dx^1 \wedge dx^J \right) + d(h \, dx^J)$$

$$= - \sum_{j=2}^{n} \frac{\partial h}{\partial x^j} \, dx^j \wedge dx^J + \sum_{i=1}^{n} \frac{\partial h}{\partial x^i} \, dx^i \wedge dx^J$$

$$= \frac{\partial h}{\partial x^1} \, dx^1 \wedge dx^J$$

and (1) therefore holds for terms of both types, as claimed.

(b) Show that (1) holds if $Y \equiv 0$ and $\varphi$ any differential form.

If $Y \equiv 0$, then its flow is the identity, and $\mathcal{L}_Y \varphi = 0$ for any $\varphi$. But in this case, the right-hand side of (1) vanishes, too, and the formula therefore holds.

(c) If $Y$ is any smooth vector field and $\varphi$ is any smooth differential form on a smooth manifold $M$, use (a) and (b) to show that (1) holds on all of $M$ by first showing that it holds on a dense open subset of $M$.

Let $U \subset M$ be the set where $Y \neq 0$, and let $V \subset M$ be the interior of the set where $Y = 0$. Thus $M = \overline{U} \cup V$. However, every point of $U$ has a neighborhood on which we can find a coordinate system in which $Y = \partial / \partial x^1$; and every point of $V$ has a neighborhood on which $Y \equiv 0$. Hence (1) holds on $U$ by part (a), and (1) holds on $V$ by part (b). Thus, the difference of the two sides of (1) vanishes on $U \cup V$. By continuity, it therefore also vanishes on $U \cup V = M$, and we have therefore shown that (1) holds everywhere.

6. Let $Y$ be the vector field on $\mathbb{R}^3$ appearing in problem 5 (Misprint! Should have read problem 4!), and let

$$\varphi = x^1 dx^2 \wedge dx^3.$$
Compute the Lie derivative $L_Y \varphi$ in two ways:

(a) from the definition; and

Using the flow found in Problem 4,

$$\psi_t^* \varphi = (x^1 \cos t - x^2 \sin t)d(x^1 \sin t + x^2 \cos t) \wedge dx^3$$
$$= (x^1 \cos t - x^2 \sin t)(\sin tdx^1 + \cos tdx^2) \wedge dx^3$$
$$= \left( \frac{1}{2}x^1 \sin 2t - x^2 \sin^2 t \right)dx^1 \wedge dx^3 + (x^1 \cos^2 t - \frac{1}{2}x^2 \sin 2t)dx^2 \wedge dx^3$$

Thus

$$\frac{d}{dt}\psi_t^* \varphi = (x^1 \cos 2t - 2x^2 \sin t \cos t)dx^1 \wedge dx^3 - (2x^1 \cos t \sin t + x^2 \cos 2t)dx^2 \wedge dx^3$$

and hence

$$L_Y \varphi = \frac{d}{dt}\psi_t^* \varphi|_{t=0} = x^1 dx^1 \wedge dx^3 - x^2 dx^2 \wedge dx^3$$

(b) by means of (1).

$$L_Y \varphi = Y \lrcorner d\varphi + d(Y \lrcorner \varphi)$$
$$= \left( x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} \right) \lrcorner dx^1 \wedge dx^2 \wedge dx^3 + d \left[ \left( x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} \right) \lrcorner x^1 dx^2 \wedge dx^3 \right]$$
$$= -x^1 dx^1 \wedge dx^3 - x^2 dx^2 \wedge dx^3 + d \left[ (x^1)^2 dx^3 \right]$$
$$= -x^1 dx^1 \wedge dx^3 - x^2 dx^2 \wedge dx^3 + 2x^1 dx^1 \wedge dx^3$$
$$= x^1 dx^1 \wedge dx^3 - x^2 dx^2 \wedge dx^3$$

Then verify that your two answers agree.

My apologies for the confusion caused by the misprint!

7. Let $M$ be a smooth $n$-manifold, and let $\omega \in \Omega^n(M)$ be a smooth $n$-form on $M$. Let $p \in M$ be any point such that $\omega \neq 0$ at $p$. Prove that there is a coordinate system $(x^1, \ldots, x^n)$ on a neighborhood $U$ of $p$ in which

$$\omega = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$
In an arbitrary coordinate system in which \( p \) corresponds to the origin,

\[
\omega = f(y^1, y^2, \ldots, y^n)dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n.
\]

Setting

\[
x^1 = \int_0^{y^1} f(t, y^2, \ldots, y^n)dt,
\]

we then have

\[
dx^1 = f \ dy^1 + \text{terms involving } dy^2, \ldots, dy^n
\]

and so

\[
\omega = dx^1 \wedge dy^2 \wedge \cdots \wedge dy^n.
\]

Setting \( y^j = x^j \) for \( j = 2, \ldots, n \), we thus have

\[
\omega = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.
\]

Moreover, the inverse function theorem then guarantees that \( (x^1, \ldots, x^n) \) is a coordinates system in some neighborhood of \( p \), since

\[
\det \left[ \frac{\partial x^j}{\partial y^k} \right] = f \neq 0
\]

at the origin.

8. Let \( \psi \in \Omega^{n-1}(M) \) be a smooth \((n-1)\)-form on a smooth \( n \)-manifold \( M \).

(a) Let \((y^1, \ldots, y^n) : U \to \mathbb{R}^n\) be any smooth coordinate system on \( M \). Prove that there is a unique smooth vector field \( X \) on \( U \) such that

\[
\psi = X \lrcorner (dy^1 \wedge \cdots \wedge dy^n).
\]

The general \((n-1)\)-form on \( U \) is given by

\[
\psi = \sum_{j=1}^n f_j \ dy^1 \wedge \cdots \wedge \widehat{dy^j} \wedge \cdots \wedge dy^n
\]

where the “hat” means “omit this term.” Demanding that

\[
X = \sum_j X^j \frac{\partial}{\partial y^j}
\]
satisfy
\[\psi = X \downarrow (dy^1 \wedge \cdots \wedge dy^n).\]
is thus equivalent to requiring that
\[X^j = (-1)^j f_j\]

(b) Then use this to prove that, near any \(p \in M\) where \(\psi \neq 0\), one can choose coordinates \((x^1, \ldots, x^n)\) for which
\[\psi = f \, dx^2 \wedge \cdots \wedge dx^n\]
for some smooth positive function \(f(x^1, \ldots, x^n)\).

At any such \(p\), and for any coordinate system about \(p\), one has \(X|_p \neq 0\). Now choose a new coordinate system about \(p\) in which \(X = \partial/\partial x^1\). Since
\[dy^1 \wedge \cdots \wedge dy^n = f \, dx^1 \wedge \cdots \wedge dx^n\]
in these coordinates, for some \(f \neq 0\) near \(p\), it therefore follows that
\[
\psi = X \downarrow (dy^1 \wedge \cdots \wedge dy^n)
= \frac{\partial}{\partial x^1} \downarrow (f \, dx^1 \wedge \cdots \wedge dx^n)
= f \, dx^2 \wedge \cdots \wedge dx^n,
\]
as claimed